COMPUTER ALGEBRA AND LANCZOS POTENTIAL

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ABSTRACT
We found in 2016 a few results on the mathematical structure of the conformal Killing differential sequence in arbitrary dimension n, in particular the rank and order of the successive differential operators for $n = 3, n = 4$ or $n \geq 5$. They were so striking that we did not dare to publish them before our former PhD student A. Quadrat (INRIA) could confirm them while using new computer algebra packages that he developped for studying extension modules in differential homological algebra. In the meantime, we found in 2017 the "*missing link*" justifying the doubts we had since a long time on the origin and existence of Gravitational Waves in General Relativity. We also provide an example showing how these extension modules are depending on the *structure constants* appearing in the *Vessiot structure equations* (1903), still not acknowledged after one century even if they generalize the constant Riemannian curvature integrability condition of L.P. Eisenhart (1926). The present paper is made from the transparencies we provided during a lecture at the recent 24 th conference on Applications of Computer Algebra (ACA 2018) held in Santiago de Compostela, Spain, june 18-22, 2018.

KEY WORDS
Differential sequence; Variational calculus; Differential constraint; Control theory; Killing operator; Riemann tensor; Bianchi identity; Weyl tensor; Lanczos tensor; Contact transformations; Vessiot structure equations. Differential homological algebra; Extension modules.

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GENERAL RELATIVITY $\Rightarrow$ GRAVITATIONAL WAVES

1 **METRIC**: $\omega = (\omega_{ij}) = (\omega_{ji}) \in S_2T^*, det(\omega) \neq 0 \stackrel{linearization}{\longrightarrow} \Omega \in S_2T^*$

2 **CHRISTOFFEL SYMBOLS**: $\gamma^k_{ij} = \frac{1}{2}\omega^{kr}(\partial_i\omega_{rj} + \partial_j\omega_{ir} - \partial_r\omega_{ij}) \stackrel{lin}{\longrightarrow} \Gamma \in S_2T^* \otimes T$

3 **RIEMANN TENSOR**: $\rho^k_{l,ij} = \partial_i\gamma^k_{lj} - \partial_j\gamma^k_{li} + \gamma^r_{li}\gamma^k_{rj} - \gamma^r_{lj}\gamma^k_{ri} \stackrel{lin}{\longrightarrow} R \in \wedge^2T^* \otimes T^* \otimes T$

$$\omega_{kr}\rho^r_{l,ij} = \rho_{kl,ij} = -\rho_{lk,ij} = -\rho_{kl,ji} = \rho_{ij,kl} \Rightarrow \rho^r_{r,ij} = 0$$
$$\rho^k_{l,ij} + \rho^k_{i,jl} + \rho^k_{j,li} = 0 \Rightarrow \rho^r_{i,rj} = \rho^r_{j,ri}$$

4 **KILLING OPERATOR**: $\mathcal{D} : T \rightarrow S_2T^* : \xi \rightarrow \mathcal{L}(\xi)\omega = \Omega$

$$\mathcal{D}\xi = 0, \mathcal{D}\eta = 0 \Rightarrow \mathcal{D}[\xi,\eta] = 0 \quad (Lie\ operator)$$

5 **KILLING EQUATIONS**: $(\mathcal{L}(\xi)\omega)_{ij} \equiv \Omega_{ij} \equiv \omega_{rj}\partial_i\xi^r + \omega_{ir}\partial_j\xi^r + \xi^r\partial_r\omega_{ij} = 0$

$$\omega_{rj}v^r_i + \omega_{ir}v^r_j = 0 \quad (symbol\ \ g_1 \in T^* \otimes T)$$

6 **KILLING SEQUENCE**:

$$0 \rightarrow \Theta \rightarrow T \underset{1}{\stackrel{\mathcal{D}}{\longrightarrow}} S_2T^* \underset{2}{\stackrel{\mathcal{D}_1}{\longrightarrow}} F_1 \underset{1}{\stackrel{\mathcal{D}_2}{\longrightarrow}} F_2$$

$$n \stackrel{Killing}{\longrightarrow} \frac{n(n+1)}{2} \stackrel{Riemann}{\longrightarrow} \frac{n^2(n^2-1)}{12} \stackrel{Bianchi}{\longrightarrow} \frac{n^2(n^2-1)(n-2)}{24}$$

7 **RICCI TENSOR**: $\rho_{ij} = \rho^r_{i,rj} = \rho_{ji} \stackrel{lin}{\longrightarrow} (R_{ij}) \in S_2T^*$

8 **EINSTEIN TENSOR**: $\boxed{\epsilon_{ij} = \rho_{ij} - \frac{1}{2}\omega_{ij}\omega^{rs}\rho_{rs}} \stackrel{lin}{\longrightarrow} (E_{ij}) \in S_2T^*$

9 **EINSTEIN EQUATIONS**: $(Einstein)\ \ div(E) = 0 \Rightarrow \boxed{E_{ij} \sim \Sigma_{ij}} \Leftarrow div(\Sigma) = 0\ \ (Cauchy)$

10 **WAVE EQUATIONS**: $\boxed{\bar{\Omega}_{ij} = \Omega_{ij} - \frac{1}{2}\omega_{ij}\omega^{rs}\Omega_{rs}} \Rightarrow \boxed{\Box\bar{\Omega}_{ij} + ... \sim \Sigma_{ij}}$

MATHEMATICALLY CORRECT BUT CONCEPTUALLY WRONG

DIFFERENTIAL SEQUENCE

$$\xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta$$

DIRECT PROBLEM $\mathcal{D}$ given, find $\mathcal{D}_1$: M. Janet (1920), D.C. Spencer (1970)

$\Updownarrow$

INVERSE PROBLEM $\mathcal{D}_1$ given, find $\mathcal{D}$: not always possible : Wheeler's challenge (1970)

THEOREM: A classical control system is controllable if and only if it is *parametrizable*, that is if and only if it generates the *compatibility conditions* (CC) of a previous operator describing therefore *a* parametrization.

EXAMPLE: *DOUBLE PENDULUM*

Rigid bar of length L moving along the left to right horizontal axis $0x$, $0y$ downwards vertial axis parallel to gravity g, first pendulum made by a mass m_1, having length l_1 and mouving by an angle θ_1 with respect to the vertical, second pendulum made by a mass m_2, having length l_2 and mouving by an angle θ_2 with respect to the vertical.

Control system: $\boxed{\ddot{x} + l_1\ddot{\theta}_1 + g\theta_1 = 0, \quad \ddot{x} + l_2\ddot{\theta}_2 + g\theta_2 = 0}$

Parametrization:

• $l_1 \neq l_2$:

$$\left\{ \begin{array}{rcl} -l_1 l_2 d^4\phi - g(l_1+l_2)d^2\phi - g^2\phi & = & x \\ l_2 d^4\phi + g d^2\phi & = & \theta_1 \\ l_1 d^4\phi + g d^2\phi & = & \theta_2 \end{array} \right.$$

• $l_1 = l_2 = l, \theta = \theta_1 - \theta_2 \Rightarrow l\ddot{\theta} + g\theta = 0$

$\theta(0) = 0, \dot{\theta}(0) = 0 \Rightarrow \theta(t) = 0$.

COMPUTER ALGEBRA ABSOLUTELY NEEDED for $l_1 = cst, l_2 = l_2(t)$.

COROLLARY: A classical control system defined over an ordinary differential field K by equations linearly independent over the ring $D = K[d]$ of differential operators with coefficients in K is controllable if and only if the formal adjoint of the corresponding differential operator is injective, even when D is non-commutative.

COUNTEREXAMPLE: EINSTEIN EQUATIONS CANNOT BE PARAMETRIZED

$\boxed{\textbf{DOUBLE DUALITY TEST}}$

$$
\begin{array}{ccccccc}
 & & & & & \zeta' & \boxed{5} \\
 & & & & \nearrow^{\mathcal{D}_1{}'} & & \\
\boxed{4} & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & \boxed{1} \\
 & & & & & & \\
\boxed{3} & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & \boxed{2}
\end{array}
$$

$$ad(ad(\mathcal{D})) = \mathcal{D}, \;\; ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1 \circ \mathcal{D}) = 0$$

$$\Rightarrow \mathcal{D}_1 \circ \mathcal{D} = 0 \Rightarrow \mathcal{D}_1 \; AMONG \; the \; CC \; of \; \mathcal{D}$$

Step $\boxed{5} \Rightarrow \mathcal{D}_1{}' \; GENERATES \; the \; CC \; of \; \mathcal{D} \;\; \Rightarrow \;\; \boxed{\mathcal{D}_1 \leq \mathcal{D}_1{}'}$

THEOREM : $\mathcal{D}_1$ parametrized by $\mathcal{D} \;\; \Leftrightarrow \;\; \boxed{\mathcal{D}_1 = \mathcal{D}_1{}'}$

COUNTEREXAMPLE: Contrary to the Ricci operator, the Einstein operator is SELF-ADJOINT, the sixth terms being exchanged between themselves under *ad*:

$$\lambda^{ij}(\omega^{rs}d_{ij}\Omega_{rs}) \stackrel{ad}{\longleftrightarrow} (\omega^{rs}d_{ij}\lambda^{ij})\Omega_{rs} = (\omega_{ij}d_{rs}\lambda^{rs})\Omega^{ij}$$

$$
\begin{array}{ccccc}
 & & & & Riemann \;\; 20 \\
 & & & \nearrow & \\
4 & \xrightarrow{Killing} & 10 & \xrightarrow{Einstein} & 10 \\
 & & & & \\
4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{Einstein} & 10
\end{array}
$$

VARIATIONAL CALCULUS WITH CONSTRAINTS

MOTIVATION:
Suppose that $\mathcal{D}_1$ generates the CC of $\mathcal{D}$ *AND* that $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$.

$\boxed{\mathcal{D}\xi = \eta}$

$$\begin{aligned}\Phi = \int_V \varphi(\eta)dx \Rightarrow \delta\Phi &= \int \frac{\partial\varphi}{\partial\eta}\delta\eta dx \\ &= \int \frac{\partial\varphi}{\partial\eta}\mathcal{D}\delta\xi dx \\ &= \int (ad(\mathcal{D})\frac{\partial\varphi}{\partial\eta})\delta\xi dx + ... \\ &\Rightarrow \quad Cauchy\end{aligned}$$

$\boxed{\mathcal{D}_1\eta = 0}$

$$\begin{aligned}\Phi = \int_V (\varphi(\eta) - \lambda\mathcal{D}_1\eta)dx \Rightarrow \delta\Phi &= \int (\frac{\partial\varphi}{\partial\eta}\delta\eta - \lambda\mathcal{D}_1\delta\eta)dx \\ &= \int (\frac{\partial\varphi}{\partial\eta} - ad(\mathcal{D}_1)\lambda)\delta\eta dx + ...\end{aligned}$$

$$\Rightarrow \mu = \frac{\partial\varphi}{\partial\eta} = ad(\mathcal{D}_1)\lambda(parametrization\ by\ \lambda) \overset{ad(\mathcal{D})}{\longrightarrow} ad(\mathcal{D})\mu = 0(elimination\ of\ \lambda)$$

EXAMPLE: $\boxed{n = 2}$ Airy parametrization (1863)

$$2 \overset{Killing}{\longrightarrow} 3 \overset{Riemann}{\longrightarrow} 1 \rightarrow 0$$

$$2 \overset{Cauchy}{\longleftarrow} 3 \overset{Airy}{\longleftarrow} 1$$

KEY RESULT $\quad Cauchy = ad(Killing), \quad Airy = ad(Riemann)$

$$\lambda(d_{22}\Omega_{11} - \boxed{2}d_{12}\Omega_{12} + d_{11}\Omega_{22}) = (d_{22}\lambda\Omega_{11} - \boxed{2}d_{12}\lambda\Omega_{12} + d_{11}\lambda\Omega_{22}) + ...$$

$$\sigma^{ij}\Omega_{ij} = \sigma^{11}\Omega_{11} + \boxed{2}\sigma^{12}\Omega_{12} + \sigma^{22}\Omega_{22}$$

Cauchy $\boxed{d_1\sigma^{11} + d_2\sigma^{12} = f^1,\ \ d_1\sigma^{21} + d_2\sigma^{22} = f^2}$

Airy $\boxed{\sigma^{11} = d_{22}\lambda,\ \ \sigma^{12} = \sigma^{21} = -d_{12}\lambda,\ \ \sigma^{22} = d_{11}\lambda}$

SELF-ADJOINT OPERATOR

n=3 **BELTRAMI PARAMETRIZATION (1892)**

$$\left\{\begin{array}{c}\sigma^{11}\\ \sigma^{12}\\ \sigma^{12}\\ \sigma^{13}\\ \sigma^{23}\\ \sigma^{33}\end{array}\right\}=\left\{\begin{array}{cccccc}0&0&0&d_{33}&-2d_{23}&d_{22}\\ 0&-d_{33}&d_{23}&0&d_{13}&-d_{12}\\ 0&d_{23}&-d_{22}&-d_{13}&d_{12}&0\\ d_{33}&0&-2d_{13}&0&0&d_{11}\\ -d_{23}&d_{13}&d_{12}&0&d_{11}&0\\ d_{22}&-2d_{12}&0&d_{11}&0&0\end{array}\right\}\left\{\begin{array}{c}\phi_{11}\\ \phi_{12}\\ \phi_{13}\\ \phi_{22}\\ \phi_{23}\\ \phi_{33}\end{array}\right\}$$

$$NOT\ \ SELF-ADJOINT$$

$$\Leftrightarrow\ \ d_r\sigma^{ir}=0\ \ (Cauchy)$$

PARAMETRIZATION: $\boxed{ad(Riemann)=Beltrami}$

$$\left\{\begin{array}{c}\sigma^{11}\\ 2\sigma^{12}\\ 2\sigma^{13}\\ \sigma^{22}\\ 2\sigma^{23}\\ \sigma^{33}\end{array}\right\}=\left\{\begin{array}{cccccc}0&0&0&d_{33}&-2d_{23}&d_{22}\\ 0&-2d_{33}&2d_{23}&0&2d_{13}&-2d_{12}\\ 0&2d_{23}&-2d_{22}&-2d_{13}&2d_{12}&0\\ d_{33}&0&-2d_{13}&0&0&d_{11}\\ -2d_{23}&2d_{13}&2d_{12}&0&d_{11}&0\\ d_{22}&-2d_{12}&0&d_{11}&0&0\end{array}\right\}\left\{\begin{array}{c}\phi_{11}\\ \phi_{12}\\ \phi_{13}\\ \phi_{22}\\ \phi_{23}\\ \phi_{33}\end{array}\right\}$$

$$SELF-ADJOINT$$

TEXTBOOKS $\sigma^{ij}=2\frac{\partial\varphi}{\partial\Omega_{ij}}\Rightarrow\sigma^{ij}=\sigma^{ji}$ **WRONG**

FIRST CONTRADICTION

$\boxed{n=4}$ $\qquad ad(Killing) = Cauchy, \;\; ad(Riemann) = Beltrami$

$$
\begin{array}{ccccccccccc}
\boxed{4} & \xrightarrow{Killing} & 10 & \xrightarrow{Riemann} & 20 & \xrightarrow{Bianchi} & 20 & \longrightarrow & 6 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & & & \\
 & & 10 & \xrightarrow{Einstein} & 10 & \xrightarrow{div} & \boxed{4} & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 \leftarrow & \boxed{4} & \xleftarrow{Cauchy} & 10 & \xleftarrow{Beltrami} & 20 & \xleftarrow{ad(Bianchi)} & 20 \\
 & & & \| & \nwarrow & \uparrow & & \uparrow \\
 & & & 10 & \xleftarrow{Einstein} & 10 & \xleftarrow{ad(div)} & \boxed{4} \\
 & & & & & \uparrow & & \uparrow \\
 & & & & & 0 & & 0
\end{array}
$$

SECOND CONTRADICTION

$$E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}\omega^{rs}R_{rs} \quad \Rightarrow \quad E = C \circ R$$

EINSTEIN OPERATOR IS SELF-ADJOINT
RICCI OPERATOR IS NOT SELF-ADJOINT

$$E : \Omega \xrightarrow{C} \bar{\Omega} = \Omega - \frac{1}{2}\omega tr(\Omega) \xrightarrow{X} S_2T^*$$

$Einstein$ operator E (6 terms) $\rightarrow$ $Wave$ operator X (4 terms only)

$$E = X \circ C \Rightarrow E = ad(E) = ad(C) \circ ad(X) = C \circ ad(X) = C \circ R$$

$$\Rightarrow ad(X) = Ricci \;\; \Rightarrow \;\; \boxed{\text{X=ad(Ricci)}}$$

EINSTEIN OPERATOR IS USELESS
ONLY RICCI OPERATOR IS USEFULL

LANCZOS CONFUSION

POINCARE SEQUENCE IS SELF ADJOINT UP TO SIGN

$$\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} .... \xrightarrow{d} \wedge^n T^* \rightarrow 0$$

EXAMPLE: n=3 $\quad \wedge^0 T^* \xrightarrow{grad} \wedge^1 T^* \xrightarrow{curl} \wedge^2 T^* \xrightarrow{div} \wedge^3 T^* \rightarrow 0$

$$ad(grad) = -div, \; ad(curl) = curl, \; ad(div) = -grad$$

EXTENSION MODULES: $\exists$ GAP

$$\begin{array}{ccccc} \xi & \longrightarrow & \left\{ \begin{array}{rcl} d_{22}\xi & = & \eta^2 \\ d_{12}\xi & = & \eta^1 \end{array} \right. & \longrightarrow & d_1\eta^2 - d_2\eta^1 = \zeta \\ \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta \\ & & & & \\ \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda \\ d_{12}\mu^1 + d_{22}\mu^2 = \nu & \longleftarrow & \left\{ \begin{array}{rl} -d_1\lambda & = \mu^2 \\ d_2\lambda & = \mu^1 \end{array} \right. & \longleftarrow & \lambda \\ & \swarrow & & & \\ d_1\mu^1 + d_2\mu^2 = \nu' & & & & \end{array}$$

THEOREM: If M is the differential module defined by $\mathcal{D}$, the *extension modules* $ext^i(M)$ do not depend on the sequence used for their computation.

APPLICATION: The *Spencer sequence* for any Lie operator $\mathcal{D}$ which is coming from a Lie group of transformations is (locally) isomorphic to the tensor product of the Poincaré sequence by the corresponding finite Lie algebra $\mathcal{G}$.

COROLLARY: $\quad ext^1(M) = 0, \; ext^2(M) = 0, ... \Rightarrow$ NO GAP

CONFUSION: Lanczos has been trying in vain to do for the *Bianchi* operator what he did for the *Riemann* operator, a useless but possible

SHIFT BY ONE STEP.

LANCZOS SECRET

$$
\begin{array}{ccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
0 \rightarrow & g_4 & \rightarrow & S_4T^*\otimes T & \rightarrow & S_3T^*\otimes F_0 & \rightarrow & T^*\otimes F_1 & \rightarrow & F_2 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \parallel & & & \\
0 \rightarrow & T^*\otimes g_3 & \rightarrow & T^*\otimes S_3T^*\otimes T & \rightarrow & T^*\otimes S_2T^*\otimes F_0 & \rightarrow & T^*\otimes F_1 & \rightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
0 \rightarrow & \wedge^2T^*\otimes g_2 & \rightarrow & \wedge^2T^*\otimes S_2T^*\otimes T & \rightarrow & \wedge^2T^*\otimes T^*\otimes F_0 & \rightarrow & 0 & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & & & \\
0 \rightarrow & \wedge^3T^*\otimes g_1 & \rightarrow & \underline{\wedge^3T^*\otimes T^*\otimes T} & \rightarrow & \wedge^3T^*\otimes F_0 & \rightarrow & 0 & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & & & \\
0 \rightarrow & \wedge^4T^*\otimes T & = & \wedge^4T^*\otimes T & \rightarrow & 0 & & & & & \\
 & \downarrow & & \downarrow & & & & & & & \\
 & 0 & & 0 & & & & & & &
\end{array}
$$

Vertical down arrows are δ-maps of Spencer

$$g_1 \simeq \wedge^2T^* \subset T^*\otimes T,\ g_2 = 0 \Rightarrow g_3 = 0 \Rightarrow g_4 = 0$$

SNAKE CHASE $\Rightarrow$ $F_2 = H^3(g_1)$ *SPENCER COHOMOLOGY)*

$$
\begin{array}{ccccccccc}
0 \longrightarrow & F_2 & \longrightarrow & \wedge^3T^*\otimes g_1 & \stackrel{\delta}{\longrightarrow} & \wedge^4T^*\otimes T & \longrightarrow 0 \\
0 \longrightarrow & 20 & \longrightarrow & 24 & \stackrel{\delta}{\longrightarrow} & 4 & \longrightarrow 0
\end{array}
$$

$$
\begin{array}{lccccccc}
 & B^i_{1,234} & - & B^i_{2,341} & + & B^i_{3,412} & - & B^i_{4,123} = 0 \\
 & B_{i1,1} & - & B_{i2,2} & + & B_{i3,3} & - & B_{i4,4} = 0 \\
i = 4 \Rightarrow & B_{41,1} & - & B_{42,2} & + & B_{43,3} & = & 0 \\
 & L_{23,1} & + & L_{31,2} & + & L_{12,3} & = & 0
\end{array}
$$

LANCZOS $\boxed{L_{ij,k} + L_{ji,k} = 0, \quad L_{ij,k} + L_{jk,i} + L_{ki,j} = 0}$ (24-4=20)

CLASSICAL / CONFORMAL

CLASSICAL: $\mathcal{L}(\xi)\omega = 0$

$$
\begin{array}{llllllllllll}
n=2 & 2 & \underset{1}{\longrightarrow} & 3 & \underset{2}{\longrightarrow} & 1 & \longrightarrow & 0 & & & & \\
n=3 & 3 & \underset{1}{\longrightarrow} & 6 & \underset{2}{\longrightarrow} & 6 & \underset{1}{\longrightarrow} & 3 & \longrightarrow & 0 & & \\
n=4 & 4 & \underset{1}{\overset{K}{\longrightarrow}} & 10 & \underset{2}{\overset{R}{\longrightarrow}} & 20 & \underset{1}{\overset{B}{\longrightarrow}} & 20 & \underset{1}{\longrightarrow} & 6 & \longrightarrow & 0
\end{array}
$$

Euler-Poincaré : $4 - 10 + 20 - 20 + 6 = 0$

CONFORMAL: $\mathcal{L}(\xi)\omega = A(x)\omega$

$$\Leftrightarrow \hat{\omega}_{ij} = \omega_{ij} \mid det(\omega) \mid^{-\frac{1}{n}}, \quad \mathcal{L}(\xi)\hat{\omega} = 0$$

$$
\begin{array}{llllllllllllll}
n=3 & 3 & \underset{1}{\longrightarrow} & 5 & \underset{3}{\overset{?}{\longrightarrow}} & 5 & \underset{1}{\longrightarrow} & 3 & \longrightarrow & 0 & & & & \\
n=4 & 4 & \underset{1}{\longrightarrow} & 9 & \underset{2}{\longrightarrow} & 10 & \underset{2}{\longrightarrow} & 9 & \underset{1}{\longrightarrow} & 4 & \longrightarrow & 0 & & \\
n=5 & 5 & \underset{1}{\overset{CK}{\longrightarrow}} & 14 & \underset{2}{\overset{W}{\longrightarrow}} & 35 & \underset{1}{\overset{?}{\longrightarrow}} & 35 & \underset{2}{\longrightarrow} & 14 & \underset{1}{\longrightarrow} & 5 & \longrightarrow & 0
\end{array}
$$

Euler-Poincaré : $5 - 14 + 35 - 35 + 14 - 5 = 0$

COMPUTER ALGEBRA WANTED

VESSIOT STRUCTURE CONSTANTS

LIE PEUDOGROUP:

$$y = f(x) \in aut(\mathbb{R}^2), \Delta(x) = det(\partial_i f^k(x)) \neq 0$$

$$\Gamma = \{y^1 = f(x^1), y^2 = x^2/\partial_1 f(x^1)\}$$

SYSTEM: $y^2 dy^1 = x^2 dx^1 \Rightarrow dy^1 \wedge dy^2 = dx^1 \wedge dx^2 \Leftrightarrow \Delta = 1$

GENERAL OBJECT: $\omega = (\alpha, \beta) \in \mathcal{F} = \wedge^1 T^* \times_X \wedge^2 T^*$

LIE OPERATOR: $\mathcal{D}\xi \equiv \mathcal{L}(\xi)\omega = 0 \Leftrightarrow \{\mathcal{L}(\xi)\alpha = 0, \mathcal{L}(\xi)\beta = 0\}$

MEDOLAGHI EQUATIONS:

$$\{\alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i = 0, \ \ \beta \partial_r \xi^r + \xi^r \partial_r \beta = 0\}$$

SPECIAL OBJECT: , $\alpha = x^2 dx^1, \beta = dx^1 \wedge dx^2 \Rightarrow \omega = (x^2, 0, 1)$

VESSIOT STRUCTURE EQUATIONS: $\boxed{d\alpha = c\beta, \ \ c = cst}$

DIFFERENTIAL SEQUENCE: $\xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta$

$$\left\{ \begin{array}{l} \xi^1 \\ \xi^2 \end{array} \right. \longrightarrow \left\{ \begin{array}{rcl} \alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i & = & \eta^i \\ \beta \partial_r \xi^r + \xi^r \partial_r \beta & = & \eta^3 \end{array} \right. \longrightarrow \partial_1 \eta^2 - \partial_2 \eta^1 - c\eta^3 = \zeta$$

ADJOINT SEQUENCE: $\nu \overset{ad(\mathcal{D})}{\longleftarrow} \mu \overset{ad(\mathcal{D}_1)}{\longleftarrow} \lambda$

$$\lambda \mid \partial_1 \eta^2 - \partial_2 \eta^1 - c\eta^3$$

$$ad(\mathcal{D}_1) \left\{ \begin{array}{lcrcl} \eta^1 & \rightarrow & \partial_2 \lambda & = & \mu^1 \\ \eta^2 & \rightarrow & -\partial_1 \lambda & = & \mu^2 \\ \eta^3 & \rightarrow & -c\lambda & = & \mu^3 \end{array} \right. \qquad \boxed{\text{INJECTIVE} \Leftrightarrow c \neq 0}$$

$$ad(\mathcal{D}) \left\{ \begin{array}{lcrcl} \xi^1 & \rightarrow & -\alpha_1 \partial_r \mu^r + \beta(c\mu^2 - \partial_1 \mu^3) & = & \nu^1 \\ \xi^2 & \rightarrow & -\alpha_2 \partial_r \mu^r + \beta(-c\mu^1 - \partial_2 \mu^3) & = & \nu^2 \end{array} \right.$$

$$\left\{ \begin{array}{lll} \bullet \ c = 0 & \Rightarrow & \partial_1 \mu^1 + \partial_2 \mu^2 = 0, \mu^3 = 0 \\ \bullet \ c \neq 0 & \Rightarrow & \partial_1 \mu^1 + \partial_2 \mu^2 = 0, \partial_1 \mu^3 - c\mu^2 = 0, \partial_2 \mu^3 + c\mu^1 = 0 \end{array} \right.$$