Deciding Rational Solvability of First-Order Algebraic Ordinary Differential Equations

G. Grasegger[1], N.T. Vo[2], F. Winkler[2]

[1] *Austrian Academy of Sciences, RICAM, Linz, Austria, georg.grasegger@ricam.oeaw.ac.at*
[2] *RISC, J.Kepler University Linz, Austria, {tvongoc, franz.winkler}@risc.jku.at*

The question of determining general solutions of an algebraic ordinary differential equation (AODE) has a long history, dating back to the work of L. Fuchs and H. Poincaré. Rational general solutions have been investigated by Feng and Gao ([2]), Chen and Ma ([1]) and Ngô and Winkler ([7]). Consider $F(x,y,y')=0$, a first-order AODE, where F is an irreducible polynomial in three variables over an algebraically closed field K. Replacing y' by a new indeterminate z, we obtain an algebraic equation $F(x,y,z)=0$ which defines a plane algebraic curve $\mathscr{C} := \{(a,b) \in A^2(\overline{K(x)}) \,|\, F(x,a,b)=0\}$ over the field $\overline{K(x)}$ of algebraic functions. We call it the corresponding algebraic curve. A parametrization of $\mathscr{C}$ is a rational map $\mathscr{P} : A^1(\overline{K(x)}) \to \mathscr{C}$ such that the image of $\mathscr{P}$ is dense in $\mathscr{C}$ with respect to Zariski topology. If furthermore $\mathscr{P}$ is a birational equivalence, it is called a proper parametrization. It is well-known that exactly the curves of genus 0 are parametrizable. A parametrization is represented as a pair of rational functions, say $\mathscr{P} = (p_1(t), p_2(t))$, with coefficients in $\overline{K(x)}$. If the curve $\mathscr{C}$ is parametrizable, then it has a proper parametrization $\mathscr{P}$ whose coefficients lie in an algebraic extension field of $K(x)$ of degree at most 2 over $K(x)$. A parametrization is called optimal if its coefficients lie in an algebraic extension field of lowest algebraic extension degree. In [4, 8] we find algorithms for computing optimal parametrizations over the rational numbers Q as well as $Q(x)$. We have extended this optimal parametrization algorithm to work over $K(x)$, and we see that a rational curve defined over $K(x)$ can always be parametrized over $K(x)$.

A rational solution of the differential equation $F(x,y,y')=0$ is a rational function $y(x) \in K(x)$, such that $F(x,y(x),y'(x))=0$. A solution $y(x)$ of the AODE is called a strong rational general solution, if $y = y(x,c) \in K(x,c) \setminus K(x)$ where c is a transcendental constant over $K(x)$.

We show that if the differential equation $F(x,y,y')=0$ has a strong rational general solution, then its corresponding curve is of genus 0. Furthermore, if the corresponding algebraic curve of the differential equation $F(x,y,y')=0$ is of genus 0, and $\mathscr{P} = (p_1,p_2) \in K(x,t)^2$ is an optimal parametrization, then there is a one-to-one correspondence between strong rational general solutions of the differential equation $F(x,y,y')=0$ and strong rational general solutions of a quasi-linear associated AODE. Since the associated AODE is of first order and of first degree, we know by Fuchs [3] that it admits a strong rational general solution only if it is a linear or a Riccati equation. A linear equation of order one can be solved easily by integration. For Riccati equations, we refer to a complete algorithm for finding all rational solutions provided by Kovacic [6]. Hence, a complete decision algorithm for the existence of a strong rational general solution of a first-order AODE can be given.

Example. (Example 1.537 in Kamke [5]) We consider the differential equation $F(x,y,y') = (xy'-y)^3 + x^6y' - 2x^5y = 0$. Its corresponding curve has a strong rational parametrization

$$\mathscr{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6}\right).$$

Hence, the associated differential equation with respect to $\mathscr{P}$ is $\omega' = \frac{1}{x^2}\omega(2\omega - x)$. This is a Riccati equation and we can determine a rational general solution $\omega(x) = \frac{x}{1+cx^2}$. Hence, the differential equation $F(x,y,y') = 0$ has the rational general solution $y(x) = cx(x+c^2)$.

References

[1] G. Chen and Y. Ma. Algorithmic reduction and rational general solutions of first order algebraic differential equations. In *Differential equations with symbolic computation*, pages 201–212. Birkhäuser, Basel, 2005.

[2] R. Feng and X.-S. Gao. Rational General Solutions of Algebraic Ordinary Differential Equations. In J. Gutierrez, editor, *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, ISSAC '04, pages 155–162, New York, 2004. ACM.

[3] L. Fuchs. Über Differentialgleichungen, deren Integrale feste Verzweigungspunkte besitzen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1884:699–710, 1884.

[4] E. Hillgarter and F. Winkler. Points on algebraic curves and the parametrization problem. In *Automated Deduction in Geometry*, Lecture Notes in Computer Science, pages 189–207, New York, 1997. Springer Berlin Heidelberg.

[5] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen*. B. G. Teubner, Stuttgart, 1997.

[6] J. J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, 2:3–43, 1986.

[7] L. X. C. Ngô and F. Winkler. Rational general solutions of parametrizable AODEs. *Publicationes Mathematicae Debrecen*, 79(3–4):573–587, 2011.

[8] J. R. Sendra, F. Winkler, and S. Pérez-Díaz. *Rational Algebraic Curves, A Computer Algebra Approach*, volume 22 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin Heidelberg, 2008.