The indicial equation of the product of linear ordinary differential operators

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Abstract

The roots of the indicial equation, constructed for a given linear ordinary differential operator, provide an important information on the solutions of the corresponding homogeneous differential equation. Operators are considered whose coefficients are formal Laurent series. The structure of the indicial equation of the product of given differential operators is described.

1 Introduction

A *formal Laurent series* over a field K of zero characteristic is an expression $s(x)$ of the form

$$\sum_{n=-\infty}^{\infty} c_n x^n, \tag{1}$$

where all c_n, i.e. *coefficients* of the series, belong to K, and there exists $m \in \mathbb{Z}$ such that $c_n = 0$ for $n < m$. For formal Laurent series, the basic arithmetic operations and differentiation of D with respect to x are defined. The ring of formal Laurent series in x over K is denoted by $K((x))$.

The differential operators considered below have the form

$$a_r(x)D^r + \cdots + a_1(x)D + a_0(x), \tag{2}$$

$a_0, \ldots, a_r \in K((x))$. Let $s(x) \in K((x)) \setminus \{0\}$ and ν be the smallest integer such that the coefficient c_ν of the series $s(x)$ is nonzero. We will denote this ν by $\operatorname{val} s(x)$ and call it the *valuation* of the series $s(x)$. The coefficient c_ν is called the *trailing coefficient* of the series $s(x)$ and is denoted by $\operatorname{tc} s(x)$. By definition, it is assumed that $\operatorname{val}(0) = \infty$, $\operatorname{tc}(0) = 0$.

As it is known, for a nonzero L of the form (2) there exist an integer ω_L (the *increment*) and a polynomial $I_L(\lambda) \in K[\lambda]$ (the *indicial polynomial*) such that for any $s(x) \in K((x))$ we have

$$L(s(x)) = c_\nu I_L(\nu) x^{\nu+\omega_L} + O(x^{m+\omega_L+1}),$$

where $\nu = \operatorname{val} s(x)$ and $c_\nu = \operatorname{tc}(s(x))$. (The notation $O(x^t)$, where $t \in \mathbb{Z}$, is used for an explicitly unspecified formal series, whose valuation is greater than or equal to t). Thus, for a nonzero $s(x)$, the equality $I_L(\operatorname{val} s(x)) = 0$ is a necessary condition for $L(s(x)) = 0$.

The algebraic equation $I_L(n) = 0$ is called the *indicial equation* [1] of the operator L. For L of the form (2) we have

$$\omega_L = \min_{0 \le j \le r} (\operatorname{val} a_j - j), \qquad I_L(n) = \sum_{\substack{0 \le j \le r \\ \operatorname{val} a_j - j = \omega_L}} \operatorname{tc}(a_j) n^{\underline{j}},$$

where $n^{\underline{j}} = n(n-1)\cdots(n-j+1)$.

2 The product of operators

Consider the problem of finding the indicial polynomial and the increment of the product of given operators $L_1,\ L_2$, for each of which the indicial polynomial and increment are known. Is it possible to do this without the calculation of the product $L_1 L_1$ itself? The answer is the following:

$$\omega_{L_1 L_2} = \omega_{L_1} + \omega_{L_2}, \quad I_{L_1 L_2}(\lambda) = I_{L_1}(\lambda + \omega_{L_2})\ I_{L_2}(\lambda). \tag{3}$$

The indicated multiplicative property of increments and indicial polynomials is preserved when passing from formal series to convergent ones.

The proof of (3) can be found, e.g., in [2]. In [3, Sect.2.3] a similar result, along with a more general discussion on the Newton polygon and polynomials of a product of two differential operators are proposed.

The author is grateful to A.A. Ryabenko for her help in preparing this note.

References

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