Separated Variables on Plane Algebraic Curves

Manfred Buchacher
Johannes Kepler Universität Linz
Austria
manfredi.buchacher@gmail.com

Abstract

We consider equations of the form

$$r(x,y)+q(x,y)p(x,y)=f(x)-g(y),$$

for rational functions $r(x,y)$, $q(x,y)$, $p(x,y)$, $f(x)$ and $g(y)$ in x and y over $\mathbb{K}$, and explain how they can be solved based on the ideas developed in [1, 2, 3]. The procedure we present reduces the non-linear problem to a linear one. However, the procedure is just a semi-algorithm. It terminates, whenever the equation has a non-trivial solution, but it may not, if there is none. Termination depends on a dynamical system on the curve associated with p and the location of the poles of r thereon. It is still an open question how the semi-algorithm could be turned into an algorithm.

The problem has a field theoretic interpretation. Let $\mathbb{K}(x,y)$ be the field generated by elements x and y satisfying the (only) relation $p(x,y)=0$, and let $\mathbb{K}(x)$ and $\mathbb{K}(y)$ be the subfields generated by x and y, respectively. Then the above equation has a (non-trivial) solution if and only if $r(x,y)$ is an element of $\mathbb{K}(x)+\mathbb{K}(y)$. There are two particular cases that are interesting in themselves: the case $r=0$, and the case $g=0$. The former corresponds to the problem of computing the intersection of $\mathbb{K}(x)$ and $\mathbb{K}(y)$, the latter to the problem of deciding whether $r(x,y)$ is an element of $\mathbb{K}(x)$ and finding all representations thereof in terms of x.

The problem arises in enumerative combinatorics, when solving discrete differential equations by reducing partial DDEs to systems of ordinary ODDEs [4]. It also arises in parameter-identification problems in ODE models [5], and in problems of image recognition [6].

References

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