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Rational Solutions of First-Order Algebraic Ordinary Difference Equations

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An algebraic ordinary difference equation (AOΔE) is a difference equation of the form

$$F(x, y(x), y(x+1), \cdots, y(x+m)) = 0,$$

where F is a nonzero polynomial in $x, y(x), y(x+1), \cdots, y(x+m)$ with coefficients in an algebraically closed field $\mathbb{K}$ of characteristic zero, and $m \in \mathbb{N}$. We say that an AOΔE is *autonomous* if the independent variable x does not appear in it explicitly. For computational purpose, we may choose $\mathbb{K} = \bar{\mathbb{Q}}$, the field of algebraic numbers. AOΔEs naturally appear from various problems, such as symbolic summation [2], factorization of linear difference operators [1], analysis of time or space complexity of computer programs with recursive calls [3]. Thus, to determine (closed form) solutions of a given AOΔE is a fundamental problem in difference algebra and is of general interest.

We are mainly interested in rational solutions of first-order AOΔEs. In [3], Feng, Gao and Huang proposed an algorithm for computing a rational solution for a first-order autonomous AOΔE provided that a bound for the degree of the rational solution is given. They also pointed out that they could not bound the degrees of rational solutions through the parametrization technique because the difference version of Theorem 3.7 in [2] is not always true (see Example 4.1 in [3]). We overcome this missing part and present an algorithm for computing such a degree bound, and thus derive a complete algorithm for computing corresponding rational solutions.

Keywords
algebraic ordinary difference equations; strong rational general solutions; parametrization; separable difference equation; resultant theory; algorithms

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