Difference-Differential Dimension Polynomials and their Invariants

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Let K be an inversive difference-differential field of zero characteristic with basic sets of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ and automorphisms $\sigma = \{\alpha_1, \dots, \alpha_n\}$ (any two mapping from the set $\Delta \cup \sigma$ commute). Let Θ and Γ denote the free commutative semigroup generated by Δ and free commutative group generated by σ, respectively. The *orders* of elements $\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta$ and $\gamma = \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Gamma$ are defined as

$$\operatorname{ord} \theta = \sum_{i=1}^{m} k_i \text{ and } \operatorname{ord} \gamma = \sum_{j=1}^{n} |l_j|,$$

respectively. Furthermore, for any $r \in \mathbb{N}$, we set

$$\Theta(r) = \{\theta \in \Theta \,|\, \operatorname{ord} \theta \leq r\} \text{ and } \Gamma(r) = \{\gamma \in \Gamma \,|\, \operatorname{ord} \gamma \leq r\}.$$

Let Λ be the semigroup of all power products $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ ($k_i \in \mathbb{N}$, $l_j \in \mathbb{Z}$). We define the orders of λ with respect to the sets Δ and σ as $\operatorname{ord}_\Delta \lambda = \sum_{i=1}^{m} k_i$ and $\operatorname{ord}_\sigma \lambda = \sum_{j=1}^{n} |l_j|$, respectively, and set

$$\Lambda(r,s) = \{\lambda \in \Lambda \,|\, \operatorname{ord}_\Delta \lambda \leq r, \operatorname{ord}_\sigma \lambda \leq s\} \qquad (r,s \in \mathbb{N}).$$

In what follows, "difference" always means "inversive difference" (that is, we consider both positive and negative powers of the basic translations α_i). Furthermore, we will use prefixes Δ-, σ- and Δ-σ- instead of adjectives "differential", "difference" and "difference-differential" , respectively. If $\eta = \{\eta_1, \dots, \eta_p\}$ is a finite subset of a Δ-σ-field extension of K, we write $K\langle\eta\rangle$ for the Δ-σ-field extension of K generated by η (as a field, $K\langle\eta\rangle = K(\{\lambda(\eta_i) \,|\, \lambda \in \Lambda, 1 \leq i \leq p\})$). The differential (Δ-) and inversive difference (σ-) field extensions of K generated by the set η are denoted by $K\langle\eta\rangle_\Delta$ and $K\langle\eta\rangle_\sigma$, respectively.

With the above notation, as it is shown in [4], there exist polynomials $\chi^{\Delta}_{\eta|K}(t), \chi^{\sigma}_{\eta|K}(t) \in \mathbb{Q}[t]$ such that

$$\chi^{\Delta}_{\eta|K}(r) = \sigma\text{-}\operatorname{tr.deg}_K K\langle\Theta(r)\eta\rangle_\sigma \text{ and } \chi^{\Delta}_{\eta|K}(r) = \Delta\text{-}\operatorname{tr.deg}_K K\langle\Gamma(r)\eta\rangle_\Delta$$

for all sufficiently large $r \in \mathbb{N}$. (If M is a subset of Θ or Γ, then $M\eta$ denotes the set $\{\mu(\eta_i) \,|\, \mu \in M, 1 \leq i \leq p\}$.) Furthermore, it is proved in [1] that there exists a polynomial in two variables $\psi_{\eta|K}(t_1,t_2) \in \mathbb{Q}[t_1,t_2]$ such that

$$\psi_{\eta|K}(r,s) = \operatorname{tr.deg}_K K(\{\lambda(\eta_i) \,|\, \operatorname{ord}_\Delta \lambda \leq r, \operatorname{ord}_\sigma \lambda \leq s, 1 \leq i \leq p\})$$

for all sufficiently large $r, s \in \mathbb{N}$.

These polynomials (called *dimension polynomials* of the Δ-σ-field extension $L = K\langle\eta\rangle$ associated with the system of Δ-σ-generators η), generally speaking, depend on the set η. However, they carry certain invariants that are independent of η and therefore characterize the extension L/K itself. In this talk we will describe these invariants in terms of differential, difference and difference-differential transcendence degrees and Krull-type dimension of field extensions. We will also discuss methods of computation of dimension polynomials and generalizations of the results on their invariants to the case of a difference-differential field extension with arbitrary partition of basic sets of derivations and translations. (The existence of the corresponding multivariate dimension polynomials was proved in [2] and [3, Section 4.2].)

References

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