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Solution of non homogenous Ordinary Differential equations using Parametric Integral Method

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The solution of non homogenous ordinary differential equation (ODE) is an important research subject appearing in numerous engineering fields. When the ODE is associated with boundary conditions (BC), the problem is referred to as a Boundary Value Problem (BVP). Numerical schemes such as finite differences and finite elements have been used for the solution of such problem.

A general homogeneous ODE may be expressed as:

$$\begin{cases} \sum\limits_{n=0}^{n=p} a_n(x) \frac{d^{(n)}y}{dx^n} = 0 \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{1}$$

This equation may be decomposed into the homogenous part and a non homogenous part, using a MacLaurin expansion of each coefficient $a_n(x)$. For any $n \in \{1, ..., p\}$, we have:

$$a_n(x) = a_n(0) + a_n'(0)x + \frac{x^2}{2} a_n''(0) + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} a^{(n)}(0) \tag{2}$$

Inserting this last identity into Equation (1) leads to:

$$\begin{cases} L_0(y) = -L(y) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{3}$$

where the differential operator L is defined by:

$$L = \sum_{n=0}^{n=p} \left[\sum_{n=1}^{\infty} \frac{x^n}{n!} a^{(n)}(0) \right] \frac{d^{(n)}}{dx^n} \tag{4}$$

The operator L_0 is defined as :

$$L_0 = \sum_{n=0}^{n=p} a_n(0) \frac{d^{(n)}}{dx^n} \tag{5}$$

In this contribution we present a general methodology based on the Adomian decomposition method (ADM) as described in [3]), where the inverse operator L^{-1} is expressed in terms of eigenvectors and eigenvalues expansion. The ADM is a systematic method for solution of either linear or nonlinear operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It enables to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) (see [5]) without physical restrictive assumptions, such as those required by linearization, perturbation, ad hoc assumptions, and guessing the initial term or a set of basis functions.

Using ADM, we denote a possible solution by $y(x) = \sum_{m=0}^{\infty} y_m(x)$. A general solution of the non homogenous ODE may be found in an iterative way as follows:

- Solve for $y_0(x)$:

$$\begin{cases} L_0(y_0) = 0 \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{6}$$

- Solve for $y_m(x); m = 1, 2, ...$

$$\begin{cases} L_0(y_m) = -L(y_{m-1}) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{7}$$

After solving for $y_0(x)$, the general solution for Equations (7) may be derived using the Green function associated with the operator L_0.

$$\begin{cases} L_0(G(x,\xi)) = \delta(x-\xi) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases}$$

Using Equation (8) and suitable boundary conditions for $G(x, \xi)$, we obtain an iterative solution for $m \geq 1$:

$$y_m(x) = \int_a^b G(x,\xi) L(y_{m-1}(\xi) d\xi \tag{8}$$

In a large class of boundary value problems, the Green function $G(x, \xi)$ may be expressed as an eigenfunction expansion as follows:

$$G(x,\xi) = \sum_{r=1}^{r=q} \frac{\phi_r(x)\phi_r(\xi)}{\lambda_r} \tag{9}$$

where λ_r is the eigenvalue associated with the eigenfunction $\phi_r(x)$ which is the solution of the following ODE:

$$\begin{cases} L_0(\phi_r) = \lambda_r \phi_r \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{10}$$

So finally the iterative Adomian solution of Equation (7) may be written as:

$$y_m(x) = \sum_{r=1}^{r=q} \frac{\phi_r(x)}{\lambda_r} \int_a^b \phi_r(\xi) L(y_{m-1}(\xi) d\xi \tag{11}$$

In this talk, this last expression will be used to generate different types of iterative algorithms for the solution of the BVP. This iterative algorithm generates an iterative algorithm which can be implemented in a CAS. As examples, we will present solutions of groundwater flow through non homogenous formations using parametric integral solutions. This type of integrals have been already analysed by the authors in [1, 2, 4].

Keywords: parametric integral, non homogenous ODE, Adomian decomposition method

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