Computation of all Factorizations in Certain Non-Commutative Rings.

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There exist many classifications of commutative integral domains with respect to factorization properties of their elements. Anderson et al. [AA92, AAZ90, AM96, And97] have coined certain terminology like finite factorization domain (FFD), half factorization domain (HFD), idf-domain and bi-factorization domain (BFD). Furthermore, they studied the connection between all these different types of domains with respect to implication. A recent paper [BS15] generalizes these notions to non-commutative rings, with applications to maximal orders in central simple algebras and the semigroup of non zero-divisors of the ring of $n \times n$ upper triangular matrices over a commutative domain. No algorithm for factorization has been proposed.

We study noncommutative domains with a view towards algorithms for factorizing concrete elements. In a recent publication [BHL14], a generalization of the term finite factorization domain, which applies to non-commutative rings, has been established. Namely, we consider factorizations up to multiplication with central units of the algebra. Necessary conditions on a given ring to be an FFD have also been formulated, leading to the result that among many other, the ubiquitous G-algebras (which are Noetherian domains) are FFDs. As a consequence, the problem formulation "find all distinct factorizations of an element in a G-algebra" becomes viable, since the output is expected to be finite.

An algorithm that finds all possible factorizations for an element in a G-algebra $\mathscr{G}$ with minor assumptions on the underlying field K has been established [HL16]. With this, one is able to generalize other algorithms, which were exclusively used for commuatative rings before. An example is the "Factorized Gröbner Basis" algorithm. However, the structure of the output of this algorithm in the non-commutative case has no direct interpretation as in the commutative case. We conjecture a strong connection to a decomposition of a solution space, when viewing elements in G-algebras as operator equations.

Our presentation will serve as an introduction into non-commutative finite factorization domains, provide an overview of the status quo of the factorization problem for G-algebras and its applications, and finishes with a vision on future exploration possibilities in the area of characterizing factorization properties in non-commutative integral domains.

References

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