Polylogarithms at the multi-indices of non-positive integers

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Abstract

We extend the definition and construct several bases for polylogarithms Li_T, where T are recognizable, by a finite state (multiplicity) automaton and of alphabet $X = \{x_0, x_1\}$[1]. The kernel of the "polylogarithmic map" $\mathrm{Li}_\bullet$ is also characterized and provides a rewriting process which terminates to a normal form. We mostly concentrate on the algebraic aspects of this extension.

As a matter of fact, the interest of rational series is twofold: algebraic and analytic. Firstly (from the algebraic point of view) they are closed under shuffle products and the shuffle exponential of letters (and their linear combinations, see the paragraph about the algebra "star of the plane") is precisely their Kleene star. On the other hand the growth of their coefficients is tame[2] [9, 23, 24] and, as such, their associated polylogarithms can be (in the domain $Dom(\mathrm{Li})$) rightfully computed [15, 19, 21, 22]. Doing this, we recover many functions (as the simple polynomials), forgotten in the straight algebra of polylogarithms. Let us, now, go into details, using the notations of [1, 26].

1. We denote $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ the closure of $\mathbb{C}\langle X\rangle$ by rational operations $\{+, \mathtt{conc}, {}^*\}$ [1]. This space is closed by shuffle product and, for any $a_0, a_1 \in \mathbb{C}$, one has

$$\begin{aligned} (a_0x_0 + a_1x_1)^* &= (a_0x_0)^* ⧢ (a_1x_1)^* \\ \Delta_{⧢}((a_0x_0 + a_1x_1)^*) &= (a_0x_0 + a_1x_1)^* \otimes (a_0x_0 + a_1x_1)^*. \end{aligned}$$

   In here, we denote $S^* = \sum_{n\geq 0} S^n, \forall S \in \mathbb{C}\langle X\rangle$. By the Kleene-Schützenberger theorem, a power series S belongs to $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ iff it is *recognizable* by an automaton admitting *a linear representation* (β, μ, γ) of dimension $n \geq 1$, with $\beta \in \mathscr{M}_{n,1}(\mathbb{C}), \gamma \in \mathscr{M}_{1,n}(\mathbb{C}), \mu : X^* \mapsto \mathscr{M}_{n,n}(\mathbb{C})$ (a multiplicative morphism). For any $w \in X^*$, one has $\langle S \mid w\rangle = \beta\mu(w)\gamma$ (see [1, 6]).

[1] The space of rational series considered here is $(\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle, ⧢, 1_{X^*})$, it is a subspace of $Dom(\mathrm{Li})$.

[2] *i.e.* for such a rational series S over X, their exists a real $K > 0$ and a positive real morphism χ such that, for any word w over the monoid X^*, the coefficient $|\langle S \mid w\rangle|$ is majorated by $K \times \chi(w)$ [6, 9, 23, 24].

2. We consider also the differential forms $\omega_0(z) = z^{-1}dz$, $\omega_1(z) = (1-z)^{-1}dz$ and denote Ω the cleft plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $\lambda(z)$ the rational fraction $z(1-z)^{-1}$ belonging to the differential ring $\mathscr{C} := \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$, endowed with the differential operator $\partial_z := d/dz$ and with the element $1_\Omega : \Omega \to \mathbb{C}$, as unit (*i.e.,* for any $z \in \Omega, 1_\Omega(z) = 1$).

In continuation of [7, 9], the principal object of the present work is the *polylogarithm* function, well defined for any r-uplet $(s_1, \ldots, s_r) \in \mathbb{C}^r, r \in \mathbb{N}_+$ and for any $z \in \mathbb{C}$ such that $|z| < 1$, as follows

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \quad \text{and} \quad \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N)\, z^N,$$

where the arithmetic function $\mathrm{H}_{s_1,\ldots,s_r} : \mathbb{N} \to \mathbb{Q}$ is called *harmonic sum* and is expressed by $\mathrm{H}_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$.

By analytic continuation [11, 27] and after a theorem by Abel, for any $r \geq 1$, if $(s_1, \ldots, s_r) \in \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r; \sum_{i=0}^m \Re(s_i) > m\}$, one obtains *polyzeta* values as follows

$$\zeta(s_1, \ldots, s_r) := \lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to \infty} \mathrm{H}_{s_1,\ldots,s_r}(N).$$

This is no more valid in the divergent cases and requires the renormalization of the corresponding divergent polyzetas. It is already done for the case of polyzetas at positive multi-indices [3, 4, 5, 23] and it is done [10, 13, 25] and completed in [7, 9] for the case of non-positive multi-indices.

To study the polylogarithms at non- positive (negative) multi-indices, one relies on [7, 9] (resp. [18, 20]). Let $Y = \{y_k\}_{k \geq 0}$ and $Y_0 = \{y_0\} \sqcup Y$ be the alphabets.

1. the (one-to-one) correspondence between the multi-indices $(s_1, \ldots, s_r) \in \mathbb{Z}_{\leq 0}^r$ (resp. $\mathbb{N}_{\geq 1}^r$) and the words $y_{s_1} \ldots y_{s_r}$ (resp. $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$) in Y_0^* (resp. $X^*x_1 + 1_{X^*}$),
2. indexing polylogarithms by words $y_{s_1} \ldots y_{s_r} \in Y_0^*$: $\mathrm{Li}^-_{y_{s_1} \ldots y_{s_r}} = \mathrm{Li}_{-s_1,\ldots,-s_r}$.

Moreover, one obtains the polylogarithms at positive indices as image by the following isomorphism of the shuffle algebra [18], $\mathrm{Li}_\bullet : (\mathbb{C}\langle X \rangle, ⧢, 1_{X^*}) \longrightarrow (\mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}, \times, 1_\Omega)$, such that[3],

$$x_0^n \longmapsto \log^n(z)/n!, \quad x_1^n \longmapsto \log^n((1-z)^{-1})/n!, \quad x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \longmapsto \mathrm{Li}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1}.$$

Extending over the set of rational power series on non commutative variables when possible, see discussion after (1), as follows

$$\text{with} \qquad S \quad = \quad \sum_{n \geq 0} \langle S \mid x_0^n \rangle x_0^n + \sum_{k \geq 1} \sum_{w \in (x_0^* x_1)^k x_0^*} \langle S \mid w \rangle\, w,$$

[3] With the section chosen below, one has $x_0^n \longmapsto \Big(\log(z) - \log(z_0)\Big)^n/n!$.

$$\text{one defines}\qquad \mathrm{Li}_S(z) = \sum_{n\geq 0}\langle S \mid x_0^n\rangle \frac{\log^n(z)}{n!} + \sum_{k\geq 1}\sum_{w\in(x_0^*x_1)^k x_0^*}\langle S \mid w\rangle \mathrm{Li}_w. \qquad (1)$$

Some of these sums do not converge for the topology of compact convergence [4] and we will call $Dom(\mathrm{Li})$, the space of series for which $\langle|\mathrm{Li}|\mid|S|\rangle$ is convergent [5]. The morphism $\mathrm{Li}_\bullet$ is not injective, but $\{\mathrm{Li}_w\}_{w\in X^*}$ are still linearly independant over $\mathscr{C}$ [22, 23].

Example 1 *i.* $1_\Omega = \mathrm{Li}_{1_{X^*}} = \mathrm{Li}_{x_1^* - x_0^* ⧢ x_1^*}$.

ii. $\lambda = \mathrm{Li}_{(x_0+x_1)^*} = \mathrm{Li}_{x_0^* ⧢ x_1^*} = \mathrm{Li}_{x_1^* - 1}$.

iii. $\mathscr{C} = \mathbb{C}[\mathrm{Li}_{x_0^*}, \mathrm{Li}_{(-x_0)^*}, \mathrm{Li}_{x_1^*}]$.

iv. $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*} = \{\mathrm{Li}_S \mid S \in \mathbb{C}[x_0^*] ⧢ \mathbb{C}[(-x_0)^*] ⧢ \mathbb{C}[x_1^*] ⧢ \mathbb{C}\langle X\rangle\}$.

Let us consider also the operators, acting on $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ [24]:

$$\partial_z = d/dz, \theta_0 = zd/dz, \theta_1 = (1-z)d/dz,$$
$$\forall f \in \mathscr{C},\qquad \iota_0^{(z_0)}(f) = \int_{z_0}^{z} f(s)\omega_0(s) \quad\text{and}\quad \iota_1(f) = \int_0^z f(s)\omega_1(s).$$

Here, the operator $\iota_0^{(z_0)}$ is well-defined then one can check easily that [7, 9, 21, 22]

1. The subspace $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the action of $\{\theta_0, \theta_1\}$ and $\{\iota_0, \iota_1\}$ [6].
2. The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy in particular,

$$\theta_1 + \theta_0 = [\theta_1, \theta_0] = \partial_z \quad\text{and}\quad \forall k = 0,1, \theta_k\iota_k = \mathrm{Id},$$
$$[\theta_0\iota_1, \theta_1\iota_0] = 0 \quad\text{and}\quad (\theta_0\iota_1)(\theta_1\iota_0) = (\theta_1\iota_0)(\theta_0\iota_1) = \mathrm{Id}.$$

3. $\theta_0\iota_1$ and $\theta_1\iota_0$ are scalar operators within $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$, respectively with eigenvalues λ and $1/\lambda$, *i.e.* $(\theta_0\iota_1)f = \lambda f$, and $(\theta_1\iota_0)f = (1/\lambda)f$.
4. Let $w = y_{s_1}\dots y_{s_r} \in Y^*$ (then $\pi_X(w) = x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1$) and $u = y_{t_1}\dots y_{t_r} \in Y_0^*$. The functions Li_w and Li_u^- satisfy

$$\mathrm{Li}_w = (\iota_0^{s_1-1}\iota_1\dots\iota_0^{s_r-1}\iota_1)1_\Omega, \qquad \mathrm{Li}_u^- = (\theta_0^{t_1+1}\iota_1\dots\theta_0^{t_r+1}\iota_1)1_\Omega,$$
$$\iota_0\,\mathrm{Li}_{\pi_X(w)} = \mathrm{Li}_{x_0\pi_X(w)}, \qquad \iota_1\,\mathrm{Li}_w = \mathrm{Li}_{x_1\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_0\pi_X(w)} = \mathrm{Li}_{\pi_X(w)}, \qquad \theta_1\,\mathrm{Li}_{x_1\pi_X(w)} = \mathrm{Li}_{\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_1\pi_X(w)} = \lambda\,\mathrm{Li}_{\pi_X(w)}, \qquad \theta_0\,\mathrm{Li}_{x_1\pi_X(w)} = \mathrm{Li}_{\pi_X(w)}/\lambda.$$

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[4] For example, (1) gives a series $\langle Li \mid x_0^* x_1\rangle$ which is not convergent.

[5] we denote by $|T|$, the extension, term by term of the function $z \mapsto |z|$, i.e.,

$$|T| = \sum_{w\in X^*} |\langle T \mid w\rangle|\, w.$$

[6] Here, we state the identities for indifferently for $\iota_0 = \iota_0^{(z_0)}$ or the classic ι_0, see [7, 9, 14].

Here, we explain the whole project of extension of $\mathrm{Li}_\bullet$, study different aspects of it, in particular what is desired of $\iota_i, i = 0,1$. The interesting problem in here is to study what is expected of the sections $\iota_i, i = 0,1$. In fact, we will use this construction to extend $\mathrm{Li}_\bullet$ to $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and, after that, we extend it to a much larger rational algebra.

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