Applications of Computer Algebra – ACA 2021
Virtual. Online | July 23-27, 2021
Session on *"Algebraic and Algorithmic Aspects of Differential and Integral Operators (AADIOS)"*

On sequences associated to the invariant theory of rank two simple Lie algebras

***Alin Bostan*[1]*, Jordan Tirrell*[2]*, Bruce W. Westbury*[3]*, Yi Zhang*[4]** [Yi.Zhang03@xjtlu.edu.cn]

[1] Inria, Université Paris-Saclay, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France
[2] Department of Mathematics and Computer Science, Washington College, USA
[3] Department of Mathematical Sciences, The Unversity of Texas at Dallas, USA
[4] Department of Applied Mathematics, School of Science, Xi'an Jiaotong-Liverpool University, Suzhou, 215123, China

The representation theory of simple Lie algebras is a cornerstone of algebraic and enumerative combinatorics, giving rise to combinatorial objects such as tableaux, symmetric functions, quantum groups, crystal graphs, and so on. We are interested in two families of sequences in OEIS. Sequences in the first family of sequences are called *octant sequences*. For example, sequence A059710, which is the first octant sequence, is defined to be a sequence associated to fundamental representations of the exceptional simple Lie algebra G_2, of rank two and dimension fourteen [4]. The second octant sequence is A108307 [5], and it is defined to be the cardinality of the set of set partitions of $[n]$ with no enhanced 3-crossing. Our first contribution is to prove that sequences A059710 and A108307 are tightly related by a binomial transform.

Theorem 0.1. *Let $T_3(n)$ and $E_3(n)$ be the n-th terms of A059710 and A108307, respectively. Then E_3 is the binomial transform of T_3, , for $n \geq 0$,*

$$E_3(n) = \sum_{k=0}^{n} \binom{n}{k} T_3(k).$$

Theorem 0.1 provides an unexpected connection between invariant theory of G_2 and combinatorics of set partitions. In the same spirit, [5] and [3] prove a binomial relation between E_3 and A108304, which is the third octant sequence, respectively. In summary, these two results show that the octant sequences are associated to representations of G_2.

Based on Theorem 0.1, as well as on results by Bousquet-Mélou and Xin [1], our second contribution is to give two independent proofs of a recurrence equation for T_3 conjectured by Mihailovs [6], which was the initial motivation for our study:

Theorem 0.2. *The sequence T_3 is determined by the initial conditions $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and the recurrence relation that for $n \geq 0$,*

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0. \quad (1)$$

Moreover, we give an alternative proof of Theorem 0.2, using the interpretation of T_3 in terms of G_2 walks and using algorithms for computing Picard-Fuchs differential equations for algebraic residues. As a consequence, closed formulae for the generating function of T_3 are obtained in terms of the classical Gaussian hypergeometric function.

We consider a second family of sequences, called *quadrant sequences*. These are defined to be sequences associated to representations of G_2 restricted to $SL(3)$. By invariant theory, these sequences are also are related by binomial transforms. Based on this, we derive a uniform recurrence equation holding *for each quadrant sequence*. Furthermore, we show that sequences in the second family are identical to quadrant sequences because they satisfy the same initial conditions and recurrence equations. They are related to the octant sequences by the branching rules [1] for the maximal subgroup $SL(3)$ of G_2.

Keywords
representation theory, binomial transforms, recurrence equations

References

[1] M. BOUSQUET-MÉLOU; G. XIN, On partitions avoiding 3-crossings. *Sém. Lothar. Combin..* **21**(54), Art. B54e (2005).
[2] R. GASKELL; R. T. SHARP, Generating functions for G_2 characters and subgroup branching rules. *J. Math. Phys.* **22**(12), 2736–2739 (1981).
[3] J. B. GIL; J. O. TIRRELL, A simple bijection for enhanced, classical, and 2-distant k-noncrossing partitions. *Disc. Math.*, **343**(6), 111705 (2020)
[4] G. KUPERBERG, Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, **180**(1), 109–151 (1996)
[5] Z. LIN, Restricted inversion sequences and enhanced 3-noncrossing partitions. *European J. Combin..*, **70**, 202–211 (2018)
[6] B. W. WESTBURY, Enumeration of non-positive planar trivalent graphs. *J. Algebraic Combin.*, **25**(4), 357–373 (2007)