New algorithm for differential elimination based on support bound

Yulia Mukhina
LIX, CNRS
École Polytechnique, Institute Polytechnique de Paris
Palaiseau, France, 91120
yulia.mukhina@polytechnique.edu

Abstract

Differential elimination refers to finding consequences of a system of differential equations depending only on a chosen subset of variables. In the context of dynamical modeling, one often starts with a polynomial dynamical system of the form $\mathbf{x}' = \mathbf{g}(\mathbf{x})$ and is interested to obtain the minimal equation satisfied by a single component of $\mathbf{x}$ (for example, x_1). Based on the degrees of the polynomials in $\mathbf{g}$, we give an upper bound on the support of such minimal equation which can be further used, for example, for computing this equations using an ansatz. We show that our bound is sharp in “more than half the cases”

1 Introduction

Differential elimination is a differential analogue of elimination for polynomial systems and Gaussian elimination from linear algebra. Its study has been initiated by Ritt [8], the founder of differential algebra, in the 1930s. He developed the foundations of the characteristic set approach, which has been made fully constructive by Seidenberg [9]. The algorithmic aspect of this research culminated in the Rosenfeld-Gröbner algorithm [3, 6] implemented in the BLAD library [4]. We will focus on a special case of differential elimination of practical importance. More precisely, given a polynomial dynamical system, that is, an ODE system of the form

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}), \tag{1}$$

compute an equation of the minimal order satisfied by x_1-component of every solution of (1). This question naturally arises, for example, if experimental data is not available for other variables [5].

2 Preliminaries

In order to state the main problem more precisely, we fix a field $\mathbb{K}$ of characteristic zero. For a set of variables $\mathbf{x} = [x_1, \ldots, x_n]^T$, one can define a *ring of differential polynomial in* $\mathbf{x}$ as a ring of infinitely many variables in $\mathbf{x}$ and its formal derivatives:

$$R = \mathbb{K}[\mathbf{x}^{(\infty)}] = \mathbb{K}[x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ 0 \leqslant j].$$

This ring can be endowed with a structure of a differential ring by defining a derivation $'$ to be zero on $\mathbb{K}$ and $(x_i^{(j)})' = x_i^{(j+1)}$ for every $1 \leqslant i \leqslant n$ and $0 \leqslant j$. For a polynomial $p \in \mathbb{K}[\mathbf{x}^{(\infty)}]$ and $1 \leqslant i \leqslant n$, we will call the largest j such that $x_i^{(j)}$ appears in p the *order* of p respect to x_i, and if p does not involve x_i, we set the order equal to -1.

An ideal $I \subset R$ is called a *differential ideal* if $a' \in I$ for every $a \in I$. For any $f_1, \ldots, f_s \in R$, we denote by $(f_1, \ldots, f_s)^{(\infty)}$ the differential ideal

$$(f_1^{(\infty)}, \ldots, f_s^{(\infty)}).$$

To a system of polynomial ODEs (1), with $\mathbf{g} \in \mathbb{K}[\mathbf{x}]^n$, one can assign a differential ideal $(\mathbf{x}' - \mathbf{g}(\mathbf{x}))^{(\infty)}$. Then, by the differential Nullstellensatz [7, Proposition 2.4], the equations satisfied by the x_1-component of every (formal power series) solution of (1) are exactly the elimination ideal

$$(\mathbf{x}' - \mathbf{g}(\mathbf{x}))^{(\infty)} \cap \mathbb{K}[x_1^{(\infty)}].$$

It is known that this ideal is prime [7, Proposition 1.24] and that it is completely defined its minimal polynomial [7, Proposition 1.15] (polynomials are first ordered by the order and then by the total degree) which we will denote by $f_{\min}$.

3 Main results

Theorem 1 (Bound for the support) *Let $g_1, \ldots, g_n$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\mathbf{x}]$ such that $d := \deg g_1 > 0$ and $D := \max_{2 \leqslant i \leqslant n} \deg g_i > 0$. Let $I := (\mathbf{x}' - \mathbf{g})^{(\infty)}$ and let $f_{\min} \in \mathbb{K}[x_1^{(\infty)}]$ be the minimal polynomial of $I \cap \mathbb{K}[x_1^{(\infty)}]$. Consider a positive integer ν such that $\operatorname{ord} f_{\min} \leqslant \nu$ ($\nu = n$ can be always used).*

Then for every monomial $x_1^{e_0}(x_1')^{e_1} \ldots (x_1^{(\nu)})^{e_\nu}$ in $f_{\min}$ the following inequalities hold

1. *If $d \leqslant D$, then*

$$e_0 + \sum_{k=1}^{\nu} \big(d + (k-1)(D-1)\big) e_k \leqslant \prod_{k=1}^{\nu} \big(d + (k-1)(D-1)\big); \tag{2}$$

2. *If $d > D$, then for every $0 \leqslant \ell < \nu$, we have*

$$\begin{aligned} \sum_{k=0}^{\ell} \big(k(D-1)+1\big) e_k + \sum_{i=1}^{\nu-\ell} \big(i(d-1) + \ell(D-1) + 1\big) e_{i+\ell} \leqslant \\ \leqslant \prod_{k=1}^{\ell} \big(d + (k-1)(D-1)\big) \prod_{i=1}^{\nu-\ell} \big(i(d-1) + \ell(D-1) + 1\big). \end{aligned} \tag{3}$$

We denote by $V_{n,d}$ the space of polynomials of degree at most d in the variables $\mathbf{x} = [x_1, \ldots, x_n]^T$ over the field $\mathbb{K}$.

Theorem 2 (Generic sharpness for $d \leqslant D$) *Let d, D, n be positive integers such that $d \leqslant D$. Then there exists a nonempty Zariski open subset $U \subset V_{n,d} \times V_{n,D}^{n-1}$ such that, for every $\mathbf{g} \in U$, the Newton polytope of the minimal polynomial of $(\mathbf{x}' - \mathbf{g})^{(\infty)} \cap \mathbb{K}[x_1^{(\infty)}]$ is the one given by Theorem 1 with $\nu = n$.*

4 Challenging examples

The bound given by Theorem 1 can be turned into an algorithm [1]. We demonstrate that our implementation of the algorithm can tackle problems which are out of reach for the state-of-the-art software for differential elimination on the example of a model BlueSky exhibiting the blue-sky catastrophe phenomenon (for details about the model and runtimes, see [1, Section 9]):

Name	StructualIdentifiability.jl	Maple(Diff.Thomas)	BLAD	Our Algorithm
BlueSky	> 50h	> 50h	OOM	317 min

Table 1: Comparison with other approaches
OOM = "out of memory"

Rather than using the bound given in Theorem 1 one can also algorithmically compute the support with [2, Section 5], which can offer computational advantages in the case of sparse input systems.

References

[1] Mukhina Y., Pogudin G. Projecting dynamical systems via a support bound. *arXiv preprint arXiv:2501.13680*, 2025.

[2] Mohr R., Mukhina Y. On the Computation of Newton Polytopes of Eliminants. *arXiv preprint arXiv:2502.05015*, 2025.

[3] Boulier F., Lazard D., Ollivier F., Petitot M. *Computing representations for radicals of finitely generated differential ideals.* Applicable Algebra in Engineering, Communication and Computing, **20** (2009), pp. 73–121.

[4] Boulier F. *BLAD.* Bibliothèques Lilloises d'Algèbre Différentielle. URL:`https://pro.univ-lille.fr/francois-boulier/logiciels/blad/`

[5] Dong R., Goodbrake C., Harrington H.A., Pogudin G. *Differential elimination for dynamical models via projections with applications to structural identifiability.* SIAM Journal on Applied Algebra and Geometry, **7** (2023), issue 1.

[6] Hubert E. *Notes on triangular sets and triangulation-decomposition algorithms II: Differential systems. Lecture Notes in Computer Science.* Springer Berlin Heidelberg, (2003), 40–87.

[7] Pogudin G. *Lecture notes on differential algebra.* URL: `http://www.lix.polytechnique.fr/Labo/Gleb.POGUDIN/files/da_notes.pdf`. 2020.

[8] Ritt J.F. *Differential Equations from the Algebraic Standpoint. Colloquium Publications.* American Mathematical Society, Providence, 1932.

[9] Seidenberg A. *An elimination theory for differential algebra.* University of California publications in Mathematics, **3** (1956), pp. 31–66.