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An efficient algorithm for the simultaneous triangularization of a finite set of matrices

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In the study of linear differential systems, one can be interested in deciding whether a set of m given square matrices $A_1, \ldots, A_m$ are simultaneously triangularizable or not. If the answer is yes, then we sometimes need to compute effectively an invertible matrix P such that, for all $i \in \{1, \ldots, m\}$, the matrix $P^{-1} A_i P$ is upper triangular. See, for instance, the recent paper [1].

The classical approach consists in using Lie algebra theory to test whether the matrix Lie algebra spanned by the A_i's is solvable (e.g., using the so-called derived series) and if so, find a basis in which all matrices of the Lie algebra are upper triangular using a constructive version of Lie's theorem on solvable algebras for computing common eigenvectors. See [2].

In this presentation, we will rather consider the following result due to McCoy [4]: matrices $A_1, \ldots, A_m$ are simultaneously triangularizable if and only if, for every scalar polynomial $p(x_1, \ldots, x_m)$ in the (non-commutative) variables $x_1, \ldots, x_m$, each of the matrices $p(A_1, \ldots, A_m)[A_i, A_j] = p(A_1, \ldots, A_m)(A_i\, A_j - A_j\, A_i)$ $(i, j = 1, \ldots, m)$ is nilpotent. We shall show that the proof of this result provided in [3] can be turned into an efficient algorithm for computing particular common eigenvectors of $A_1, \ldots, A_m$. As a consequence, this yields an efficient algorithm for the simultaneous triangularization problem. Note that this new approach does not require the construction of the Lie algebra spanned by the matrices A_i's. The algorithm has been implemented in Maple and we will show comparisons to the implementation of the "Lie algebra method" included in the `DifferentialGeometry/LieAlgebras` package of Maple.

Keywords
computer algebra, algorithms, linear algebra, Lie algebras

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