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Parametric b-functions for some hypergeometric ideals*

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We denote by $D := \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ the Weyl algebra over the field $\mathbb{C}$.

The aim of this note is to study the b–function associated with a class of hypergeometric ideals $H_A(\beta) \subseteq D$ following [9, Section 5.1]. Let us recall the definition of $H_A(\beta)$. Given $A = (a_{ij})$ a $d \times n$ matrix of rank d with integer coefficients, we first consider the associated toric ideal $I_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \ldots, \partial_n]$

$$I_A := \mathbb{C}[\partial]\{\partial^u - \partial^v \mid u, v \in \mathbb{N}^n, Au = Av\}.$$

Moreover we consider the Euler operators, for $1 \leq i \leq d$

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n.$$

Then for any parameter vector $\beta \in \mathbb{C}^d$ the hypergeometric ideal is defined as

$$H_A(\beta) = D \cdot I_A + \sum_{1 \leq i \leq d} D(E_i - \beta_i).$$

Given a holonomic left ideal I in D and a nonzero weight vector $\omega \in \mathbb{R}^n$, we denote $in_{(-\omega,\omega)}(I) \subset D$ the initial ideal of I with respect to the filtration $(F_p)_{p\in\mathbb{R}}$ induced on D by the vector $(-\omega, \omega) \in \mathbb{R}^{2n}$. The $\mathbb{C}$–vector space F_p is defined as follows:

$$F_p := \mathbb{C}\{x^\alpha \partial^\beta \mid -\omega\alpha + \omega\beta \leq p\} \quad \text{for } p \in \mathbb{R}.$$

Kashiwara has introduced in (*On the Holonomic Systems of Linear Differential Equations, II.* Inventiones Math. 49, 121–135, 1978) the b–function $b_{I,\omega}(s)$ associated with the pair (I, ω), as the monic generator of the ideal

$$in_{(-\omega,\omega)}(I) \cap \mathbb{C}[s] \tag{1}$$

where $s := \sum_{i=1}^n \omega_i x_i \partial_i$. It is proven in *loc. cit. Theorem 2.7* that the ideal in (1) is nonzero. In this note we follow the presentation and notations of [9, §5] on this subject.

Definition 1. The polynomial $b_{I,\omega}(s)$ is called the b–function of the holonomic ideal $I \subset D$ with respect to the weight vector ω.

Previous b-functions are closely related to the classical notion of Bernstein polynomial (also called Bernstein-Sato polynomial) $b_f(s)$ associated with a given nonzero polynomial $f \in \mathbb{C}[x]$ (see e.g. [9, Lemma 5.3.11]). Bernstein polynomials have been introduced in [2] and [8] and represent fundamental invariants in singularity theory. There are several algorithms for computing Bernstein polynomials. Some of them are described in [5], [6], [4], and [1]. These and other algorithms have been implemented in the computer algebra systems `Asir`, `Macaulay2` and `Singular` among others. Nevertheless, in practice $b_f(s)$ is hard

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to compute even in the case of a polynomial f in two variables. In [3] the authors propose the algorithm `checkRoot` which, given a rational number α checks if it is a root of the Bernstein polynomial $b_f(s)$, and computes its multiplicity.

We simply denote $b_{\omega,\beta}(s) := b_{H_A(\beta),\omega}(s)$. We refer to [9] for the main results on hypergeometric ideals and the corresponding b–functions $b_{\omega,\beta}(s)$ *for generic parameters* w *and* β (see below for details). In [7] the authors describe bounds for the roots of $b_{\omega,\beta}(s)$.

In this paper we restrict ourselves to matrices of the form $A = (1, p, q)$ with integers $1 < p < q$ and p and q coprime. The first step is to describe the Gröbner fan of the toric ideal I_A, as defined in (T. Mora; L. Robbiano, The Gröbner fan of an ideal. *J. Symbolic Comput.* **6**(2-3) 183–208 (1988)) and in (B. Sturmfels, *Gröbner bases and convex polytopes*. University Lecture Series, 8. Providence RI, 1995.) We define a finite family of disjoint regions $R_i^{(k)} \subset \mathbb{R}^3$ which are the intersection of two half-spaces with the line $(1, p, q)\mathbb{R}$ in common (see Example 3). The possible integers k and i depend on the extended Euclidean division of q over p. We prove an equality $\mathbb{R}^3 = \bigcup_{i,k} \overline{R_i^{(k)}}$ such that for each $\omega \in R_i^{(k)}$, the initial ideal $in_\omega(I_A)$ is a monomial ideal and it is independent of ω.

In [9, Proposition 5.1.9.] there is a description of $b_{\omega,\beta}(s)$ for Zariski generic β and generic ω In (M.C. Fernández-Fernández, *Soluciones Gevrey de sistemas hipergeométricos asociados a una curva monomial lisa.* DEA, U. Sevilla, 2008.), the polynomial $b_{\omega,\beta}(s)$ is described for $\omega = (1, 0, 0)$ and β generic. Our main result is:

Theorem 2. *Given $R_i^{(k)}$, a facet of the Gröbner fan of I_A, there is a proper Zariski closed set $C_i^{(k)} \subset R_i^{(k)}$ such that if $\omega \in R_i^{(k)} \setminus C_i^{(k)}$ and β is generic the b–function is*

$$b_{\omega,\beta}(s) = \prod_{\alpha \in F_i^{(k)}} (s - \alpha)$$

for certain finite set $F_i^{(k)} \subseteq \mathbb{C}$. Moreover, if $\omega \in C_i^{(k)}$ or β is non-generic, the right hand side of previous equality gives a multiple of the b–function.

The set $F_i^{(k)}$ is explicitly described in terms of standard monomials of $in_{(-\omega,\omega)}(H_A(\beta))$. In the following example we sum up our results.

Example 3. Consider the matrix $A = (1, 3, 5)$. The Gröbner fan of $I_A \subset \mathbb{C}[\partial_x, \partial_y, \partial_z]$ consists of seven facets. Let us focus in one of them, namely $R_1^{(2)} = \{\omega \in \mathbb{R}^3 \mid 2\omega_1 + \omega_2 > \omega_3, \omega_1 + 3\omega_2 < 2\omega_3\}$. For any $\omega \in R_1^{(2)}$

$$in_\omega(I_A) = D\left(\partial_x^3, \partial_x^2\partial_y, \partial_x\partial_z, \partial_z^2\right).$$

Any complex number $\beta \neq 2$ is generic, and we have that

$$in_{(-\omega,\omega)}(H_A(\beta)) = D\left(\partial_x^2, \partial_x\partial_z, \partial_z^2, E - \beta\right).$$

We have $C_1^{(2)} = R_1^{(2)} \cap \{3\omega_1 + 4\omega_2 = 3\omega_3\}$. The b–function for $\omega \in R_1^{(2)} \setminus C_1^{(2)}$ and $\beta \neq 2$ is

$$b_{\omega,\beta}(s) = (s - \frac{\beta}{3}\omega_2)(s - \omega_1 - \frac{\beta - 1}{3}\omega_2)(s - \frac{\beta - 5}{3}\omega_2 - \omega_3).$$

If $\omega \in C_1^{(2)}$ and $\beta \neq 2$, the polynomial

$$(s - \frac{\beta}{3}\omega_2)(s - \omega_1 - \frac{\beta - 1}{3}\omega_2)$$

is a multiple of the b–function. With `Singular` we check that in this case we obtain the true b–function and not just a multiple. If $\omega \in R_1^{(2)}$ but $\beta = 2$ we have the following multiple of the b–function:

$$\begin{cases} (s - \frac{2}{3}\omega_2)(s - \omega_1 - \frac{1}{3}\omega_2)(s - 2\omega_1)(s + \omega_2 - \omega_3) & \text{if } \omega \notin C_1^{(2)} \\ (s - \frac{2}{3}\omega_2)(s - \omega_1 - \frac{1}{3}\omega_2)(s - 2\omega_1) & \text{otherwise.} \end{cases}$$

Again, with `Singular` we check that this is indeed $b_{\omega,2}(s)$. However, if we consider the region $R_2^{(2)} = \{\omega \in \mathbb{R}^3 \mid \omega_1 + 3\omega_2 > 2\omega_3, 3\omega_3 > 5\omega_2\}$, we have $\beta = 1, 2, 4, 7$ as non-generic values, and for $\omega \in R_2^{(2)}$ and $\beta = 2$ we give a polynomial with five roots, and only four of them are the roots of $b_{\omega,2}(s)$.

If $\omega \in \mathbb{R}^3 \setminus \bigcup_{i,k} R_i^{(k)}$ the study of $b_{\omega,\beta}(s)$ is a work in progress.

Keywords: b–function, hypergeometric ideal.

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