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A decision algorithm for strong rational general solutions of algebraic ordinary differential equations

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We consider first-order algebraic ordinary differential equations (AODEs) and study their rational general solutions. A rational general solution of a first-order AODE contains an arbitrary constant. In case the constant appears rationally, we call the solution strong. We present an algorithm for deciding the existence of a strong rational general solution of a first-order AODE, and in the positive case, compute such a solution. The problem of computing a rational general solution of first-order AODEs has not yet been solved in full generality. Our method is based on optimal parametrizations of algebraic curves over the field of rational functions.

Consider the first-order AODE,

$$F(x, y, y') = 0 ,$$

where F is an irreducible polynomial in three variables over an algebraically closed field $\mathbb{K}$. Replacing y' by a new indeterminate z, we obtain an algebraic equation $F(x, y, z) = 0$. This algebraic equation defines a plane algebraic curve

$$\mathcal{C} := \{(a, b) \in \mathbb{A}^2(\overline{\mathbb{K}(x)}) \,|\, F(x, a, b) = 0\}$$

over the field $\overline{\mathbb{K}(x)}$ of algebraic functions. We call it the corresponding algebraic curve. A parametrization of $\mathcal{C}$ is a rational map

$$\mathcal{P} : \mathbb{A}^1(\overline{\mathbb{K}(x)}) \to \mathcal{C} \subset \mathbb{A}^2(\overline{\mathbb{K}(x)}) ,$$

such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ with respect to the Zariski topology. If furthermore $\mathcal{P}$ is a birational equivalence, it is called a proper parametrization. A parametrization is represented as a pair of rational functions, say $\mathcal{P} = (p_1(t), p_2(t))$, with coefficients in $\overline{\mathbb{K}(x)}$. The field which extends $\mathbb{K}(x)$ by coefficients of $\mathcal{P}$ is called the field of coefficients of $\mathcal{P}$. In case the degree of the field of coefficients over $\mathbb{K}(x)$ is as small as possible, we call $\mathcal{P}$ an optimal parametrization. It is well known that the field of coefficients of an optimal parametrization has at most algebraic extension degree 2 over $\mathbb{K}(x)$.

A rational solution of the differential equation $F(x, y, y') = 0$ is a rational function $y(x) \in \mathbb{K}(x)$, such that $F(x, y(x), y'(x)) = 0$. According to Ritt [1] the radical differential ideal $\{F\}$ can be decomposed as

$$\{F\} = \underbrace{\left(\{F\} : \frac{\partial F}{\partial y'}\right)}_{general\ component} \cap \underbrace{\left\{F, \frac{\partial F}{\partial y'}\right\}}_{singular\ component} .$$

S is the separant of F, i.e., the derivative of F w.r.t. $y^{(n)}$). Ritt shows that the general component is a prime differential ideal; its generic zero is called a *general solution* of the AODE $F(x, y, y') = 0$. Such a general solution must contain a transcendental constant c. In [2, 3] we have presented a method for determining rational general solutions of first-order AODEs. This method is based on rational parametrization of surfaces. Whereas it can determine rational general solutions for almost all parametrizable first-order AODEs, it is not a decicion algorithm.

Here we are a little more modest, and we aim at determining so-called strong rational general solutions. A solution $y(x)$ of the AODE is called a *strong rational general solution*, if $y = y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$, where c is a transcendental constant over $\mathbb{K}(x)$. So a strong rational general solution is a proper rational function in x and c over $\mathbb{K}$.

The key fact with allows to decide the existence of strong rational general solutions, and in the positive case compute them, is the following:

Theorem *Let $F \in \mathbb{K}(x)[y, z]$ be an irreducible polynomial. If the algebraic curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F = 0$ is a rational curve, then the coefficient field of its optimal parametrization is always $\mathbb{K}(x)$.*

A full description of this decision method can be found in [4].

Keywords
ordinary differential equation , algebraic curve, rational parametrization, rational general solution

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