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The Roland and Astrid Dana-Picard Chair for Mathematics, Education and Judaism

ACA Santiago de Compostela 2018

OROT ISRAEL COLLEGE OF EDUCATION
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AADIOS

Solution of non homogenous stiff Ordinary Differential Equations using a Parametric Integral Method

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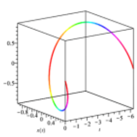

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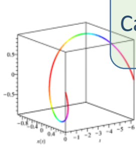
Parametric integrals (Ref: DP,Z)

- Ubiquitous in engineering
- Build bridges between Integrals, Series, Combinatorics, etc.
- Examples:

$$\int_0^1 x^p\,(\ln x)^q\,dx = \frac{(-1)^q\,q!}{(p+1)^{q+1}}$$

$$\int_0^{\sqrt{n}}\left(1-\frac{x^2}{n}\right)^n dx = \sqrt{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}$$

Catalan numbers: $C_p = \dfrac{4^{p-1}\cdot\pi}{a^{2p+1}(p+1)}\displaystyle\int_{-a}^{a} x^{2p}\sqrt{\frac{a+x}{a-x}}\,dx$



An example from soil mechanics (DPZ in iJMEST 2017)

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

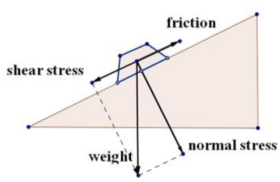
If k is a non-negative integer, then the following holds:

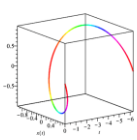
$$I_{4k} = \frac{\pi}{4} + \sum_{l=1}^{2k} \frac{(-1)^l}{2l-1}$$

$$I_{4k+1} = \frac{\ln 2}{2} + \sum_{l=1}^{2k} \frac{(-1)^l}{2l}$$

$$I_{4k+2} = -\frac{\pi}{4} - \sum_{l=1}^{2k+1} \frac{(-1)^l}{2l-1}$$

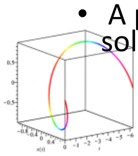
$$I_{4k+3} = -\frac{\ln 2}{2} - \sum_{l=1}^{2k+1} \frac{(-1)^l}{2l}$$





Stiff ODEs – different definitions

- A stiff problem is one for which no solution component is unstable (i.e. no eigenvalue of the Jacobian matrix has a real part which is at all large and positive) and at least one component is very stable (i.e. at least one eigenvalue has a real part which is large and negative).
- A problem is **stiff** if the solution being sought varies slowly but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results.
- **Stiffness** occurs when some components decay more rapidly than others.
- The matrix A in the linear system of differential equations $\frac{du}{dt} = Au, t \in [0,T]$ has negative eigenvalues.
- A problem is **stiff** if explicit methods fail to provide solutions or works extremely slowly.



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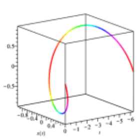
The topic

- Stiff ordinary differential equations arise frequently in the study of chemical kinetics, electrical circuits, vibrations, control systems and so on.
- It is a difficult and important concept in the study of differential equations.
- It depends on the differential equation itself, the initial conditions, and the numerical method.



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Example: Van de Pol equation for relaxation oscillation

The **Van der Pol oscillator** is a non-conservative oscillator with non-linear damping.

> $\mu := 1000$

$\mu := 1000$

> $ode := \frac{d^2}{dx^2} y(x) - \mu\,(1 - y(x)^2)\left(\frac{d}{dx} y(x)\right) + y(x) = 0$

$ode := \frac{d^2}{dx^2} y(x) - 1000\,(1 - y(x)^2)\left(\frac{d}{dx} y(x)\right) + y(x) = 0$

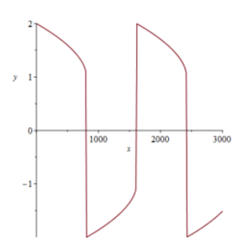
> $ic := \{y(0) = 2, D(y)(0) = 0\}$

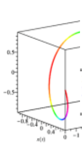
$ic := \{y(0) = 2, D(y)(0) = 0\}$

> sol_ode := dsolve({ode} ∪ ic, numeric, range = 0 ..3000, stiff = true)

sol_ode := **proc**(x_rosenbrock) ... **end proc**

> plots[odeplot](sol_ode, [x, y(x)])





> sol2_ode := dsolve({ode} ∪ ic, numeric, range = 0 ..3000, stiff = false)

```
Warning, cannot evaluate the solution further right of 5.2538547, maxfun limit exceeded (see ?dsolve,maxfun for details)
```

[Length of output exceeds limit of 1000000]

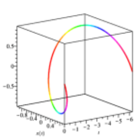
Stability of the solution

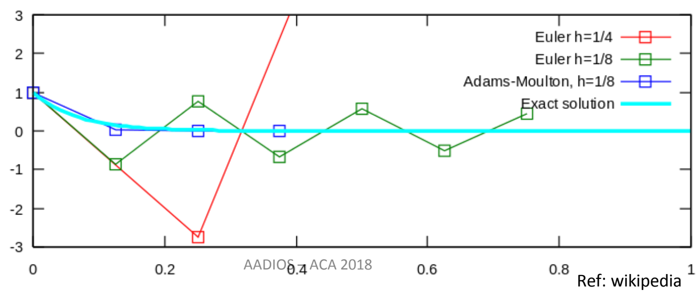
$$\begin{cases} y'(t) = -15y(t) \\ y(0) = 1 \\ t > 0 \end{cases}$$

1.Euler's method with a step size of *h=1/4* oscillates wildly and quickly exits the range of the graph - shown in red.
2.Euler's method with h=1/8 produces a solution within the graph boundaries, but oscillates about zero - shown in green.
3.The trapezoidal method (that is, the two-stage Adams–Moulton method) is given by

$$y_{n+1} = y_n + \frac{1}{2}h\left(f\left(t_n, y_n\right) + f\left(t_{n+1}, y_{n+1}\right)\right)$$

where $y' = f(t, y)$.
Applying this method instead of Euler's method gives a much better result (blue). The numerical results decrease monotonically to zero, just as the exact solution does .





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Ref: wikipedia

General ODE and Boundary Value Problem

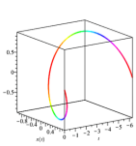
A general ODE may be expressed as follows:

$$\begin{cases} \sum_{n=0}^{n=p} a_n(x)\frac{d^{(n)}y}{dx^n} = f(x) \\ a \leq x \leq b \\ BCatx = ax = b \end{cases}$$

Homogeneous if f≡0

Decompose it into homogeneous and non-homogeneous part using MacLaurin developments:

$$a_k(x) = a_k(0) + a_k'(0)x + \frac{x^2}{2}a_k''(0) + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}a^{(k)}(0).$$



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General ODE and Boundary Value Problem

By substitution, we obtain:

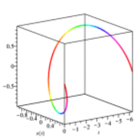
$$\begin{cases} L_0(y) = -L(y) \\ a \leq x \leq b \\ BCatx = ax = b \end{cases}$$

Where the differential operator L is defined by:

$$L = \sum_{n=0}^{n=p} \left[\sum_{n=1}^{\infty} \frac{x^n}{n!} a^{(n)}(0) \right] \frac{d^{(n)}}{dx^n}.$$

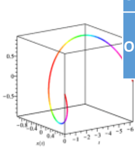
And

$$L_0 = \sum_{n=0}^{n=p} a_n(0) \frac{d^{(n)}}{dx^n}$$



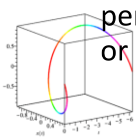
Matlab Library

Solver	Kind of Problem	Base Algorithm
Ode45	Non-stiff differential equations	Runge-Kutta
Ode23	Non-stiff differential equations	Runge-Kutta
Ode113	Non-stiff differential equations	Adams-Bashfort-Moulton
Ode15s	Stiff differential equations	Numerical Differentiation Formulas (Backward Differentiation Formulas)
Ode23s	Stiff differential equations	Rosenbrock
Ode23t	Moderately stiff differential equations	Trapezoidal Rule
Ode23tb	Stiff differential equations	TR-BDF2
Ode15i	Fully implicit differential equations	BDFs



Methodology based on the Adomian decomposition method

- The **Adomian decomposition method** (ADM) is systematic method for solution of either linear or nonlinear operator equations, including ODEs, PDEs, integral equations, integro-differential equations, etc.
- The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering.
- It allows to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) without physical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions.



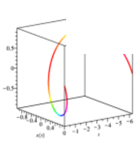
Using an Adomian decomposition method ([1]), we assume a solution $y(x) = \sum_{m=0}^{\infty} y_m(x)$. Then the general solution of the non homogeneous ODE may be solved in an iterative way as follows:

- Solve for $y_0(x)$:

$$
\begin{aligned}
L_0(y_0) &= 0 \\
a \leq x &\leq b \\
BCatx = ax &= b
\end{aligned}
$$

- Solve for $y_m(x); m = 1, 2, \ldots.$

$$
\begin{aligned}
L_0(y_m) &= -L(y_{m-1}) \\
a \leq x &\leq b \\
BCatx = ax &= b
\end{aligned}
$$



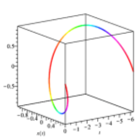
Eigenvalues expansion

After solving for $y_0(x)$, the general solution of the equations (8) may be derived using the Green function associated with the operator L_0.

$$\begin{aligned} L_0(G(x,\xi)) &= \delta(x-\xi) \\ a \leq x &\leq b \\ BCatx &= ax = b \end{aligned}$$

Using this and suitable boundary conditions for $G(x,\xi)$, one obtains an iterative solution for m≥1:

$$y_m(x) = \int_a^b G(x,\xi) L(y_{m-1}(\xi) d\xi$$



Iterative solution

In a large class of boundary value problems, the Green function $G(x,\xi)$ may expressed as an eigenfunction expansion as follows:

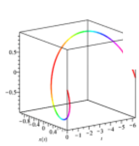
$$G(x,\xi) = \sum_{r=1}^{r=q} \frac{\phi_r(x)\phi_r(\xi)}{\lambda_r},$$

where λ_r is the eigenvalue associated with the eigenfunction $\phi_r(x)$ which is the solution of the following ODE:

$$\begin{aligned} L_0(\phi_r) &= \lambda_r \phi_r \\ a \leq x &\leq b \\ BCatx &= ax = b \end{aligned}$$

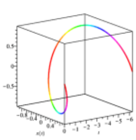
So finally the iterative Adomian solution of equation (8) may be written as:

$$y_m(x) = \sum_{r=1}^{r=q} \frac{\phi_r(x)}{\lambda_r} \int_a^b \phi_r(\xi) L(y_{m-1}(\xi) d\xi.$$



Remarks

- This last expression will be used to generate different types of iterative algorithms for the solution of the BVP.
- This iterative algorithm generates an iterative algorithm which can be implemented in a CAS.
- We wish to mention solutions of groundwater flow through non homogeneous formations using **parametric integral** solutions.



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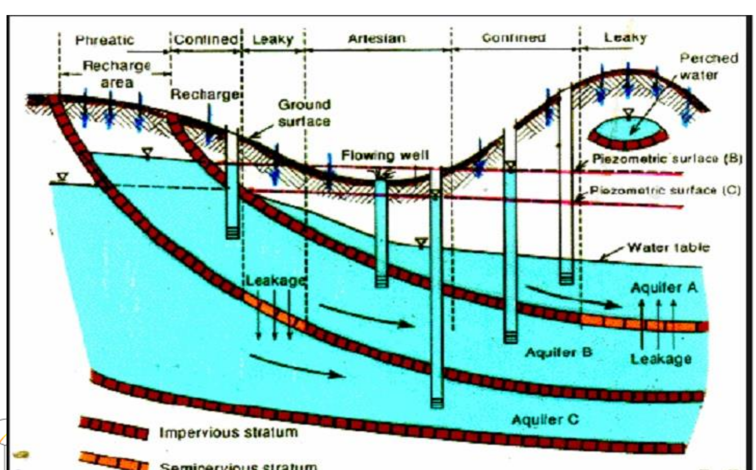
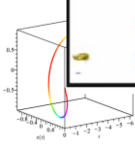
Motivation:
Artificial recharge of Confined aquifers





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Non Homogenous Problems

Consider, as an example, the well known two dimensional non homogeneous boundary value problem on the interval $[0, L]$:

$$\frac{d^2y}{dx^2} + k^2y = f(x) \tag{19}$$

(20)

The boundary conditions are: $y(0) = y(L) = 0$

To simply the calculations assume that $L = \pi$. The differential operator is $L_0 = d^2/dx^2 + k^2$ and we seek an eigenfunction expansion satisfying the Sturm Liouville equation:

$$\frac{d^2y_n}{dx^2} + k^2y_n + \lambda_n y = 0) \tag{21}$$

$$y_n(0) = y_n(\pi) = 0 \tag{22}$$

(23)

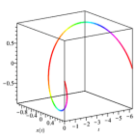
In general $y_n = A_n sin(nx) + B_n cos(nx)$ and the corresponding eigenvalues λ_n is given by $\lambda_n == n^2 - k^2$. The boundary conditions in 23 requires $B_n = 0$ and $y_n(x) = \sqrt{\frac{2}{\pi}} sin(nx)$. The functions $sin(nx)$ are the eigenfunctions of the ODE equations 23 The general solution of the non homogeneous equation 20 is given by: $y(x) = \sum_{n=1}^{\infty} a_n y_n(x)$ where $a_n = -\frac{1}{n^2-k^2} \int_0^{\pi} (\frac{2}{\pi})^{1/2} \sin(nz)dz$

Finally the solution of the non homogeneous equation 20 is given by:

$$y(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2 - k^2} \int_0^{\pi} f(z) \sin(nz)dz \tag{24}$$

When using a string of length L, this last equation reduces to:

$$y(x) = -\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi x}{L})}{(\frac{n\pi}{L})^2 - k^2} \int_0^{L} f(z) \sin(\frac{n\pi z}{L})dz \tag{25}$$



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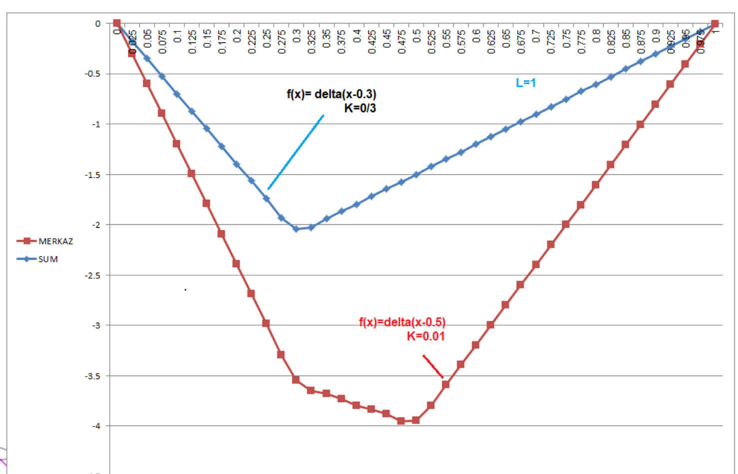
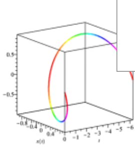
Non Homogenous Problems





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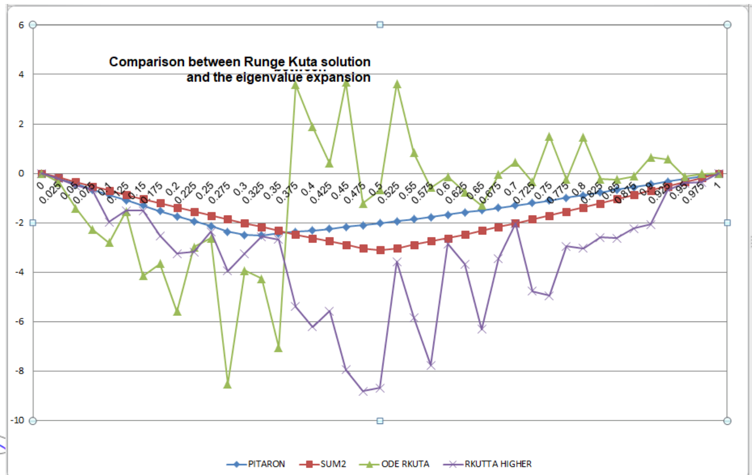
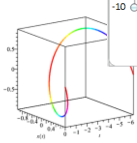
Non Homogenous Problems





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Heterogeneous problems

Consider the well known 2-dimensional heterogeneous boundary value problem appearing in groundwater hydrology on the interval $[0, L]$:

$$\frac{d}{dx}\left[T(x)\frac{dy}{dx}\right] + k^2 y = 0 \tag{19}$$

The boundary conditions are: $y(0) = y(L) = 0$

By expanding $T(x)$ using MacLaurin expansion or more simply $T(x) = T_0 + D(x)$. Then the heterogeneous equation may be approximated as:

$$\frac{d}{dx}\left[T_0)\frac{dy}{dx}\right] + k^2 y = -\frac{d}{dx}\left[D(x)\frac{dy}{dx}\right] \tag{20}$$

If we define $K^2 = \frac{k^2}{T_0}$, 20 may be rewritten as:

$$\frac{d^2y}{dx^2} + K^2 y = -\frac{1}{K_0}\frac{d}{dx}\left[D(x)\frac{dy}{dx}\right] \tag{21}$$

Using the Adomian method and expanding $y(x)$ as

$$y(x) = y_0(x) + y_1(x) + y_2(x) + ... = \sum_{l=0}^{\infty} y_l(x), \tag{22}$$

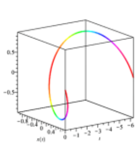
the equations (21) may be solved iteratively as:

1. Solution for $y_0(x)$

$$\begin{cases} \frac{d^2y_0}{dx^2} + K^2 y_0 = 0 \\ y_0(0) = y_0(L) = 0 \end{cases} \tag{23}$$

2. Solution for $y_n(x); n \geq 1$

$$\frac{d^2y_n}{dx^2} + K^2 y_n = -\frac{1}{K_0}\frac{d}{dx}\left[D(x)\frac{dy_{n-1}}{dx}\right] \tag{24}$$



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Heterogeneous problems

Using the solution of the non homogeneous equation (12) on can get:

$$y_n(x) = \frac{2}{LK_0} \sum_{n=1}^{\infty} \frac{sin(\frac{n\pi x}{L})}{(\frac{n\pi}{L})^2 - k^2} \int_0^{\pi} \left[\frac{d}{dz}[D(z)\frac{dy_{n-1}}{dz}\right] sin(\frac{n\pi z}{L})dz \quad (25)$$

Using integration by parts, one obtains:

$$\int_0^{\pi} [\frac{d}{dz} \left[D(z)\frac{dy_{n-1}}{dz}\right] sin(\frac{n\pi z}{L})dz = \quad (26)$$

$$\left[D(z)\frac{dy_{n-1}}{dz}\right] \sin(\frac{n\pi z}{L}]_0^{\pi} - \frac{n\pi}{L} \int_0^{\pi} \left[D(z)\frac{dy_{n-1}}{dz}\right] \cos(\frac{n\pi z}{L})dz \quad (27)$$

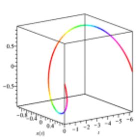
Then

$$J_n = \int_0^{\pi} [\frac{d}{dz} \left[D(z)\frac{dy_{n-1}}{dz}\right] sin(\frac{n\pi z}{L})dz = \quad (28)$$

$$-\frac{n\pi}{L} \int^{\pi} \left[D(z)\frac{dy_{n-1}}{dz}\right] cos(\frac{n\pi z}{L})dz \quad (29)$$

So finally, we have a generic relation:

$$y_n(x) = -\frac{2}{LK_0} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi x}{L})}{(\frac{n\pi}{L})^2 - k^2} \frac{n\pi}{L} \int_0^{\pi} \left[D(z)\frac{dy_{n-1}}{dz}\right] \cos(\frac{n\pi z}{L})dz \quad (30)$$



When $D(z) = \delta z - x_0$, then a series solution of the stiff ODE is given by:

$$y_n(x) = -\frac{2}{LK_0} \sum_{n=1}^{\infty} \frac{sin(\frac{n\pi x}{L})}{(\frac{n\pi}{L})^2 - k^2} \frac{n\pi}{L} \left[D(x_0)\frac{dy_{n-1}}{dz}\right] (x_0)cos(\frac{n\pi x_0}{L}) \quad (31)$$

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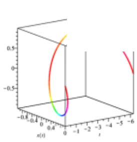
Heterogeneous problems – an algorithm

The above equations are the basis of an algorithm to plot the solution of equation 34. The different steps are:

- Initialization: Choose N points in the range $[0, L]$.
- Compute $y_0(x_i) i = 1 \cdots N$.
- Step 1: $n = 1$.
- For $i = 1$ to N_{max} ; Compute $\frac{dy_{n-1}}{dz}(x_i) i = 1, \cdots N$.
- Compute J_n numerically.
- $i = 1, \cdots N$, Compute $y_n(x_i)$.
- $y(x_i) = \sum_0^n y_n(x_i)$.
- If $|y_n(x_i)| \leq \epsilon$ then STOP.
- $n = n + 1$, then GOTO Step 1.



Conclusions

- A general presentation of the use eigenfunction expansion for non homogenous and hetrogenous ODE has been presented.
- This different methods presented may be added to existing libraries in CAS software such as Matlab and Maple.
- We wish to mention solutions of groundwater flow through non homogeneous formations using **parametric integral** solutions.