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Solving and factoring boundary problems for linear ordinary differential equations in differential algebras☆

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Abstract

We present a new approach for expressing and solving boundary problems for linear ordinary differential equations in the language of differential algebras. Starting from an algebra with a derivation and integration operator, we construct an algebra of linear integro-differential operators that is expressive enough for specifying regular boundary problems with arbitrary Stieltjes boundary conditions as well as their solution operators.

On the basis of these structures, we define a new multiplication on regular boundary problems in such a way that the resulting Green's operator is the reverse composition of the constituent Green's operators. We provide also a method for lifting any factorization of the underlying differential operator to the level of boundary problems. Since this method only needs the computation of initial value problems, it can be used as an effective alternative for computing Green's operators in the case where one knows how to factor the given differential operators.

Keywords: Linear boundary value problems; Ordinary differential equations; Green's operators; Factorization; Differential algebra; Noncommutative Gröbner bases

1. Introduction

In this paper, we develop a new approach for handling *boundary problems* in the language of differential algebras, restricting ourselves to the case of linear boundary problems for

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ordinary differential equations. (We reserve the traditional term "boundary value problem" for the particular type of boundary problems that have only point evaluations, i.e. point conditions in the terminology of Section 5.) The algebraic language that we build up allows us

- to state boundary problems in a natural algebraic language,
- to express their solution operators in the same language,
- to compute the solution operators from a fundamental system,
- to multiply boundary problems corresponding to the solution operators,
- to lift factorizations of differential operators to boundary problems.

The present paper extends the ideas from Rosenkranz (2005) and Rosenkranz et al. (2003) in several aspects: Boundary problems can now be formulated and solved in any differential algebra that meets some natural conditions (Theorem 26), the case of variable coefficients is fully included, and a new monoid structure on boundary problems provides an elegant description and an alternative computation method for the corresponding solution operators.

For developing an appropriate notion of a boundary problem in a given differential algebra, it will be useful to have a look at the *classical setting* of Stakgold (1979, p. 203) dealing with a two-point boundary problem on a finite interval $[a, b]$. Disregarding weak solutions and ill-posed problems for simplicity, the general idea is that a differential equation

$$u^{(n)}(x) + c_{n-1}(x)\,u^{(n-1)}(x) + \cdots + c_1(x)\,u'(x) + c_0(x)\,u(x) = f(x) \tag{1}$$

with coefficient functions $c_{n-1}, \ldots, c_1, c_0 \in C^\infty[a, b]$ and forcing function $f \in C^\infty[a, b]$ is supplemented with additional conditions that determine the solution $u \in C^\infty[a, b]$ uniquely. In certain cases, these may be initial conditions, but in general one has to deal with constraints that combine the values and derivatives of u at both endpoints a and b. In the context of a linear differential equation like (1), it is natural to restrict oneself to linear conditions of the form

$$p_{n-1}\,u^{(n-1)}(a) + \cdots + p_0\,u(a) + q_{n-1}\,u^{(n-1)}(b) + \cdots + q_0\,u(b) = e, \tag{2}$$

where the p_i, q_i and e are given complex numbers. For obvious reasons, boundary conditions of the form (2) are known as two-point boundary conditions; note that they include initial conditions as the special case where all the q_i vanish. In order to obtain a regular boundary problem, one imposes n suitable linear boundary conditions (2) on a given linear differential equation (1). Since all differential equations, operators and conditions will be linear in this paper, we will from now on drop the attribute "linear".

Classical boundary problems (1), (2) have a rich structure. First of all, it is clear that one can decompose the solution of (1), (2) into a solution of the semi-inhomogeneous problem (obtained from (2) by setting all $e = 0$) and a solution of the semi-homogeneous problem (obtained from (1) by setting $f = 0$). Since we assume that fundamental systems are available, the latter problem reduces to linear algebra, and we can concentrate on the semi-inhomogeneous problem. Thus we assume from now on homogeneous boundary conditions.

A second crucial observation is that the solution u depends linearly on the forcing function f. In fact, the assumption of a regular boundary problem means (Definition 25) that there is a unique u for every given f, so there is a solution operator $G\colon C^\infty[a, b] \to C^\infty[a, b]$ with $u = Gf$. This so-called *Green's operator* G is linear.

Taking advantage of the linear structure, it is possible to compute the Green's operator G rather than a particular solution u belonging to a fixed forcing function f. We may view this as solving the parametrized differential equation (1) together with boundary conditions (2). There

is also a practical reason why it is useful to have the Green's operator: The forcing function f is often more likely to change (e.g. as the "source term" in heat conduction), while the shape of the differential equation (its left-hand side) and the boundary conditions remain fixed. In the classical setting, the Green's operator $G\colon C^\infty[a,b] \to C^\infty[a,b]$ can be represented in the form of an integral operator

$$Gf(x) = \int_a^b g(x,\xi)\, f(\xi)\, \mathrm{d}\xi$$

with a uniquely determined *Green's function* $g \in C^{n-2}[a,b]^2$. So once g is found, one can compute each desired solution u in a single integration.

Now let us describe our strategy of rebuilding this scenario in a (moderately general) *differential algebra*. In the place of $C^\infty[a,b]$, we take a differential algebra $\mathcal{F}$ as our starting point. Obviously, a differential equation (1) is then given by

$$Tu = f \tag{3}$$

with a differential operator $T \in \mathcal{F}[\partial]$, and one has to find the solution $u \in \mathcal{F}$ in terms of a given forcing function $f \in \mathcal{F}$. (In order to gain flexibility, we will actually consider differential operators $T \in \mathcal{F}_0[\partial]$ for a suitable subalgebra $\mathcal{F}_0 \leq \mathcal{F}$; see Definition 18.) Boundary conditions can be given by

$$\beta_1 u = \cdots = \beta_n u = 0 \tag{4}$$

for suitable functionals $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$, where $\mathcal{F}^*$ denotes the dual space of $\mathcal{F}$. We will allow rather general boundary conditions of the so-called Stieltjes type (see Definition 14), including not only two-point conditions like (2) but also global conditions involving integrals.

At this point, we would like to make a general remark on *point evaluation* in differential algebra. This is a topic not often considered (within the given algebraic setting), despite its undisputed importance in the applications. The problem is that the elements of a differential algebra (or differential ring or differential field) are abstractions of functions that are not meant to be "evaluated". Robinson (1961) has addressed this discrepancy by introducing what he called localized differential rings. Working in the much wider scope of polynomial differential equations, he has developed a solvability criterion for initial value problems. To our knowledge, his ideas have not found much resonance. For a more practical perspective on initial value problems for differential–algebraic equations, see the recent survey by Pritchard and Sit (in press), containing a method for determining admissible initial conditions. Our own approach is to consider boundary conditions in their natural context: as functionals of the aforementioned type.

This is why we require a differential algebra—they provide a vector space structure together with the structure of a differential ring. In fact, we need more than that (Section 2): Since we want to express the Green's operator of a boundary problem (3), (4), we need a linear operator $\int$ denoting *integration*, just like ∂ is used for differentiation. We stipulate that $\int$ is a section (right inverse) of ∂, meaning that

$$\partial \int = 1.$$

Further analysis will make it clear that we must also require $\int$ to satisfy a version of the Baxter axiom, an algebraic formulation of integration by parts. As we shall see, this necessarily excludes differential fields from the admissible differential algebras $\mathcal{F}$. We are thus led to the following

crucial observation (Proposition 6): Despite their extremely useful role e.g. in the Galois theory of linear differential equations (van der Put and Singer, 2003), differential fields are inadequate for treating initial/boundary conditions along with the differential equations. In some sense, this result is to be expected: Point evaluations correspond to maximal ideals, which are not available in fields.

We call the resulting structure $(\mathcal{F}, \partial, \int)$ an integro-differential algebra. They induce a natural algebra of *integro-differential operators* $\mathcal{F}[\partial, \int]$, just like $(\mathcal{F}, \partial)$ alone induces the algebra of differential operators $\mathcal{F}[\partial]$. We introduce a suitable rewrite system (Baader and Nipkow, 1998) for these operators (Section 3), enabling their convenient symbolic manipulation. Our rewrite system is both Noetherian and confluent (Proposition 13), and the corresponding normal forms have a natural description (Proposition 17). The advantage of the $\mathcal{F}[\partial, \int]$ language is that it provides a uniform frame for stating initial/boundary problems as well as deriving and expressing their Green's operators.

The departure from differential fields has the consequence that inhomogeneous differential equations cannot be reduced to homogeneous ones in the way explained by van der Put and Singer (2003, Exercise 1.14.1). Hence we have to resort to an algebraic version of the familiar method of "variation of the constant" for solving even *initial value problems* (Section 4), and this necessitates a condition on solutions of inhomogeneous first-order differential equations. It essentially requires that exponential solutions exist and behave as normal: they have a reciprocal.

For treating *boundary problems* (3), (4) in a convenient fashion, we specify them as pairs:

$$(T, \mathcal{B}) \quad \text{with} \quad T \in \mathcal{F}_0[\partial] \quad \text{and} \quad \mathcal{B} = [\beta_1, \ldots, \beta_n] \leq \mathcal{F}^*.$$

Using this setup, we will show (Section 5) that they have a Green's operator that can be expressed in $\mathcal{F}[\partial, \int]$, and we sketch how one can compute it. For a concrete implementation in the classical C^∞ setting, see the previous article (Rosenkranz, 2005). Generalizing the idea of a boundary problem as "a surjective linear map with linear functionals as side conditions", we have also developed an abstract treatment for general vector spaces in our forthcoming paper Regensburger and Rosenkranz (in press). This approach allows us to apply the ideas of Sections 6 and 7, e.g. to linear partial differential equations or systems of linear ordinary differential equations.

The algebraic treatment of boundary problems applied in this paper not only allows for a symbolic solution, it is also a natural setting for exposing an important structure connecting boundary problems amongst themselves (Section 6): It turns out that the composition structure of Green's operators is reflected in a *monoid structure* on the boundary problems, arising as a semi-direct product of $\mathcal{F}_0[\partial]$ and the additive structure of subspaces in $\mathcal{F}^*$.

Finally (Section 7), we will show how to factor a given boundary problem $(T, \mathcal{B})$ into smaller ones. While *factorization* of linear ordinary differential operators is an important topic in symbolic computation (Grigoriev, 1990; van der Put and Singer, 2003; Schwarz, 1989; Tsarev, 1996), it neglects the presence of boundary conditions (possibly addressed in a post-processing step). We will show how every factorization of the differential operator T gives rise to various factorizations of $(T, \mathcal{B})$, whose full classification is stated. In order to lift a factorization of T to the level of boundary problems, one only needs to solve an initial value problem. Hence one may employ factorization as a tool for computing the Green's operator G. In the extreme case of splitting T into linear factors, one obtains G as a composition of first-order Green's operators, which can be computed easily. (In practical examples, one will often be content with a partial factorization.)

Some remarks on *notation*. We write $\mathbb{N}$ for the set of all natural numbers including zero. The variable n ranges over $\mathbb{N}$. All algebras are assumed to be commutative with identity. The zero-

dimensional subspace of any vector space will be denoted by $O = \{0\}$. We write $[f_1, \ldots, f_n]$ for the subspace generated by the vectors $f_1, \ldots, f_n$ of some vector space $\mathcal{F}$. For subsets $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{F}^*$, the so-called orthogonal is defined as

$$\mathcal{A}^\perp = \{\varphi \in \mathcal{F}^* \mid \forall_{f\in\mathcal{A}}\, \varphi(f) = 0\} \leq \mathcal{F}^*,$$
$$\mathcal{B}^\perp = \{f \in \mathcal{F} \mid \forall_{\varphi\in\mathcal{B}}\, \varphi(f) = 0\} \leq \mathcal{F};$$

see Section 5 for more details.

2. Integration in differential algebras

Let $(\mathcal{F}, \partial)$ be a *differential algebra* over a field K, so $\partial\colon \mathcal{F} \to \mathcal{F}$ is a K-linear map fulfilling the Leibniz rule $\partial(fg) = f\,\partial(g) + g\,\partial(f)$. For convenience, we may assume $K \leq \mathcal{F}$, and we write f' as shorthand for $\partial(f)$. Furthermore, we will assume that K has characteristic zero (even though some definitions and results would make sense in positive characteristic), except when stated otherwise. Then we may also assume $\mathbb{Q} \leq K$, so that $\mathcal{F}$ is what is sometimes called a Ritt algebra (Kaplansky, 1957, p. 12).

The algebra of (formal) *differential operators* over the differential algebra $\mathcal{F}$ is denoted by $\mathcal{F}[\partial]$, as e.g. in van der Put and Singer (2003). Addition in $\mathcal{F}[\partial]$ is obvious, while multiplication is determined by the rule $\partial f = f\partial + f'$. Each $T \in \mathcal{F}[\partial]$ acts on $\mathcal{F}$ as an (actual) differential operator $T\colon \mathcal{F} \to \mathcal{F}$. The identity operator of $\mathcal{F}[\partial]$ is denoted by $\partial^0 = 1$ just like the unit element $1 \in \mathcal{F}$; it will be clear from the context which is meant.

Our goal is to solve inhomogeneous differential equations by using Green's operators. The simplest such equation is $u' = f$, and its solution operators $\int\colon f \mapsto u$ are exactly the sections of the differential operator ∂. A derivation need not have any sections; e.g. in the algebra of univariate differential polynomials, the indeterminate cannot be a derivative. But if it does, their description follows from linear algebra.

Proposition 1. *Every section $\int\colon \mathcal{F} \to \mathcal{F}$ of the derivation $\partial\colon \mathcal{F} \to \mathcal{F}$ corresponds to a unique projector $P\colon \mathcal{F} \to \mathcal{F}$ with $P = 1 - \int\partial$, and to a unique direct sum $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$ with $\mathcal{C} = \operatorname{Ker}(\partial) = \operatorname{Im}(P)$ and $\mathcal{I} = \operatorname{Im}(\int) = \operatorname{Ker}(P)$.*
If $\int$ is any fixed section of ∂, every projector P with $\operatorname{Im}(P) = \operatorname{Ker}(\partial)$ induces a section $(1-P)\int$, and every section of ∂ arises uniquely in this way.

Proof. See Nashed and Votruba (1976, p. 17) or Regensburger and Rosenkranz (in press). □

We refer to the elements of $\mathcal{I} = \operatorname{Im}(\int)$ as the *initialized* functions (with respect to $\int$), while those of $\mathcal{C} = \operatorname{Ker}(\partial)$ are usually known as the *constants* (with respect to ∂). In the prototypical case of $\mathcal{F} = C^\infty(\mathbb{R})$, the initialized functions are those that can be written as $F(x) = \int_\alpha^x f(\xi)\,\mathrm{d}\xi$ for an integrand $f \in C^\infty(\mathbb{R})$ and an initialization point $\alpha \in \mathbb{R}$; hence F is exactly that antiderivative of f that fulfills the initial condition $F(\alpha) = 0$.

For solving inhomogeneous differential equations $Tu = f$ of higher order, one must expect to iterate the section $\int$. In general, this could lead to "nested integrals" of arbitrary complexity. But we know from the classical C^∞ setting (see Section 1) that the Green's operator G can always be expressed using a single integration, with the so-called *Green's function* g as its integral kernel. The essential role of Green's functions is to resolve nested integrals, whereas the passage from an operator $G\colon C^\infty[a,b] \to C^\infty[a,b]$ to a function $g \in C^{n-2}[a,b]^2$ is immaterial from our viewpoint.

In order to capture this behavior, we need an identity for resolving nested integrals (eventually leading to the $\int f \int$ rule in Table 1). Such an identity is given by the so-called *Baxter axiom* (of weight zero), asserting

$$(\int f)(\int g) = \int (f \int g) + \int (g \int f) \tag{5}$$

for all $f, g \in \mathcal{F}$; see Guo (2002), Baxter (1960) and Rota (1969) for more details. One sees immediately that (5) is an algebraic version of integration by parts, rewritten in such a way that it need not refer to any derivation. A Baxter algebra $(\mathcal{F}, \int)$ is then a K-algebra $\mathcal{F}$ with a K-linear operation $\int$ fulfilling the Baxter axiom (5).

If $\int$ is again a section of a derivation ∂ on $\mathcal{F}$, we note an important consequence of (5). Writing x as an abbreviation for $\int 1$, we obtain $x^2/2 = \int\int 1$ and inductively $x^n/n! = \int \cdots \int 1$ with n iterates of $\int$. Hence the powers $u = x^k$ with $k < n$ are solutions of $u^{(n)} = 0$, and one checks immediately that they are all linearly independent. This means that $\operatorname{Ker}(\partial^n)$ contains $[1, x, \ldots, x^{n-1}]$ as an n-dimensional subspace. So we see that $\mathcal{F}$ contains (an isomorphic copy of) the *polynomial ring* $K[x]$ and is thus infinite dimensional. Note that $K[x] \leq \mathcal{F}$ is simultaneously a differential algebra under ∂ and a Baxter algebra under $\int$, so $(K[x], \partial, \int)$ is an integro-differential algebra in the sense of Definition 4.

What we shall actually need is the *differential Baxter axiom*, requiring

$$\int f g = f \int g - \int (f' \int g) \tag{6}$$

for all $f, g \in \mathcal{F}$. Note that this is what most people do when they actually apply integration by parts (eventually leading to the $\int f \partial$ rule in Table 1), but (6) cannot be stated in pure Baxter algebras. The variant (5) follows immediately by substituting $\int f$ for f in (6), and often the two versions are actually equivalent (especially in the cases relevant for us—see after Definition 8). For seeing that in general (6) is stronger than (5), we need a somewhat artificial construction (Example 3). In fact, we can easily characterize what makes the *differential Baxter axiom* stronger than the pure one.

Lemma 2. *A section $\int$ of ∂ fulfills the differential Baxter axiom (6) iff it fulfills the pure Baxter axiom (5) and the homogeneity condition $\int cf = c \int f$ for all $c \in \mathcal{C}$ and $f \in \mathcal{F}$.*

Proof. Assume $\int$ fulfills (6). Then $\int$ also fulfills (5) as observed above, while substituting a constant $c \in \mathcal{C}$ for f in (6) gives homogeneity. Conversely, assume that $\int$ fulfills (5) and the homogeneity condition. The latter hypothesis means that (6) is satisfied if $f \in \mathcal{C}$. Now consider $f \in \mathcal{I}$ so that $\int f' = f$. Substituting f' for f in (5), we see that (6) is also satisfied for these $f \in \mathcal{I}$. But then the general case of $f \in \mathcal{F}$ follows via the direct sum $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$. □

Example 3. Let K be a field of characteristic zero. Then $(R[x], \partial)$ with $R = K[y]/y^4$ and $\partial f = f_x$ is a differential algebra over K. Defining

$$\int f = \int_0^x f(\xi, y)\, \mathrm{d}\xi + f(0,0)\, y^2, \tag{7}$$

we obtain a K-linear map $\int\colon R[x] \to R[x]$. Since the second term vanishes under ∂, we see immediately that $\int$ is a section of ∂. For verifying the Baxter axiom (5), let us write $\fint$ for the ordinary integral in (7) and compute

$$(\int f)(\int g) = (\fint f)(\fint g) + y^2 \fint \big(g(0,0)\, f + f(0,0)\, g\big) + f(0,0)\, g(0,0)\, y^4,$$
$$\int (f \int g) = \int f (\fint g + g(0,0)\, y^2) = \fint (f \fint g) + y^2 \fint \big(g(0,0)\, f\big).$$

Since $y^4 \equiv 0$ and the ordinary integral $\int$ fulfills the Baxter axiom (5), this implies immediately that $\int$ does also. However, it does not fulfill the stronger axiom (6), because the homogeneity condition is violated: Observe that $\operatorname{Ker}(\partial) = R$, so in particular we should have $\int y \cdot 1 = y \cdot \int 1$. But one checks immediately that the left-hand side yields xy, while the right-hand side yields $xy + y^3$.

For excluding cases like the preceding example, we will insist that "integral operators" must satisfy the *differential Baxter axiom*.

Definition 4. Let $(\mathcal{F}, \partial)$ be a differential algebra. A section $\int$ of ∂ is called an *integral* if it satisfies the differential Baxter axiom (6). In this case, we call $(\mathcal{F}, \partial, \int)$ an *integro-differential algebra*.

Example 5. As an *example*, detailed in Rosenkranz (2005, p. 176), take $\mathcal{F} = C^\infty[a, b]$ with its usual derivation ∂ and integral operators

$$\int^*\colon f \mapsto \int_a^x f(\xi)\,\mathrm{d}\xi \quad \text{and} \quad \int_*\colon f \mapsto \int_x^b f(\xi)\,\mathrm{d}\xi.$$

Then both $(\mathcal{F}, \partial, \int^*)$ and $(\mathcal{F}, \partial, -\int_*)$ are integro-differential algebras. By contrast, the operator

$$f \mapsto \int_a^b \int_\tau^x f(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

used for regularizing an ill-posed problem in Rosenkranz (2005, p. 192), is just a section for ∂, but not an integral.

Using Proposition 1, we can characterize integrals by their projectors and direct sums. In the above example, we observe that the projectors $f \mapsto f(a)$ and $f \mapsto f(b)$, corresponding respectively to the integrals $\int^*$ and $-\int_*$, are *multiplicative*, whereas the projector $\int_a^b$ for the third operator is not. This behavior is the key to their characterization.

Proposition 6. *A section $\int\colon \mathcal{F} \to \mathcal{F}$ of the derivation $\partial\colon \mathcal{F} \to \mathcal{F}$ is an integral iff its projector $P\colon \mathcal{F} \to \mathcal{F}$ is multiplicative iff $\mathcal{I} = \operatorname{Im}(\int)$ is an ideal.*

Proof. Assume first that $\int$ is an integral for ∂, let $P = 1 - \int\partial$ be its projector and $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$ the corresponding direct sum with $\mathcal{C} = \operatorname{Ker}(\partial)$ and $\mathcal{I} = \operatorname{Im}(\int)$, according to Proposition 1. We must prove $P(fg) = P(f)\,P(g)$ for all $f, g \in \mathcal{F}$. Substituting g' for g in (6), we obtain

$$\begin{aligned} 0 &= \textstyle\int fg' - f\int g' + \int(f'\int g') = \int fg' - f(g - Pg) + \int(f'(g - Pg)) \\ &= \textstyle\int fg' + \int f'g - fg + f\,Pg - (\int f')\,Pg, \end{aligned}$$

where we have used the homogeneity of $\int$ in the last step. But then

$$P(fg) = fg - \textstyle\int(f'g + fg') = (f - \int f')\,Pg = Pf\,Pg,$$

as claimed. Assume conversely that P is multiplicative, and take $f, G \in \mathcal{F}$ arbitrary. Expanding the definition of P and using the Leibniz law gives

$$P(fG) = (1 - \textstyle\int\partial)fG = fG - \int f'G - \int fG'$$

and

$$Pf\;PG = (f - \textstyle\int f')(G - \int G') = fG - G\int f' - f\int G' + (\int f')(\int G');$$

equating the two expressions, we obtain

$$(\int f')(\int G') + \int f'G + \int fG' = G\int f' + f\int G',$$

which yields indeed (6) by specializing to $G = \int g$.

Let us now prove that $\mathcal{I}$ is an ideal under the assumption that P is multiplicative. Since P is a projector along $\mathcal{I}$, we have $PG = 0$ iff $G \in \mathcal{I}$. Hence for all $f \in \mathcal{F}$ and $G \in \mathcal{I}$ we have $P(fG) = Pf\,PG = 0$, and $fG \in \mathcal{I}$ as claimed. Finally, we assume that $\mathcal{I}$ is an ideal and prove that P is multiplicative. Taking $f, g \in \mathcal{F}$ arbitrary, we set $f_0 = Pf \in \mathcal{C}$ and $g_0 = Pg \in \mathcal{C}$. Then $f_1 = f - f_0 \in \mathcal{I}$ and likewise $g_1 = g - g_0 \in \mathcal{I}$, so we obtain

$$P(fg) = P(f_0g_0) + P(f_0g_1) + P(f_1g_0) + P(f_1g_1) = f_0g_0 = Pf\,Pg$$

since all of f_0g_1, f_1g_0, $f_1g_1 \in \mathcal{I}$ vanish under P, while $f_0g_0 \in \mathcal{C}$ is fixed by P. □

For the operators $\int^*$ and $\int_*$ in Example 5, the *Baxter axiom* is of course known to hold. In the following example, where this is not obvious, we can take advantage of Proposition 6.

Example 7. Consider $\mathcal{F} = C^\infty(\mathbb{R}^2)$ with the derivation $\partial u = u_x + u_y$. Finding sections for ∂ means solving the partial differential equation $u_x + u_y = f$. Its general solution is given by

$$u(x, y) = \int_\alpha^x f(t, t - x + y)\,\mathrm{d}t + g(y - x),$$

where $g \in C^\infty(\mathbb{R})$ and $\alpha \in \mathbb{R}$ are arbitrary. In order to ensure a linear section, one has to choose $g = 0$, arriving at

$$\int f = \int_\alpha^x f(t, t - x + y)\,\mathrm{d}t.$$

Using a change of variables, one may verify that $\int$ satisfies the Baxter axiom (5), so $(\mathcal{F}, \int)$ is a Baxter algebra. We see also that $\mathcal{C} = \mathrm{Ker}(\partial)$ is given by the functions $(x, y) \mapsto g(x - y)$ with arbitrary $g \in C^\infty(\mathbb{R})$, while $\mathcal{I} = \mathrm{Im}(\int)$ consists of the functions $f \in \mathcal{F}$ satisfying $f(\alpha, y) = 0$ for all $y \in \mathbb{R}$. The projector $P\colon \mathcal{F} \to \mathcal{F}$ maps a function f to the function $(x, y) \mapsto f(\alpha, \alpha - x + y)$. Since the homogeneity condition is obviously satisfied, we conclude that $(\mathcal{F}, \partial, \int)$ is an integro-differential algebra. But with Proposition 6, we could have derived this result immediately since P is multiplicative and $\mathcal{I}$ an ideal.

As we see from the above example, the space of constants for an integro-differential algebra may be infinite dimensional. Since we want to treat boundary problems for *ordinary differential equations*, we will exclude these cases. Note that in the following definition our terminology deviates from that of Kolchin (1973, p. 58), which simply requires having a single derivation. So in Kolchin's sense, the differential algebra of Example 7 would be addressed as "ordinary".

Definition 8. A differential algebra $(\mathcal{F}, \partial)$ is called *ordinary* if $\dim \mathrm{Ker}(\partial) = 1$.

Having an ordinary differential algebra $\mathcal{F}$ has several important *consequences*. First of all, it is clear that we have $K = \mathcal{C}$, so $\mathcal{F}$ is an algebra over its own field of constants. But then a section is automatically homogeneous over $\mathcal{C}$, so the pure Baxter axiom (5) and its differential version (6) coincide. Furthermore, we obtain the familiar relation

$$\mathrm{Ker}(\partial^n) = [1, x, \ldots, x^{n-1}], \tag{8}$$

which can be seen thus: As mentioned above, the Baxter axiom implies the inclusion $\supseteq$. Equality follows from $\dim \operatorname{Ker}(\partial^n) = n$, which is a consequence of the identity

$$\operatorname{Ker}(T^2) = G\ \operatorname{Ker}(T) \dotplus \operatorname{Ker}(T)$$

in Regensburger and Rosenkranz (in press), generally valid for epimorphisms T and sections G of T.

One knows from linear algebra that a projector P onto a one-dimensional subspace $[w]$ of a K-vector space V can be written as $P(v) = \varphi(v)\, w$, where φ is a unique functional with $\varphi(w) = 1$. If V is moreover a K-algebra, a projector onto $K = [1]$ is canonically described by the functional φ with normalization $\varphi(1) = 1$. Hence in an ordinary differential algebra, the projectors corresponding (via Proposition 1) to sections of the derivation can be regarded as *normalized functionals*.

In an ordinary *integro*-differential algebra $(\mathcal{F}, \partial, \int)$, the normalized functional corresponding to the integral $\int$ is moreover *multiplicative* by Proposition 6. Since this will be a crucial ingredient for our later development, it deserves a special name.

Definition 9. Let $(\mathcal{F}, \partial, \int)$ be an ordinary integro-differential algebra. Then we call the multiplicative functional $\mathtt{E} = 1 - \int\partial$ its *evaluation*.

The terminology stems from the *standard model* described in Example 5, where $\mathtt{E}$ is a point evaluation. For boundary problems on a finite interval, it is natural to treat both endpoints specially, leading to a pair of evaluations and integrals. This is the situation described in Rosenkranz (2005, p. 182) by the concept of "analytic algebra".

Example 10. An *analytic algebra* $(\mathcal{F}, \partial, \int^*, \int_*)$ is equivalent to a pair of ordinary integro-differential algebras $(\mathcal{F}, \partial, \int^*)$ and $(\mathcal{F}, \partial, -\int_*)$. Writing as in the above reference $f \mapsto f^{\leftarrow}$ and $f \mapsto f^{\rightarrow}$ for the evaluations of respectively $\int^*$ and $\int_*$, one finds that

$$(\textstyle\int^* f)^{\rightarrow} = \int^* f + \int_* f = (\int_* f)^{\leftarrow}.$$

This relation implies (after some calculation) that $\int_*$ is the adjoint of $\int^*$, with respect to the inner product $\langle|\rangle\colon \mathcal{F} \times \mathcal{F} \to \mathcal{C}$ given by

$$\langle f|g\rangle = (\textstyle\int^* + \int_*)\, fg.$$

In the standard model $\mathcal{F} = C^\infty[a, b]$, we have $f^{\leftarrow} = f(a)$ and $f^{\rightarrow} = f(b)$, yielding the L^2 inner product $\langle f|g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$.

The multiplicative functionals on an algebra are known as its *characters* (note that all characters are normalized). We write $\mathcal{M}(\mathcal{F})$ for the vector space of all characters on an ordinary integro-differential algebra $(\mathcal{F}, \partial, \int)$. The evaluation of $\mathcal{F}$ is a distinguished character $\mathtt{E} \in \mathcal{M}(\mathcal{F})$ whose kernel $\mathcal{I}$ is an ideal with $\mathcal{F} = K \dotplus \mathcal{I}$ according to Proposition 6.

One calls a K-algebra *augmented* if there exists a character on it. Its kernel $\mathcal{I}$ is then known as an *augmentation ideal* and forms a direct summand of K; see Cohn (2003, p. 132). Augmentation ideals are always maximal ideals (generalizing the $C^\infty[0, 1]$ case) since the direct sum $\mathcal{F} = K \dotplus \mathcal{I}$ induces a ring isomorphism $\mathcal{F}/\mathcal{I} \cong K$. Reformulating Proposition 6, we obtain now a characterization of integrals in ordinary differential algebras.

Corollary 11. *In an ordinary differential algebra $(\mathcal{F}, \partial)$, a section $\int$ of ∂ is an integral iff its normalized functional is a character iff $\mathcal{I} = \operatorname{Im}(\int)$ is an augmentation ideal.*

Note that the augmentation ideal $\mathcal{I}$ corresponding to an integral is in general not a differential ideal of $\mathcal{F}$. We can see this e.g. for $\int^*$ in Example 5, where $\mathcal{I}$ consists of all $f \in \mathcal{F}$ with $f(0) = 0$, so that $\mathcal{I}$ is *not differentially closed* since $(x \mapsto x) \in \mathcal{I}$ but $(x \mapsto 1) \notin \mathcal{I}$.

We have now gathered the main ingredients needed for treating boundary problems, namely integro-differential algebras. Similar structures are introduced under the name *Rota–Baxter algebras* in the recent preprint by Guo and Keigher (2007), which came to our attention only after completing this article. The situation considered there is more general in four respects: The algebras are over unital commutative rings rather than fields, they may be noncommutative, they may have nonzero weight, and they satisfy the pure Baxter axiom (5) rather than the differential version (6). Their interest stems mainly from combinatorial investigations of tree-like structures, where the weight is usually nonzero.

3. Integro-differential operators

From here onwards, let $(\mathcal{F}, \partial, \int)$ be an ordinary integro-differential algebra over a field K with evaluation E. We introduce now an algebra of operators on $\mathcal{F}$ using rewrite systems (Baader and Nipkow, 1998) in the spirit of Bergman (1978). The *integro-differential operators* $\mathcal{F}[\partial, \int]$ are defined as the K-algebra generated by the symbols ∂ and $\int$, the "functions" $f \in \mathcal{F}$ and the multiplicative "functionals" $\varphi \in \mathcal{M}(\mathcal{F})$, modulo the rewrite rules given in Table 1. We will use the variables f, g for elements of $\mathcal{F}$ and the variables φ, ψ, χ for elements of $\mathcal{M}(\mathcal{F})$. Every integro-differential operator can be written as a sum of "monomials", every monomial as a coefficient times a "term".

In the rules of Table 1 as well as in the rest of this paper, we use the notation $U \cdot f$ for the *action* of U on a function f, where U is an element of the free algebra in the above generators. It is an easy matter to check that the rewrite rules of Table 1 are fulfilled in $(\mathcal{F}, \partial, \int)$, so we may regard $\cdot$ as an action of $\mathcal{F}[\partial, \int]$ on $\mathcal{F}$. In particular, $f \cdot g$ now denotes the product of functions $f, g \in \mathcal{F}$.

We remark that Table 1 is to be understood as including *implicit rules* for $\int\int$, $\int\partial$ and $\int\varphi$ by substituting $f = 1$ in the rules for $\int f \int$, $\int f \partial$ and $\int f \varphi$, respectively. Moreover, one obtains the *derived rule* $\mathrm{E}\int = 0$ from the definition of the evaluation E. Note that $\mathcal{F}[\partial]$ is a subalgebra of $\mathcal{F}[\partial, \int]$ with the same induced action on $\mathcal{F}$.

Example 12. The *analytic polynomials* of Rosenkranz (2005, p. 176) are also an important special case of integro-differential operators (the restriction to $K = \mathbb{C}$ imposed there is not essential). They are constructed on top of an analytic algebra $(\mathcal{F}, \partial, \int^*, \int_*)$ with evaluations $f \mapsto f^{\leftarrow}$ and $f \mapsto f^{\rightarrow}$, as explained in Example 10. As usual, we can express one integral using the other, yielding either $-\int_* = (1-^{\rightarrow})\int^*$ or $-\int^* = (1-^{\leftarrow})\int_*$. Choosing randomly the first alternative, we work with the integro-differential algebra $(\mathcal{F}, \partial, \int^*)$. Up to notational details, the analytic polynomials over $(\mathcal{F}, \partial, \int^*, \int_*)$ are then the subalgebra of $\mathcal{F}[\partial, \int^*]$ generated by the operators

$$\begin{aligned} D &= \partial, & L &= {}^{\leftarrow}, \\ A &= \textstyle\int^*, & R &= {}^{\rightarrow}, \\ B &= \textstyle\int_*, = (1-^{\leftarrow})\int^* & \lceil f \rceil &= f, \end{aligned}$$

using the same names as in the cited article. We use also the abbreviation $F = A + B$ for the operator of definite integration.

Table 1
Rewrite rules for integro-differential operators

fg	$\to$	$f \cdot g$	∂f	$\to$	$\partial \cdot f + f\partial$	$\int f \int$	$\to$	$(\int \cdot f)\int - \int(\int \cdot f)$
$\varphi\psi$	$\to$	ψ	$\partial\varphi$	$\to$	0	$\int f \partial$	$\to$	$f - \int(\partial \cdot f) - (\mathtt{E} \cdot f)\,\mathtt{E}$
φf	$\to$	$(\varphi \cdot f)\,\varphi$	$\partial\int$	$\to$	1	$\int f \varphi$	$\to$	$(\int \cdot f)\,\varphi$

Note that for analytic polynomials, the multiplication operators $\lceil f \rceil$ are restricted to *basis elements* $f \in \mathcal{F}$; similar restrictions could be made here. The point is that a system of normal forms on $\mathcal{F}[\partial, \int]$ presupposes a canonical simplifier on the free algebra generated by ∂ and $\int$, the functions $f \in \mathcal{F}$ and the functionals $\varphi \in \mathcal{M}(\mathcal{F})$. Expansion with respect to fixed bases of $\mathcal{F}$ and $\mathcal{M}(\mathcal{F})$ provides such a canonical simplifier, but there may also be others. In Rosenkranz (2005), we have implemented a ground simplifier via such a basis expansion (where $\mathcal{F}$ was given by the exponential polynomials). In the present paper, we take the viewpoint that the free algebra is equipped with *some* canonical simplifier (the "ground simplifier"), and the confluence result of the following proposition has to be understood relative to such a ground simplifier.

Proposition 13. *The rewrite system of Table 1 is Noetherian and confluent.*

Proof. By the Diamond Lemma 1.2 from Bergman (1978), it suffices to ensure the following two facts: First we must construct a partial well-order $>$ on the word monoid in the generators of $\mathcal{F}[\partial, \int]$ such that $>$ is compatible with the monoid structure and the rewrite system in Table 1. Second we have to prove that all ambiguities of the rewrite system are resolvable. For defining the partial well-order, we put $\partial > f$ for all functions f and extend this to words by the graded lexicographic construction. The resulting partial order is clearly well-founded (since it is on the generators) and compatible with the monoid structure (by its grading). It is also compatible with the rewrite system because all rules reduce the word length except for the Leibniz rule, which is compatible because $\partial > f$.

For proving that the ambiguities of Table 1 are resolvable, note first that we have no inclusion ambiguities while there are exactly 14 overlap ambiguities. For overlapping rules $ww_1 \to p_1$ and $w_2w \to p_2$ to be resolvable, their S-polynomial $p_2w_1 - w_2p_1$ must reduce to zero. This is indeed the case, as one can check by an easy calculation (using also the axioms of integro-differential algebras for $\mathcal{F}$). As a representative example, let us reassure ourselves that the S-polynomial from the rules for $ww_1 = \int f \partial$ and $w_2w = \int g \int$ does indeed reduce to

$$
\begin{aligned}
&(\int \cdot g)\int f\partial - \int(\int \cdot g)\, f\partial - \int gf + \int g \int f' + \int g\,(\mathtt{E} \cdot g)\,\mathtt{E} \\
&= (\int \cdot g) f - (\int \cdot g)\int f' - (\int \cdot g)\,(\mathtt{E} \cdot f)\,\mathtt{E} - (\int \cdot g) f + \int \partial \cdot ((\int \cdot g) \cdot f) \\
&\qquad + (\mathtt{E} \cdot ((\int \cdot g) \cdot f)\,\mathtt{E} - \int(g \cdot f) + (\int \cdot g)\int f' - \int(\int \cdot g) f' + (\mathtt{E} \cdot f)(\int \cdot g)\,\mathtt{E} \\
&= \int \partial \cdot ((\int \cdot g) \cdot f) + (\mathtt{E} \cdot ((\int \cdot g) \cdot f)\,\mathtt{E} - \int(g \cdot f) - \int(\int \cdot g) f' \\
&= \int(g \cdot f) + \int(\int \cdot g) f' + 0 - \int(g \cdot f) - \int(\int \cdot g) f' \\
&= 0,
\end{aligned}
$$

as it should. □

In other words, the polynomials given by the difference between the left-hand and right-hand sides of Table 1 form a *two-sided noncommutative Gröbner basis*. For the theory of Gröbner bases, we refer the reader to Buchberger (1965, 1970, 1998), for its noncommutative extension to Mora (1986), Mora (1994) and Ufnarovski (1998).

Comparing the analytic polynomials in Rosenkranz (2005, p. 183) with the rewrite system of Table 1, we would like to emphasize the gain in *simplicity and economy*: Despite their higher generality, the integro-differential operators of $\mathcal{F}[\partial, \int]$ require just 9 instead of 36 identities! Consequently, their confluence proof (resolving 14 overlaps) can still be produced by hand, while the automatically generated confluence proof for the analytic polynomials (resolving 233 overlaps) contains 2000 lines; see Rosenkranz (2005, p. 184f) for a small fragment of it.

Having a Noetherian and confluent rewrite system, every integro-differential operator has a *unique normal form* (Baader and Nipkow, 1998, p. 12). In order to describe these normal forms explicitly, it is useful to single out a particular portion of the operators that will also turn out to play a distinguished role in specifying boundary conditions (see Section 5).

Definition 14. The elements of the right ideal

$$\mathcal{S}(\mathcal{F}) = \mathcal{M}(\mathcal{F})\,\mathcal{F}[\partial, \textstyle\int]$$

are called *Stieltjes boundary conditions* over $\mathcal{F}$; if there is no danger of ambiguity, we will henceforth just speak of "boundary conditions".

We will now describe the *normal forms* in $\mathcal{F}[\partial, \int]$, starting with a simple observation on reducibility (in general not describing normal forms), which is afterwards used for characterizing the normal forms of boundary conditions.

Lemma 15. *Every integro-differential operator in $\mathcal{F}[\partial, \int]$ can be reduced to a linear combination of monomials $f\varphi\int g\psi\partial^i$, where $i \geq 0$ and each of $f, \varphi, \int, g, \psi$ may also be absent.*

Proof. Call a monomial consisting only of functions and functionals "algebraic". Using the left column of Table 1, it is immediately clear that all such monomials can be reduced to f or φ or $f\varphi$. Now let w be an arbitrary monomial in the generators of $\mathcal{F}[\partial, \int]$. By using the middle column of Table 1, we may assume that all occurrences of ∂ are moved to the right, so that all monomials have the form $w = w_1 \cdots w_n \partial^i$ with $i \geq 0$ and each of $w_1, \ldots, w_n$ either a function, a functional or $\int$. We may further assume that there is at most one occurrence of $\int$ among the $w_1, \ldots, w_n$. Otherwise the monomials $w_1 \cdots w_n$ contain $\int \tilde{w} \int$, where each $\tilde{w} = f\varphi$ is an algebraic monomial. But then we can reduce

$$\textstyle\int \tilde{w} \int = (\int f\varphi)\int = (\int \cdot\, f)\varphi \int$$

by using the corresponding rule of Table 1. Applying these rules repeatedly, we arrive at algebraic monomials left and right of $\int$ (or just a single algebraic monomial if $\int$ is absent). □

Proposition 16. *Every boundary condition of $\mathcal{S}(\mathcal{F})$ has the normal form*

$$\sum_{\varphi \in \mathcal{M}(\mathcal{F})} \left(\sum_{i \in \mathbb{N}} a_{\varphi,i}\, \varphi \partial^i + \varphi \textstyle\int f_\varphi \right)$$

with $a_{\varphi,i} \in K$ and $f_\varphi \in \mathcal{F}$ almost all zero.

Proof. By Lemma 15, every boundary condition of $\mathcal{S}(\mathcal{F})$ is a linear combination of monomials having the form

$$w = \chi f \varphi \textstyle\int g \psi \partial^i \quad \text{or} \quad w = \chi f \varphi \partial^i \tag{9}$$

where each of f, g, φ, ψ may also be missing. Using the left column of Table 1, the prefix $\chi f \varphi$ can be reduced to a scalar multiple of a functional, so we may as well assume that f and φ are not

present; this finishes the right-hand case of (9). For the remaining case $w = \chi \int g\psi\partial^i$, assume first that ψ is present. Then we have

$$\chi\,(\textstyle\int g\psi) = \chi\,(\int\cdot\, g)\,\psi = (\chi\textstyle\int\cdot\, g)\,\chi\psi = (\chi\textstyle\int\cdot\, g)\,\psi,$$

so w is again a scalar multiple of $\psi\partial^i$, and we are done. Finally, assume we have $w = \chi\int g\partial^i$. If $i = 0$, this is already a normal form. Otherwise we obtain

$$w = \chi\,(\textstyle\int g\partial)\,\partial^{i-1} = (\chi\cdot g)\,\chi\partial^{i-1} - \chi\textstyle\int g'\partial^{i-1} - (\mathtt{E}\cdot g)\,\mathtt{E}\partial^{i-1},$$

where the first and the last summand are in the required normal form, while the middle summand is to be reduced recursively, eventually leading to a middle term in normal form $\pm\chi\int g'\partial^0 = \pm\chi\int g'$. □

The Stieltjes boundary conditions have the additional benefit of allowing a simple description of the normal forms for all integro-differential operators. Just as we obtain the *differential operators* $\mathcal{F}[\partial] \subset \mathcal{F}[\partial, \int]$ with their usual normal forms, we write also $\mathcal{F}[\int] \subset \mathcal{F}[\partial, \int]$ for the subalgebra of *integral operators*, generated by the functions and $\int$ modulo the Baxter rule (uppermost in the right column of Table 1). Using Lemma 15, it is clear that the normal forms of integral operators are linear combinations of $f\int g$ with $f, g \in \mathcal{F}$.

Finally, we write $\mathcal{F}[\mathtt{E}]$ for the left $\mathcal{F}$-submodule generated by $\mathcal{S}(\mathcal{F})$ and call them *Stieltjes boundary operators* (briefly "boundary operators"). Note that $\mathcal{F}[\mathtt{E}]$ includes $\mathcal{S}(\mathcal{F})$ as well as all finite dimensional projectors P along Stieltjes boundary conditions. The latter can all be described as follows: If $u_1, \ldots, u_n \in \mathcal{F}$ and $\beta_1, \ldots, \beta_n \in \mathcal{S}(\mathcal{F})$ are biorthogonal in the sense that $\beta_i(u_j) = \delta_{ij}$, then

$$P = \sum_{i=1}^{n} u_i\,\beta_i, \tag{10}$$

is the projector onto $[u_1, \ldots, u_n]$ along $[\beta_1, \ldots, \beta_n]^\perp$; see for example Köthe (1969, p. 71) and Regensburger and Rosenkranz (in press, Prop. 2). From the representation (10) it is immediately clear that $P \in \mathcal{F}[\mathtt{E}]$. All elements of $\mathcal{F}[\mathtt{E}]$ have the normal form (10), except that the (u_j) need not be biorthogonal to the (β_i).

It turns out now that every monomial of an integro-differential operator is either a *differential operator* or an *integral operator* or a *boundary operator*.

Proposition 17. *Up to ordering the summands, every normal form of $\mathcal{F}[\partial, \int]$ with respect to the rewrite system of Table 1 can be written uniquely as a sum $T + G + B$ having the following normal-form summands: a differential operator $T \in \mathcal{F}[\partial]$, an integral operator $G \in \mathcal{F}[\int]$, and a boundary operator $B \in \mathcal{F}[\mathtt{E}]$.*

Proof. Inspection of Table 1 confirms that all integro-differential operators having the described sum representation $T + G + P$ are indeed in normal form. Let us now prove that every integro-differential operator of $\mathcal{F}[\partial, \int]$ has such a representation. It is sufficient to consider its monomials w. If w starts with a functional, we obtain a boundary condition by Proposition 16; so assume this is not the case. From Lemma 15 we know that

$$w = f\varphi\textstyle\int g\psi\partial^i \quad \text{or} \quad w = f\varphi\partial^i,$$

where each of φ, g, ψ may be absent. But $w \in \mathcal{F}[\mathtt{E}]$ unless φ is absent, so we may actually assume

$$w = f \int g \psi \partial^i \quad \text{or} \quad w = f \partial^i.$$

The right-hand case yields $w \in \mathcal{F}[\partial]$. If ψ is present in the other case, we may reduce $\int g\psi$ to $(\int \cdot\, g)\,\psi$, and we obtain again $w \in \mathcal{F}[\mathtt{E}]$. Hence we are left with $w = f \int g \partial^i$, and we may assume $i > 0$ since otherwise we have $w \in \mathcal{F}[\int]$ immediately. But then we can reduce

$$\begin{aligned} w &= f\,(\textstyle\int g \partial)\,\partial^{i-1} = f\Big(g - \int (\partial \cdot g) - (\mathtt{E} \cdot g)\,\mathtt{E}\Big)\partial^{i-1} \\ &= (fg)\,\partial^{i-1} - f \textstyle\int (\partial \cdot g)\,\partial^{i-1} - (\mathtt{E} \cdot g)\, f\, \mathtt{E}\, \partial^{i-1}, \end{aligned}$$

where the first term is obviously in $\mathcal{F}[\partial]$ and the last one in $\mathcal{F}[\mathtt{E}]$. The middle term may be reduced recursively until the exponent of ∂ has dropped to zero, leading to a term in $\mathcal{F}[\int]$. □

4. Initial value problems

Up to now we have not discussed the existence of solutions for differential equations, except for two particularly simple cases: the homogeneous differential equation $u^{(n)} = 0$ whose solution space is given by $[1, x, \ldots, x^{n-1}]$ as stated in (8), and the inhomogeneous equation $u' = f$ with $\int f$ as particular solution. In order to have some finer control on which differential equations we want to have solutions, we will allow specifying the *coefficients* of the pertinent linear differential operators. (In differential Galois theory, one usually works with differential fields, where one can study extensions in a much more convenient manner. As we have seen above, though, this route is not accessible for us here.)

Definition 18. A differential subalgebra $\mathcal{F}_0 \leq \mathcal{F}$ is called *saturated* for a differential algebra $\mathcal{F}$ if $\dim \operatorname{Ker}(T) = n$ for every monic $T \in \mathcal{F}_0[\partial]$ with $\deg T = n$ and if all nonzero solutions u of $u' = au$, with $a \in \mathcal{F}_0$, are invertible in $\mathcal{F}$. In this context, we call $\mathcal{F}$ the *ground algebra* and $\mathcal{F}_0$ the *coefficient algebra*. If $\mathcal{F}_0$ coincides with $\mathcal{F}$, we simply speak of a saturated integro-differential algebra.

Some remarks on this definition are in order. First of all, we point out that we need $\mathcal{F}_0$ to be differentially closed such that we can multiply within $\mathcal{F}_0[\partial]$, which will be needed for multiplying boundary problems in Section 6. The first condition on solvability ensures that *homogeneous equations* $Tu = 0$ have a fundamental system with the appropriate number of solutions, while the second condition means that *exponentials* behave as usual. Note also that $\mathcal{F}$ is an ordinary differential algebra as soon as it possesses a saturated coefficient algebra.

Not every integro-differential algebra has a saturated coefficient algebra; e.g. the polynomial algebra $(K[x], \partial, \int)$ does not. We do not know any useful criteria for settling this question. However, there are several important *examples* of integro-differential algebras with saturated coefficient algebras:

Example 19. The prototypical example is furnished by $C^\infty[a, b]$ where $[a, b]$ is a finite interval of $\mathbb{R}$. As a coefficient algebra, one may take either $C^\infty[a, b]$ itself or any differential subalgebra like $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{C}[x]$. Similarly, one may take analytic functions $C^\omega[a, b]$ and its differential subalgebras. Less demanding but practically important, the exponential polynomials, as defined in Rosenkranz (2005, p. 176), can be taken as a ground algebra with $\mathbb{C}$ as a coefficient algebra.

Example 20. For any field K of characteristic 0, the formal power series $K[[z]]$ are a saturated integro-differential algebra, with derivation and integration defined as usual. This may also be inferred from the next example by the isomorphism described there.

Example 21. Let K be an arbitrary field (note that we are explicitly including the case of positive characteristic in this example). Then the algebra $H(K)$ Hurwitz series (Keigher, 1997) over K is defined as the K-vector space of infinite K-sequences with the multiplication defined as

$$(a_n) \cdot (b_n) = \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right)_n$$

for all $(a_n), (b_n) \in H(K)$. If one introduces derivation and integration through

$$\partial\,(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots),$$
$$\textstyle\int (a_0, a_1, \dots) = (0, a_0, a_1, \dots),$$

the Hurwitz series form an integro-differential algebra $(H(K), \partial, \int)$, as explained by Keigher and Pritchard (2000) and Guo (2002).

Note that as an additive group, $H(K)$ coincides with the formal power series $K[[z]]$, but its multiplicative structure differs: We have an isomorphism

$$\sum_{n=0}^{\infty} a_n z^n \;\mapsto\; (n!\, a_n)$$

from $K[[z]]$ to $H(K)$ if and only if K has characteristic zero. The point is that one can integrate every element of $H(K)$, whereas the formal power series z^{p-1} does not have an antiderivative in $K[[z]]$ if K has characteristic p.

Defining the exponential function $\exp = (1, 1, 1, \dots)$, we obtain immediately $\partial \exp = \exp$. One can introduce a composition $f \circ g$ for $f, g \in H(K)$ whenever g has vanishing constant term, and the usual chain rule is satisfied for this composition (Keigher and Pritchard, 2000). Then the first-order homogeneous equation $u' = au$ with $a \in H(K)$ is solved by

$$u = c\ \exp \circ (\textstyle\int a),$$

which is easily seen to be invertible in $H(K)$. By Corollary 4.3 in Keigher and Pritchard (2000), we know also that all monic homogeneous differential equations of order n have an n-dimensional kernel. Hence $H(K)$ is a saturated integro-differential algebra.

Throughout the rest of this paper, we assume that $(\mathcal{F}, \partial, \int)$ is an integro-differential algebra with a saturated coefficient algebra $\mathcal{F}_0$. As before, we write E for its evaluation. Having integrals, it is natural to expect that we can also solve *inhomogeneous equations*. As we shall see now, it is always possible to find a particular solution, but we can be more specific than that.

We formulate the *initial value problem* for a monic differential operator $T \in \mathcal{F}_0[\partial]$ and character $\eta \in \mathcal{M}(\mathcal{F})$ as follows: Given a forcing function $f \in \mathcal{F}$, find $u \in \mathcal{F}$ such that

$$\begin{aligned} Tu &= f \\ \eta u = \eta u' &= \cdots = \eta u^{(n-1)} = 0, \end{aligned} \tag{11}$$

where $\deg T = n$. Problems of this kind can be solved uniquely.

Proposition 22. *For every monic $T \in \mathcal{F}_0[\partial]$ and $\eta \in \mathcal{M}(\mathcal{F})$, the initial value problem of the form (11) has a unique solution $u \in \mathcal{F}$ for given $f \in \mathcal{F}$.*

Proof. We can use the usual technique of reformulating (11) as a system of linear first-order differential equations with companion matrix $A \in \mathcal{F}_0^{n\times n}$; then we apply the familiar variation-of-constants formula, as described e.g. by Coddington and Levinson (1955, p. 74). To this end, we pick a fundamental system $u_1, \dots, u_n \in \mathcal{F}$ for T and compute the Wronskian matrix

$$W = \begin{pmatrix} u_1 & \dots & u_n \\ u_1' & \dots & u_n' \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)} \end{pmatrix}.$$

Observe that $d = \det W$ satisfies the first-order differential equation $d' = ad$, where a is the trace of $A \in \mathcal{F}_0$; see for example Exercise 1.14.5 in van der Put and Singer (2003), but note that we do not need a differential field. Since $\mathcal{F}_0$ is saturated for $\mathcal{F}$, the determinant d must be invertible and hence W a regular matrix.

By Proposition 1 and Corollary 11, the operator $\int\!\!\!\!\!-\, = (1-\eta)\int$ is the integral having the evaluation $\eta = 1 - \int\!\!\!\!\!-\,\partial$. We extend the action of the operators $\int\!\!\!\!\!-, \partial, \eta$ componentwise to $\mathcal{F}^n$. Setting now

$$\hat{u} = (W \int\!\!\!\!\!-\, W^{-1})\, \hat{f}$$

with $\hat{f} = (0, \dots, 0, f)^\top \in \mathcal{F}^n$, one may readily check that $\hat{u} \in \mathcal{F}^n$ is a solution of the first-order system $\hat{u}' = A\hat{u} + \hat{f}$ with initial condition $\eta\hat{u} = 0$. Writing u for the first component of $\hat{u}$, we have a solution of (11).

For proving uniqueness, assume u is a solution of (11) for $f = 0$; we must show $u = 0$. We may expand $u = c_1u_1 + \cdots + c_nu_n$ in terms of the fundamental system $u_1, \dots, u_n$ with suitable coefficients $c_1, \dots, c_n \in K$. Then the initial conditions of (11) may be summarized by $\eta(Wc) = 0$ with the coefficient vector $c = (c_1, \dots, c_n)^\top \in K^n$. But $\eta(Wc) = \eta(W)c$ because η is linear, and $\det \eta(W) = \eta(\det W)$ because it is moreover multiplicative. Since $\det W \in \mathcal{F}$ is invertible, this implies that $\eta(W) \in K^{n\times n}$ is regular, so $c = \eta(W)^{-1}0 = 0$ and $u = 0$. □

As mentioned after Example 10, every integro-differential algebra $(\mathcal{F}, \partial, \int)$ comes with a distinguished character: the evaluation $\eta = \mathtt{E}$. Hence we may speak of the initial value problem associated with a monic $T \in \mathcal{F}_0[\partial]$. If $u \in \mathcal{F}$ is the unique solution to such an initial value problem with forcing function f, we obtain an operator $T^\blacklozenge\colon \mathcal{F} \to \mathcal{F}$ with $u = T^\blacklozenge f$, which we shall call the *fundamental right inverse* for T. The notation and terminology are in accordance with Rosenkranz (2005), where the evaluation $\mathtt{E}\colon C^\infty[a,b] \to C^\infty[a,b]$ is given by $u \mapsto u(a)$. We observe also that $T^\blacklozenge$ is a particular case of a Green's operator.

Proposition 23. *For every monic $T \in \mathcal{F}_0[\partial]$, the fundamental right inverse can be realized as an integro-differential operator $T^\blacklozenge \in \mathcal{F}[\partial, \int]$.*

Proof. Inspecting the proof of Proposition 22, one can see that u may in fact be obtained from f by the *operation of an integro-differential operator* from $\mathcal{F}[\partial, \int]$. This holds in particular for the initial value problem with $\eta = \mathtt{E}$. □

5. Boundary problems

The main purpose of $\mathcal{F}[\partial, \int]$ is to provide a unified language for expressing *boundary problems* as well as their *solutions*. As explained in Section 1, a boundary problem of order

n is typically formulated as follows: Given a forcing function $f \in \mathcal{F}$, we have to find $u \in \mathcal{F}$ such that

$$\begin{aligned} Tu &= f, \\ \beta_1 u = \cdots &= \beta_n u = 0, \end{aligned} \tag{12}$$

for a monic differential operator $T \in \mathcal{F}_0[\partial]$ with $\deg T = n$ and boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$. Clearly we have $T \in \mathcal{F}[\partial, \int]$, but also $\beta_1, \ldots, \beta_n \in \mathcal{F}[\partial, \int]$ if we restrict ourselves to the (relatively large) class of Stieltjes boundary conditions (Definition 14). The solution is usually expressed as $u = Gf$, where $G\colon \mathcal{F} \to \mathcal{F}$ is the so-called Green's operator of the boundary problem (12). As we shall see in Theorem 26, the Green's operator G can also be expressed as the action of an element in $\mathcal{F}[\partial, \int]$.

We think of the boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$ of (12) as specifying a *space of admissible functions*

$$\mathcal{A} = \{\beta_1, \ldots, \beta_n\}^\perp \leq \mathcal{F}.$$

Obviously we may replace the boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$ by other boundary conditions $\tilde{\beta}_1, \ldots, \tilde{\beta}_n \in \mathcal{F}^*$ such that $\tilde{\beta}_i = c_{i1}\beta_1 + \cdots + c_{in}\beta_n$ for a regular matrix $(c_{ij}) \in K^{n\times n}$, leading to the same space of admissible functions $\mathcal{A} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}^\perp$. This means that the admissible functions may be described invariantly as $\mathcal{A} = \mathcal{B}^\perp$ in terms of $\mathcal{B} = [\beta_1, \ldots, \beta_n] = [\tilde{\beta}_1, \ldots, \tilde{\beta}_n]$. Such a finite dimensional subspace $\mathcal{B} \leq \mathcal{F}^*$ will be called a *space of boundary conditions*.

The operators $\ldots^\perp$ on $\mathcal{F}$ and $\mathcal{F}^*$ create an order-reversing *lattice isomorphism* (a fortiori a Galois connection) between the modular lattices of finite codimensional subspaces of $\mathcal{F}$ and finite dimensional subspaces of $\mathcal{F}^*$. Specifically, we have

$$\mathcal{B}^\perp = \{u \in \mathcal{F} \mid \forall_{\beta\in\mathcal{B}}\ \beta(u) = 0\}$$

for the space of functions satisfying the boundary conditions in $\mathcal{B}$ and

$$\mathcal{A}^\perp = \{\beta \in \mathcal{F}^* \mid \forall_{u\in\mathcal{A}}\ \beta(u) = 0\}$$

for the space of boundary conditions satisfied by the functions in $\mathcal{A}$. The lattice isomorphism provides crucial relations for treating boundary problems (Section 6), specifically

$$(\mathcal{B}_1 \cap \mathcal{B}_2)^\perp = \mathcal{B}_1^\perp + \mathcal{B}_2^\perp \quad \text{and} \quad (\mathcal{B}_1 + \mathcal{B}_2)^\perp = \mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp \tag{13}$$

for finite dimensional subspaces $\mathcal{B}_1, \mathcal{B}_2 \leq \mathcal{F}^*$ and

$$\mathcal{K} \dotplus \mathcal{B}^\perp = \mathcal{F} \ \Leftrightarrow\ \mathcal{K}^\perp \dotplus \mathcal{B} = \mathcal{F}^* \tag{14}$$

for finite dimensional subspaces $\mathcal{K} \leq \mathcal{F}$ and finite codimensional subspaces $\mathcal{B} \leq \mathcal{F}^*$. We are thus in a similar situation to in algebraic geometry, where affine varieties correspond to subspaces of $\mathcal{F}$ while radical ideals correspond to subspaces of $\mathcal{F}^*$. (Our forthcoming article Regensburger and Rosenkranz (in press) provides an abstract approach along these lines.)

For our present purposes, however, we are interested in an *algorithmic treatment* of boundary conditions and their associated spaces of admissible functions. As indicated above, this can be achieved by working with Stieltjes boundary conditions—they are wide enough for practical applications while allowing convenient implementation of the operations expressed in the above identities. Our notion of Stieltjes boundary conditions is naturally motivated by the classical setting obtained by setting $\mathcal{F} = C^\infty[a, b]$ in Example 12.

In a traditional boundary problem (Stakgold, 1979, p. 203), one prescribes only a so-called two-point boundary condition

$$\beta u = \sum_{i=0}^{n-1} a_i\, u^{(i)}(a) + b_i\, u^{(i)}(b)$$

with $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in \mathbb{C}$. Obviously, we may view

$$\beta = \sum_{i=0}^{n-1} a_i\, LD^i + b_i\, RD^i$$

as an element of $\mathcal{F}[\partial, \int]$ since $L, R \in \mathcal{M}(\mathcal{F})$. In a general integro-differential algebra $\mathcal{F}$, we define a *point condition* as a linear combination of conditions having the form $\varphi\partial^i$ with $\varphi \in \mathcal{M}(\mathcal{F})$.

In the literature Brown and Krall (1974, 1977), one also considers boundary conditions of the form

$$\beta u = \sum_{i=0}^{n-1} a_i\, u^{(i)}(a) + b_i\, u^{(i)}(b) + \int_a^b f(\xi)\, u(\xi)\, \mathrm{d}\xi$$

under the name "Stieltjes boundary conditions". Here the sum part gives a point condition as before, while the integral kernel $f \in \mathcal{F}$ is used for prescribing an *integral condition*. Note that such boundary conditions are in the normal form described by Proposition 16, which is the reason for the terminology in Definition 14. We call a Stieltjes boundary condition global if $f \neq 0$.

There are at least three *reasons* for considering Stieltjes boundary conditions: First of all, they are interesting in themselves because certain boundary problems are naturally expressed in terms of global side conditions (for example, specifying the heat radiated through the boundary). This in also true for regularizing ill-posed problems and computing their generalized Green's function (Rosenkranz, 2005, p. 191). A second reason for introducing Stieltjes boundary conditions will become manifest in Section 7: Factoring a boundary problem leads to factor problems with global conditions, even for a problem having only point conditions (see Example 28). Finally, a third advantage of Stieltjes boundary conditions is that they have a natural algebraic characterization by Definition 14.

We write $\mathfrak{B}_n$ for the set of all subspaces $\mathcal{B} = [\beta_1, \ldots, \beta_n] \leq \mathcal{F}^*$ generated by n linearly independent Stieltjes boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{S}(\mathcal{F})$; note that $[] = O$ is the only element of $\mathfrak{B}_0$. Then $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ is closed under the operation $+$ of constructing the sum of vector spaces, thus yielding an abelian monoid $(\mathfrak{B}, +)$, which we call the *monoid of boundary conditions*. Specifically, the sum of an m-dimensional and an n-dimensional space of boundary conditions gives

$$[\beta_1, \ldots, \beta_m] + [\tilde{\beta}_1, \ldots, \tilde{\beta}_n] = [\beta_1, \ldots, \beta_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_n] = [\gamma_1, \ldots, \gamma_k],$$

with dimension $k \leq m + n$. In order to compute linearly independent boundary conditions $\gamma_1, \ldots, \gamma_k$, we can apply the following evident strategy.

Proposition 24. *There is an algorithm for computing a basis $\beta_1, \ldots, \beta_n \in \mathcal{S}(\mathcal{F})$ for an arbitrary $\mathcal{B} \in \mathfrak{B}$ given by generators $\gamma_1, \ldots, \gamma_m \in \mathcal{S}(\mathcal{F})$.*

Proof. Expand each of $\gamma_1, \ldots, \gamma_m$ in the K-basis of normal-form monomials as given by Proposition 16. Although the number of such basis elements is infinite, the expansions of

$\gamma_1, \dots, \gamma_m$ will only use finitely many of them, say, $m_1, \dots, m_r$. This yields an $m \times r$ matrix (a_{ij}) over K such that $\gamma_i = a_{i1}m_1 + \cdots + a_{ir}m_r$ for all $i \in \{1, \dots, m\}$. Reducing the matrix (a_{ij}) to row echelon and discarding the zero rows leads to the desired K-basis $\beta_1, \dots, \beta_n$ of $\mathcal{B}$. □

Let us write $\mathfrak{D}_n$ for the set of all monic $T \in \mathcal{F}_0[\partial]$ with $\deg T = n$, setting $\mathfrak{D} = \bigcup_n \mathfrak{D}_n$. In this paper, we will only be concerned with boundary problems (12) that are *regular* in the sense that they have a unique solution u for each forcing function f. Below we reformulate the condition of regularity directly in terms of the differential operator and the space of boundary conditions.

Definition 25. A *boundary problem* of order n is a pair $(T, \mathcal{B})$ with $T \in \mathfrak{D}_n$ and $\mathcal{B} \in \mathfrak{B}_n$; it is called *regular* if $\operatorname{Ker}(T) \dotplus \mathcal{B}^\perp = \mathcal{F}$. We write $\mathfrak{P}_n$ for the set of all regular boundary problems of order n, setting $\mathfrak{P} = \bigcup_n \mathfrak{P}_n$.

As explained in Regensburger and Rosenkranz (in press), the requirement of the direct sum is equivalent to $\operatorname{Ker}(T) \cap \mathcal{B}^\perp = O$ and also to $\operatorname{Ker}(T) + \mathcal{B}^\perp = \mathcal{F}$ since we have insisted on $\deg T = \dim \mathcal{B}$ in our current setting. It is moreover equivalent to regularity in the sense discussed above and to the following *algorithmic criterion*: If $u_1, \dots, u_n$ is any basis of $\operatorname{Ker}(T)$ and $\beta_1, \dots, \beta_n$ any basis of $\mathcal{B}$, the problem $(T, \mathcal{B})$ is regular iff

$$\begin{pmatrix} \beta_1(u_1) & \cdots & \beta_1(u_n) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \cdots & \beta_n(u_n) \end{pmatrix} \tag{15}$$

is regular in $K^{n\times n}$. This test may be found in Kamke (1967, p. 184) for the special case of two-point boundary conditions, but it generalizes even to the abstract setting described in Regensburger and Rosenkranz (in press). Since in this paper we consider only regular boundary problems, we will suppress the attribute "regular".

Note that we do not require well-posedness. Following Hadamard, a *well-posed* problem (Engl et al., 1996, p. 86) must be regular as well as stable (meaning that the solution u depends continuously on the data f). Our approach is purely algebraic, so we do not care about stability (which would first of all require a topology on $\mathcal{F}$). For example, the following boundary problem in $\mathcal{F} = C^\infty[0, 1]$ is regular but not well-posed, at least not when in the common setting of the Banach space $(\mathcal{F}, \|\cdot\|_\infty)$: Given f, find u such that $u' - u = f$ and $u''(0) = 0$. In this case, the solution exists and is unique; in fact, it is given by $u(x) = \int_0^x f(\xi)\,\mathrm{d}\xi - (f(0) + f'(0))\,\mathrm{e}^x$, so the Green's operator is $\mathrm{e}^x - \mathrm{e}^x L - \mathrm{e}^x LD$. Incidentally, this example illustrates another unusual feature of our setting—we do not restrict the derivatives in the boundary conditions to orders below the order of the differential equation (even though it will often be reasonable to make such a restriction).

The *Green's operator* G of a boundary problem $(T, \mathcal{B})$ is specified by the two requirements

$$TG = 1 \quad \text{and} \quad \operatorname{Im}(G) = \mathcal{B}^\perp.$$

If $\deg T = n$, the space of boundary conditions $\mathcal{B}$ can be described by n basis elements $\beta_1, \dots, \beta_n$, and we can rewrite this in the traditional form (12). Then the Green's operator G is given by the mapping $f \mapsto u$. Since every boundary problem $(T, \mathcal{B})$ has a unique Green's operator G in this sense, we can introduce the notation $(T, \mathcal{B})^{-1}$ for it.

In Rosenkranz (2005), we have explained how to compute from a fundamental system for T the Green's operator of a two-point boundary problem $(T, \mathcal{B})$ for the analytic algebra $C^\infty[a, b]$

Table 2
Outline for computing Green's operators

Input: $(T, \mathcal{B}) \in \mathfrak{P}$ with bases $\{u_j\}$ of $\operatorname{Ker}(T)$ and $\{\beta_i\}$ of $\mathcal{B}$
Output: $G \in \mathcal{F}[\partial, \int]$ such that $G = (T, \mathcal{B})^{-1}$
Determine $T^\blacklozenge \in \mathcal{F}[\partial, \int]$ as in Proposition 23, using $\{u_j\}$
Determine projector $P \in \mathcal{F}[\mathtt{E}]$ as in (16), using $\{u_j\}$ and $\{\beta_i\}$
Compute $G = (1 - P)\, T^\blacklozenge$ in $\mathcal{F}[\partial, \int]$

of Example 12. This result generalizes to our present setting; see Table 2 for an *outline of the computation* and Example 33 a sample problem (Green's operator for the left factor).

Theorem 26. *Every boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$ has a Green's operator that can be written as an integro-differential operator $G \in \mathcal{F}[\partial, \int]$.*

Proof. The decomposition method explained in Rosenkranz (2005) is also valid in our case; based on the algebraic generalized inverse (Nashed and Votruba, 1976; Engl and Nashed, 1981), it even carries over to the general setting described in Regensburger and Rosenkranz (in press). Thus we have

$$G = (1 - P)\, T^\blacklozenge,$$

where P is the projector onto $\operatorname{Ker}(T)$ along $\mathcal{B}^\perp$, and $T^\blacklozenge$ is the fundamental right inverse of T. From Proposition 23 we know that $T^\blacklozenge \in \mathcal{F}[\partial, \int]$. (In fact, we could take any right inverse of T, but $T^\blacklozenge$ is a canonical choice.) For computing the projector in the form (10), we choose a fundamental system $u_1, \ldots, u_n$ for T. If $\mathcal{B}$ is given by a basis $\beta_1, \ldots, \beta_n \in \mathcal{S}(\mathcal{F})$, we can change to a new basis $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ that is biorthogonal to $u_1, \ldots, u_n$ by setting

$$(\tilde{\beta}_1, \ldots, \tilde{\beta}_n)^\top = B^{-1}(\beta_1, \ldots, \beta_n)^\top,$$

where B is the matrix (15). Then

$$P = \sum_{i=1}^{n} u_i \tilde{\beta}_i \in \mathcal{F}[\mathtt{E}] \subseteq \mathcal{F}[\partial, \int] \tag{16}$$

is the desired projector, and we have $G = (1 - P)\, T^\blacklozenge \in \mathcal{F}[\partial, \int]$. □

The *factorization method* described in Section 7 provides an alternative approach to computing Green's operators. The crucial point will be that multiplying boundary problems corresponds to composing their Green's operators in reverse order (see Proposition 27). In the case of differential operators with constant coefficients, one can express any Green's operator as a product of first-order Green's operators, which can be described by a simple formula.

6. Multiplying boundary problems

Using actions, a *semi-direct product* may be defined for monoids just as for groups; the resulting structure is again a monoid (Cohn, 1982, p. 277). Unlike for groups, one has to distinguish semi-direct products (for left actions) and reverse semi-direct products (for right actions); see Eilenberg (1976) and also Regensburger and Rosenkranz (in press).

We define a right action as follows. Every integro-differential operator $U \in \mathcal{F}[\partial, \int]$ acts on $\mathfrak{B}$ as

$$\mathcal{B} \cdot U = \{\beta \circ U \mid \beta \in \mathcal{B}\};$$

if $\mathcal{B}$ is generated by n conditions $\gamma_1, \ldots, \gamma_n$, this gives

$$[\gamma_1, \ldots, \gamma_n] \cdot U = [\gamma_1 \circ U, \ldots, \gamma_n \circ U].$$

For a differential operators $T \in \mathcal{F}_0[\partial]$, a basis $\beta_1, \ldots, \beta_n$ of $\mathcal{B}$ is transformed into a basis $\beta_1 \circ T, \ldots, \beta_n \circ T$ of $\mathcal{B} \cdot T$ since T has a right inverse like $T^{\blacklozenge}$.

The resulting reverse semi-direct product $\mathfrak{D} \ltimes \mathfrak{B} = (\mathfrak{D} \times \mathfrak{B}, \cdot)$ then has the *multiplication* defined by

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 \cdot T_2 + \mathcal{B}_2). \tag{17}$$

The neutral element under this multiplication is given by the degenerate boundary problem $(1, O)$, which is regular by definition. (Written out in the classical notation, this is the following "problem": Given $f \in \mathcal{F}$, find $u \in \mathcal{F}$ such that $u = f$ without further boundary conditions!)

As mentioned after Theorem 26, we can compute Green's operators from the constituent Green's operators in a factorization, and in Section 7 we will present a method for producing such factorizations from a factorization of the differential operator. But of course this presupposes that the product of boundary problems *corresponds to the composition* of their Green's operators in reverse order. Let us write $\mathfrak{G}$ for the monoid generated by all Green's operators for boundary problems in $\mathfrak{P}$.

Proposition 27. *The boundary problems $\mathfrak{P} \subseteq \mathfrak{D} \times \mathfrak{B}$ form a submonoid of $\mathfrak{D} \ltimes_\Phi \mathfrak{B}$, and the transformation $(T, \mathcal{B}) \mapsto (T, \mathcal{B})^{-1}$ is an anti-isomorphism from $\mathfrak{P}$ to $\mathfrak{G}$. In other words, every Green's operator corresponds to exactly one boundary problem, and we have*

$$(\mathcal{P}_1 \mathcal{P}_2)^{-1} = \mathcal{P}_2^{-1} \mathcal{P}_1^{-1}$$

for all $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{P}$.

Proof. From the remark above we know already $(1, O) \in \mathfrak{P}$. By the definition (17) of the multiplication, we have

$$(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 \cdot T_2 + \mathcal{B}_2).$$

We first prove that the right-hand boundary problem is regular and that its Green's operator is given by $G_2 G_1$. Clearly we have

$$TG = (T_1 T_2)(G_2 G_1) = T_1 (T_2 G_2) G_1 = T_1 G_1 = 1,$$

so $G_2 G_1$ is a section of $T_1 T_2$. Hence $\operatorname{Ker}(T_1 T_2) \dotplus \operatorname{Im}(G_1 G_2) = \mathcal{F}$, and it remains to show

$$\operatorname{Im}(G_2 G_1) = (\mathcal{B}_1 \cdot T_2 + \mathcal{B}_2)^\perp.$$

Consider first $u = G_2 G_1 f$. We have $\beta(u) = 0$ for all $\beta \in \mathcal{B}_2$ since $\operatorname{Im}(G_2) = \mathcal{B}_2^\perp$, and $\beta(T_2 u) = \beta(G_1 f) = 0$ for all $\beta \in \mathcal{B}_1$ since $\operatorname{Im}(G_1) = \mathcal{B}_1^\perp$, so $u \in (\mathcal{B}_1 \cdot T_2 + \mathcal{B}_2)^\perp$. Conversely, assume $u \in (\mathcal{B}_1 \cdot T_2 + \mathcal{B}_2)^\perp$. Then $u \in (\mathcal{B}_1 \cdot T_2)^\perp$ and $u \in \mathcal{B}_2^\perp$ by (13). The latter condition means $u = G_2 v$ for some v, while the former condition implies $v \in \mathcal{B}_1^\perp$; hence $v = G_1 f$ and $u = G_2 G_1 f$ for some f.

Now for the uniqueness of the Green's operators. Consider two boundary problems $(T,\mathcal{B}),(\tilde{T},\tilde{\mathcal{B}}) \in \mathfrak{P}$ with the same Green's operator G. Then we obtain from $TG = 1$ and $\tilde{T}G = 1$ that $(T-\tilde{T})G = 0$, so $T-\tilde{T}$ vanishes on the infinite dimensional space $\mathrm{Im}(G) \leq \mathcal{F}$. Assume now $T \neq \tilde{T}$ for a contradiction. Then $T-\tilde{T}$ is a nonzero differential operator over a saturated coefficient algebra $\mathcal{F}_0$, so it has a finite dimensional kernel and cannot vanish on all of $\mathrm{Im}(G)$. Hence we have indeed $T = \tilde{T}$. Finally, we have also $\mathcal{B}^\perp = \mathrm{Im}(G) = \tilde{\mathcal{B}}^\perp$ and therefore $\mathcal{B} = \tilde{\mathcal{B}}$. □

Let us carry out a *simple multiplication* in the monoid $(\mathfrak{P},\cdot)$, working with the analytic polynomials of Example 12 over the ground algebra $\mathcal{F} = C^\infty[0,1]$.

Example 28. We claim that

$$(D,[F])\cdot(D,[L]) = (D^2,[L,R]). \tag{18}$$

Indeed, we have $[F]\cdot D = [FD] = [AD+BD] = [(1-L)+(-1+R)] = [R-L]$ and $[F]\cdot D + [L] = [L,R]$, so (18) follows. Written in classical notation, we have multiplied the boundary problems

$$\boxed{\begin{array}{l} u' = f \\ \int_0^1 u(\xi)\,\mathrm{d}\xi = 0 \end{array}} \cdot \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}} = \boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}}.$$

We see at this point that global conditions are necessary for the converse process: If we want to factor the boundary problem (see Section 7) on the right-hand side, we cannot have two-point boundary conditions in the left factor since it is unique (Proposition 31).

7. Factoring boundary problems

In this section we will study how to split boundary problems into smaller ones. In fact, it turns out that *every* factorization of a differential operator can be "lifted" to the level of boundary problems (Theorem 32).

Definition 29. A boundary problem $(T_2,\mathcal{B}_2) \in \mathfrak{P}$ is called a *right factor* of a boundary problem $(T,\mathcal{B}) \in \mathfrak{P}$ if T_2 is a right factor of T and $\mathcal{B}_2$ a subspace of $\mathcal{B}$.

Proposition 30. *Let $(T,\mathcal{B}) \in \mathfrak{P}$ be a boundary problem and $T = T_1T_2$ a factorization of its differential operator. Then $(T,\mathcal{B})$ has a right factor $(T_2,\mathcal{B}_2) \in \mathfrak{P}$.*

Proof. Set $n = \deg T_1$ and $m = \deg T_2$. Choose a basis

$$u_1,\dots,u_m,u_{m+1},\dots,u_{m+n} \in \mathcal{F}$$

of $\mathrm{Ker}(T)$ such that $u_1,\dots,u_m$ is a basis of $\mathrm{Ker}(T_2)$, and choose any basis

$$\beta_1,\dots,\beta_{m+n} \in \mathcal{S}(\mathcal{F})$$

of $\mathcal{B}$. Since $(T,\mathcal{B})$ is a regular problem, the matrix

$$B = \begin{pmatrix} \beta_1(u_1) & \dots & \beta_1(u_m) & \beta_1(u_{m+1}) & \dots & \beta_1(u_{m+n}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m+n}(u_1) & \dots & \beta_{m+n}(u_m) & \beta_{m+n}(u_{m+1}) & \dots & \beta_{m+n}(u_{m+n}) \end{pmatrix}$$

is regular. Hence we may use row operations to obtain a matrix with a regular upper left $m \times m$ block and zeros below. (We could reduce the matrix B to row echelon form, but this is more than we need at this point.) These operations are realized by left-multiplying B with a suitable matrix $P \in \mathrm{GL}(K, m+n)$ such that the upper left is transformed into a regular matrix

$$B_2 = \begin{pmatrix} \tilde{\beta}_1(u_1) & \dots & \tilde{\beta}_1(u_m) \\ \vdots & \ddots & \vdots \\ \tilde{\beta}_m(u_1) & \dots & \tilde{\beta}_m(u_m) \end{pmatrix}$$

with new boundary conditions

$$\tilde{\beta}_i = \sum_{j=1}^{m+n} P_{ij}\beta_j \qquad (i = 1, \dots, m).$$

But the regularity of B_2 means that $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ with $\mathcal{B}_2 = [\tilde{\beta}_1, \dots, \tilde{\beta}_m] \leq \mathcal{B}$. $\square$

A refined analysis of Proposition 30 leads to a full *classification of all right factors* $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ of a given boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$; see Regensburger and Rosenkranz (in press) for the detailed statement and proof in an abstract setting. The bottom line is that there is a bijection between right factors of $(T, \mathcal{B})$ and direct summands of $\mathrm{Ker}(T_2)$ in $\mathrm{Ker}(T)$. In detail, every right factor $(T_2, \mathcal{B}_2)$ corresponds to $\mathcal{L}_2 = \mathcal{B}_2^\perp \cap \mathrm{Ker}(T)$, while every direct summand $\mathcal{L}_2$ corresponds to $(T_2, \mathcal{B}_2)$ with $\mathcal{B}_2 = \mathcal{B} \cap \mathcal{L}_2^\perp$. One can also show that $(T_2, \mathcal{B}_2)$ is regular iff

$$\mathrm{Ker}(T_2)^\perp \cap \mathcal{B} \dotplus \mathcal{B}_2 = \mathcal{B}, \tag{19}$$

using the preservation of direct sums (14).

When referring to $\mathcal{P}_2 = (T_2, \mathcal{B}_2)$ as a right factor of $\mathcal{P} = (T, \mathcal{B})$, we are actually anticipating that there is also a *left factor* $\mathcal{P}_1 = (T_1, \mathcal{B}_1)$ such that their product yields $\mathcal{P}$. This is indeed the case, as we will see in Proposition 31. But what is immediately clear is that if $\mathcal{P}_1$ exists, it is uniquely determined by $\mathcal{P}$ alone. Indeed, we know from Proposition 27 that $G = G_2 G_1$, where G, G_1, G_2 denote the Green's operators respectively of $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$. But this implies $G_1 = T_2 G$ and hence $\mathcal{B}_1 = \mathrm{Im}(T_2 G)^\perp$.

Apart from the existence question, the disturbing feature of the relation $\mathcal{B}_1 = \mathrm{Im}(T_2 G)^\perp$ is that it presupposes knowledge of the Green's operator G. This defeats the plan of using factorization for determining the Green's operator from those of its factors. The next proposition remedies this flaw: it turns out that all we need is an *arbitrary right inverse* H_2 of the differential operator T_2. Of course we take $H_2 = G_2$, but this still needs the computation of a Green's operator (albeit of a smaller size). A more reasonable choice is $H_2 = T_2^\blacklozenge$, thus reducing the task of computing Green's operators to initial value problems. (The fundamental right inverse is a canonical choice here, but in specific settings it may be algorithmically advantageous to choose other right inverses of T_2.)

Proposition 31. *Given $(T, \mathcal{B}) \in \mathfrak{P}$ with $T = T_1 T_2$, there is a unique $(T_1, \mathcal{B}_1) \in \mathfrak{P}$ such that every right factor $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ of $(T, \mathcal{B})$ satisfies $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$. Moreover, we have*

$$\mathcal{B}_1 = (\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2,$$

where H_2 is any right inverse of T_2 and $\mathcal{B}_1 = \mathcal{B} \cdot G_2$ where G_2 is the Green's operator of any right factor $(T_2, \mathcal{B}_2)$.

Proof. We have already seen that if $(T_1, \mathcal{B}_1)$ exists, it is unique with $\mathcal{B}_1 = \mathrm{Im}(T_2G)^\perp$. Since T_2G is a right inverse of T_1, we have also $\mathrm{Ker}(T_1) \dotplus \mathrm{Im}(T_2G) = \mathcal{F}$. But this means $(T_1, \mathcal{B}_1) \in \mathfrak{P}$ if we can just ensure that $\mathcal{B}_1$ has a basis of Stieltjes boundary conditions. And this follows immediately once we have proved that

$$\mathrm{Im}(T_2G)^\perp = (\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2 \tag{20}$$

since when $\mathcal{B}$ is generated by Stieltjes boundary conditions, its intersection with $\mathrm{Ker}(T_2)^\perp$ is generated by certain linear combinations of them, while right-multiplication by H_2 still yields Stieltjes boundary conditions by the definition of $\mathcal{S}(\mathcal{F})$.

For proving (20), assume first $\beta(T_2Gu) = 0$ for all $u \in \mathcal{F}$. Setting $\tilde{\beta} = \beta \circ T_2$, we have $\beta = \tilde{\beta} \circ H_2$, and it suffices to show $\tilde{\beta} \in \mathrm{Ker}(T_2)^\perp$ and $\tilde{\beta} \in \mathcal{B} = \mathrm{Im}(G)^\perp$. But the former is immediate from the definition of $\tilde{\beta}$, and the latter follows since $\tilde{\beta}(Gu) = \beta(T_2Gu) = 0$ by hypothesis. Conversely, let us assume $\tilde{\beta} \in \mathrm{Ker}(T_2)^\perp \cap \mathcal{B}$ and show $\tilde{\beta} \circ H_2 \in \mathrm{Im}(T_2G)^\perp$. Indeed, we have

$$(\tilde{\beta} \circ H_2)(T_2Gu) = \tilde{\beta}(H_2T_2Gu) = \tilde{\beta}\big(Gu\big) - \tilde{\beta}\big((1 - H_2T_2)\,Gu\big) = 0$$

because the left summand vanishes by the hypothesis $\tilde{\beta} \in \mathcal{B} = \mathrm{Im}(G)^\perp$ and the right summand by the hypothesis $\tilde{\beta} \in \mathrm{Ker}(T_2)^\perp$ and the fact that $1 - H_2T_2$ is a projector onto $\mathrm{Ker}(T_2)$.

Next let us prove the product $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$. Using (20), it suffices to ensure the relation

$$(\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2T_2 = \mathrm{Ker}(T_2)^\perp \cap \mathcal{B} \tag{21}$$

since the regularity of $(T_2, \mathcal{B}_2)$ is equivalent to $\mathrm{Ker}(T_2)^\perp \cap \mathcal{B} \dotplus \mathcal{B}_2 = \mathcal{B}$ by (19). For proving (21), we apply the stronger result that $\beta \mapsto \beta \circ H_2T_2$ leaves $\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}$ pointwise invariant, which follows from the fact that $1 - H_2T_2$ is a projector onto $\mathrm{Ker}(T_2)$.

Finally, we prove $\mathcal{B}_1 = \mathcal{B} \cdot G_2$. Substituting G_2 for H_2 in the generic representation of $\mathcal{B}_1$, we show

$$\mathcal{B} \cdot G_2 = (\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot G_2.$$

Since $(T_2, \mathcal{B}_2)$ is regular, we can substitute $\mathrm{Ker}(T_2)^\perp \cap \mathcal{B} \dotplus \mathcal{B}_2$ for $\mathcal{B}$ in (19) in the left-hand side, and it remains to show that $\mathcal{B}_2 \cdot G_2 = 0$. But this follows from $\mathrm{Im}(G_2) = \mathcal{B}_2^\perp$. □

The constructive method for computing $\mathcal{B}_1 = (\mathrm{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2$ is the same as in the proof of Proposition 30. Using the row-operation matrix $P \in \mathrm{GL}(K, m + n)$ constructed there (the original version creating zeros only in the lower left block), we compute the new boundary conditions

$$\tilde{\beta}_i = \sum_{j=1}^{m+n} P_{ij}\beta_j \quad (i = m + 1, \ldots, m + n)$$

to obtain a basis $\tilde{\beta}_{m+1} \circ H_2, \ldots, \tilde{\beta}_{m+n} \circ H_2$ of $\mathcal{B}_1$.

Putting together Proposition 30 and Proposition 31, we have now established the following *Factorization Theorem for Boundary Problems*.

Theorem 32. *Given a boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$, every factorization $T = T_1T_2$ of the differential operator can be lifted to a factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ of the boundary problem with $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \in \mathfrak{P}$ and $\mathcal{B}_2 \leq \mathcal{B}$.*

We conclude this section with an example of a fourth-order boundary problem arising in mechanics; see Kamke (1967, p. 525).

Example 33. Using the language of analytic polynomials (see Example 12), we consider the boundary problem $\mathcal{P} = (D^4 + 4, [L, R, LD, RD])$, in traditional formulation

$$\boxed{\begin{aligned} &u'''' + 4u = f, \\ &u(0) = u(1) = u'(0) = u'(1) = 0. \end{aligned}}$$

We employ the natural factorization $D^4 + 4 = (D^2 - 2i)(D^2 + 2i)$. Using the basis functions $u_{\pm\pm} = \mathrm{e}^{\pm 1 \pm i}$ for the kernel of $D^4 + 4$, we choose the boundary conditions for the right factor $D^2 + 2i$ in such a way that its Green's operator G_2 has a convenient formulation (this is not necessary in principle but keeps expressions shorter). By the generic second-order formula from Stakgold (1979, p. 195), also derived in Rosenkranz (2005, p. 196), we are led to the right factor $\mathcal{P}_2 = (D^2 + 2i, [(i-1)\,L - LD, (1-i)\,R - RD])$ or

$$\boxed{\begin{aligned} &u'' + 2i\,u = f, \\ &(i-1)\,u(0) - u'(0) = (1-i)\,u(1) - u'(1) = 0 \end{aligned}}$$

in traditional formulation.

Boundary problem $\mathcal{P}_2$ can now be solved easily by the generic second-order formula. Alternatively, one could also apply the algorithm from Table 2 or a factorization into first-order problems as explained at the end of Section 5. In any case, one arrives at the Green's operator

$$G_2 = \frac{1+i}{4}\left(\lceil u_{+-}\rceil A\,\lceil u_{-+}\rceil + \lceil u_{-+}\rceil B\,\lceil u_{+-}\rceil\right),$$

acting on a function $f \in C^\infty[0,1]$ according to

$$G_2 f(x) = \frac{1+i}{4}\left(\int_0^x \mathrm{e}^{(1-i)(x-\xi)} f(\xi)\,\mathrm{d}\xi + \int_x^1 \mathrm{e}^{(i-1)(x-\xi)} f(\xi)\,\mathrm{d}\xi\right).$$

We use the Green's operator G_2 of boundary problem $\mathcal{P}_2$ for determining the boundary conditions of the (unique!) left factor $\mathcal{P}_1$ in the factorization $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$ according to Proposition 31. One may easily verify that $\mathcal{P}_1 = (D^2 - 2i, [F\lceil u_{+-}\rceil, F\lceil u_{-+}\rceil])$ or

$$\boxed{\begin{aligned} &u'' - 2i\,u = f \\ &\textstyle\int_0^1 \mathrm{e}^{(1-i)\xi} f(\xi)\,\mathrm{d}\xi = \int_0^1 \mathrm{e}^{(i-1)\xi} f(\xi)\,\mathrm{d}\xi = 0 \end{aligned}} \tag{22}$$

in traditional formulation.

Since this is not a two-point boundary problem, let us go through the algorithm of Table 2 in detail. The first step is to determine the fundamental right inverse of $D^2 - 2i$. A straightforward computation yields

$$H_1 = \frac{i-1}{4}\left(\lceil u_{--}\rceil\, A\,\lceil u_{++}\rceil - \lceil u_{++}\rceil\, A\,\lceil u_{--}\rceil\right).$$

Next we compute the projector P onto $\mathrm{Ker}(D^2 - 2i)$ along $[F\lceil u_{+-}\rceil, F\lceil u_{-+}\rceil]^\perp$. Using the representation (16), we compute a basis $(\hat{u}_{+-}, \hat{u}_{-+})$ biorthogonal to $(F\lceil u_{+-}\rceil, F\lceil u_{-+}\rceil)$, obtaining $P = \lceil \hat{u}_{+-}\rceil\, F\,\lceil u_{+-}\rceil + \lceil \hat{u}_{-+}\rceil\, F\,\lceil u_{-+}\rceil$. Carrying out the computation (which involves four definite integrals and inverting a 2×2 matrix) leads to

Table 3
Coefficients for G_1

	u_{--}	u_{-+}	u_{+-}	u_{++}
a_{--}	$(1+i)(\mathrm{e}^2-\mathrm{e}^{2i})$	$2i(1-\mathrm{e}^2)$	$2(\mathrm{e}^{2i}-1)$	$(1-i)(2-\mathrm{e}^2-\mathrm{e}^{2i})$
b_{--}	$(1+i)(\mathrm{e}^2-\mathrm{e}^{2i})$	$2i(1-\mathrm{e}^2)$	$2(\mathrm{e}^{2i}-1)$	$(1-i)(\mathrm{e}^{-2}+\mathrm{e}^{-2i}-2)$
a_{++}	$(1-i)(\mathrm{e}^{-2}+\mathrm{e}^{-2i}-2)$	$2(1-\mathrm{e}^{-2i})$	$2i(\mathrm{e}^{-2}-1)$	$(1+i)(\mathrm{e}^{-2i}-\mathrm{e}^{-2})$
b_{++}	$(1-i)(2-\mathrm{e}^{2i}-\mathrm{e}^2)$	$2(1-\mathrm{e}^{-2i})$	$2i(\mathrm{e}^{-2}-1)$	$(1+i)(\mathrm{e}^{-2i}-\mathrm{e}^{-2})$

Table 4
Coefficients for G

	u_{--}	u_{-+}	u_{+-}	u_{++}
a_{--}	$i(\mathrm{e}^{2i}-\mathrm{e}^2)$	$(1-i)(1-\mathrm{e}^2)$	$(1+i)(1-\mathrm{e}^{2i})$	$\mathrm{e}^2+\mathrm{e}^{2i}-2$
b_{--}	$i(\mathrm{e}^{2i}-\mathrm{e}^2)$	$(1-i)(1-\mathrm{e}^2)$	$(1+i)(1-\mathrm{e}^{2i})$	$2-\mathrm{e}^{-2}-\mathrm{e}^{-2i}$
a_{-+}	$(1-i)(1-\mathrm{e}^2)$	$\mathrm{e}^2-\mathrm{e}^{-2i}$	$i(2-\mathrm{e}^2-\mathrm{e}^{-2i})$	$(1+i)(\mathrm{e}^{-2i}-1)$
b_{-+}	$(1-i)(1-\mathrm{e}^2)$	$\mathrm{e}^2-\mathrm{e}^{-2i}$	$i(\mathrm{e}^{-2}+\mathrm{e}^{2i}-2)$	$(1+i)(\mathrm{e}^{-2i}-1)$
a_{+-}	$(1+i)(1-\mathrm{e}^{2i})$	$i(\mathrm{e}^{-2}+\mathrm{e}^{2i}-2)$	$\mathrm{e}^{2i}-\mathrm{e}^{-2}$	$(1-i)(\mathrm{e}^{-2}-1)$
b_{+-}	$(1+i)(1-\mathrm{e}^{2i})$	$i(2-\mathrm{e}^2-\mathrm{e}^{-2i})$	$\mathrm{e}^{2i}-\mathrm{e}^{-2}$	$(1-i)(\mathrm{e}^{-2}-1)$
a_{++}	$2-\mathrm{e}^{-2}-\mathrm{e}^{-2i}$	$(1+i)(\mathrm{e}^{-2i}-1)$	$(1-i)(\mathrm{e}^{-2}-1)$	$i(\mathrm{e}^{-2}-\mathrm{e}^{-2i})$
b_{++}	$\mathrm{e}^2+\mathrm{e}^{2i}-2$	$(1+i)(\mathrm{e}^{-2i}-1)$	$(1-i)(\mathrm{e}^{-2}-1)$	$i(\mathrm{e}^{-2}-\mathrm{e}^{-2i})$

$$\hat{u}_{+-} = \frac{(\mathrm{e}^2-1)\,u_{--} - (\mathrm{e}^{-2i}-1)i\,u_{++}}{\Delta},$$
$$\hat{u}_{-+} = \frac{(\mathrm{e}^{2i}-1)i\,u_{--} - (\mathrm{e}^{-2}-1)\,u_{++}}{\Delta},$$

where $\Delta = \cos 2 + \cosh 2 - 2$. Then we compute the Green's operator of boundary problem $\mathcal{P}_1$ as $G_1 = (1-P)\,H_1$. Using the normalization engine for analytic polynomials described in Rosenkranz (2005), we arrive at

$$G_1 = \frac{1}{8\Delta}\Big(\lceil u_{--}\rceil\, A\, \lceil a_{--}\rceil + \lceil u_{--}\rceil\, B\, \lceil b_{--}\rceil + \lceil u_{++}\rceil\, A\, \lceil a_{++}\rceil + \lceil u_{++}\rceil\, B\, \lceil b_{++}\rceil\Big),$$

where each of $a_{--}, b_{--}, a_{++}, b_{++}$ is a linear combination $u_{--}, u_{-+}, u_{+-}, u_{++}$ as indicated in Table 3.

According to Proposition 27, the Green's operator G of the full boundary problem $\mathcal{P}$ is given by G_2G_1. Its explicit form, obtained by noncommutative multiplication and subsequent normalization, is given here for reference; often one might prefer the factored representation in terms of G_2 and G_1. We have

$$G = \frac{1+i}{32\Delta}\Big(\lceil u_{--}\rceil\, A\, \lceil a_{--}\rceil + \cdots + \lceil u_{++}\rceil\, B\, \lceil b_{++}\rceil\Big),$$

similar to G_1 in structure, but now with four additional summands coming from u_{-+} and u_{+-}. The eight functions $a_{--}, \ldots, a_{++}$ are again linear combinations of the type before, with coefficients given in Table 4.

8. Conclusion

Factoring a differential equation reduces the order and thus aids in solving the given equation. Since differential equations usually come together with boundary conditions, they must be incorporated in an additional step (typically viewed as external to differential algebra). The theory presented in this paper extends the factorization techniques for linear ordinary differential equations in such a way that the boundary conditions become an integral part, leading to an algorithmic machinery for *factoring and solving boundary problems* over integro-differential algebras. The implementation of these algorithms will be described in a subsequent paper.

Let us now discuss some possibilities for *extending* our approach into various directions: partial differential equations, systems of linear ordinary differential equations, difference equations, polynomial boundary conditions, semilinear boundary problems, dual pairings and duality theory, analytical aspects, and localization.

In this paper, we have restricted ourselves to ordinary differential equations (and thus to ordinary integro-differential algebras in the sense of Definition 8). This is convenient since – relative to given fundamental systems – it allows us to compute Green's operators in closed form. But the concept of multiplying (and hence factoring) boundary problems, as defined in (17), may be transferred to a *more general setting* that allows for infinitely many "boundary conditions"; see Regensburger and Rosenkranz (in press).

It can in particular be applied to *linear partial differential equations*, where one can exploit suitable results about factoring linear partial differential operators (Grigoriev and Schwarz, 2007, 2005, 2004; Tsarev, 1998). As a prototype (Regensburger and Rosenkranz, in press), we have factored the one-dimensional inhomogeneous wave equation on a bounded interval into two first-order "boundary problems". Along these lines, we plan to develop symbolic algorithms for first-order partial differential equations (typical factor problems) in non-trivial geometries. Since factorization will normally end up with (symbolically) irreducible boundary problems, it becomes more important to address stability issues: Well-posed boundary problems should be factored into well-posed blocks (Engl et al., 1996), if possible.

Going into a different direction, one can also apply our methodology of multiplying and factoring boundary problems to *systems* of linear ordinary differential equations. We expect that the solution theory (now using "Green's matrices" instead of Green's functions) as well as the algorithms will essentially carry over to this setting.

Everything considered in this paper was directed towards the continuous case of linear differential equations, but we expect the discrete case of linear *difference equations* to be tractable in principle by the same methods, except for the well-known complications arising from a skew Leibniz rule and a Baxter axiom with weight unity instead of zero; see Example 1.6 in Guo (2002). As pointed out in Section 2, the concept of integro-differential algebras generalizes naturally to this situation (Guo and Keigher, 2007).

By contrast, the restriction to linear differential equations seems to be quite rigid: we do not see how to translate our ideas to nonlinear differential equations. What could be considered, though, is the case of linear differential equations with *polynomial boundary conditions*, a case that is also of interest in applications. (A classical example is given by the heat equation with radiation on the boundary, described by the Stefan–Boltzmann law: The normal derivative of the temperature is proportional to its fourth power.) Although the solution operator of such a problem is necessarily nonlinear, we hope that one can adapt some of our ideas by handling the boundary conditions through ideals instead of linear subspaces.

In this article, we have worked with the (algebraic) dual of the vector space structure of the underlying differential algebra. We think that our approach could in principle be transferred to a setting with a *dual pairing* instead of the canonical bilinear form; this would include important topological vector spaces like C^k and L^p. Of course, this requires a modification of the composition structure, leading to a category rather than a monoid of boundary problems as pointed out in Regensburger and Rosenkranz (in press). The advantage might be that one gains topological insights relating various operators (like the differential and Green's operators) and spaces (like images and kernels).

Speaking of duality, one should also mention that the usual *duality theory* of linear boundary problems (Coddington and Levinson, 1955, Chapter 11) can be transferred to "classical" Stieltjes boundary conditions (on real- or complex-valued functions); see for example Brown (1975). The idea is that every boundary problem should have a dual or "adjoint" problem whose solution operator is the "transpose" of the original problem. The adjoint problem is often useful for characterizing certain aspects of a given primal problem (e.g. the solvability criterion for the Fredholm alternative).

We have not yet exploited the factored representation of Green's operators for *characterizing Green's functions* (possibly restricted to the well-posed case to avoid distributions). This may be done from two different perspectives: From an algebraic viewpoint, one might proceed in a manner similar to the Galois theory of linear ordinary differential equations; from an analytic viewpoint, the singular value decomposition would be of interest.

Finally, we mention that we have also treated singular boundary problems, where one needs a modified Green's function/operator as in the example from Section 3.5 in Rosenkranz (2005). This leads to a *localization* in the algebra of Green's operators—differential operators appear as the "reciprocals" of suitable integral operators. In this manner, one obtains a noncommutative generalization of the Mikusiński calculus that allows a symbolic treatment of boundary problems just like the ordinary Mikusiński calculus does for initial value problems (Mikusiński, 1959). These ideas will be discussed in a future paper.

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References

Baader, F., Nipkow, T., 1998. Term Rewriting and All That. Cambridge University Press, Cambridge.

Baxter, G., 1960. An analytic problem whose solution follows from a simple algebraic identity. Pacific J. Math. 10, 731–742.

Bergman, G.M., 1978. The diamond lemma for ring theory. Adv. Math. 29 (2), 179–218.

Brown, R.C., 1975. Duality theory for nth order differential operators under Stieltjes boundary conditions. SIAM J. Math. Anal. 6 (5), 882–900.

Brown, R.C., Krall, A.M., 1974. Ordinary differential operators under Stieltjes boundary conditions. Trans. Amer. Math. Soc. 198, 73–92.

Brown, R.C., Krall, A.M., 1977. n-th order ordinary differential systems under Stieltjes boundary conditions. Czechoslovak Math. J. 27 (1), 119–131.

Buchberger, B., 1965, An algorithm for finding the bases elements of the residue class ring modulo a zero dimensional polynomial ideal (German). Ph.D. Thesis. Univ. of Innsbruck. English translation published in J. Symbolic Comput., 41 (3–4): 475–511, 2006.

Buchberger, B., 1970. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. Aequationes Math. 4, 374–383. English translation: An algorithmical criterion for the solvability of a system of algebraic equations. In Buchberger and Winkler (1998).

Buchberger, B., 1998. Introduction to Gröbner Bases. pp. 3–31, In Buchberger and Winkler (1998).

Buchberger, B., Winkler, F. (Eds.), 1998. Gröbner bases and applications. London Mathematical Society Lecture Note Series, vol. 251. Cambridge University Press, Cambridge. Papers from the Conference on 33 Years of Gröbner Bases held at the University of Linz, Linz, February 2–4, 1998.

Coddington, E.A., Levinson, N., 1955. Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York, Toronto, London.

Cohn, P.M., 1982. Algebra, 2nd ed. vol. 1. John Wiley & Sons, Chichester.

Cohn, P.M., 2003. Basic Algebra: Groups, Rings and Fields. Springer, London.

Eilenberg, S., 1976. Automata, languages, and machines (Volume B). In: Pure and Applied Mathematics, vol. 59. Academic Press, New York.

Engl, H.W., Hanke, M., Neubauer, A., 1996. Regularization of inverse problems. In: Mathematics and its Applications, vol. 375. Kluwer Academic Publishers Group, Dordrecht.

Engl, H.W., Nashed, M.Z., 1981. New extremal characterizations of generalized inverses of linear operators. J. Math. Anal. Appl. 82 (2), 566–586.

Grigoriev, D., Schwarz, F., 2004. Factoring and solving linear partial differential equations. Computing 73 (2), 179–197.

Grigoriev, D., Schwarz, F., 2005. Generalized Loewy-decomposition of D-modules. In: Kauers, M. (Ed.), ISSAC '05: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation. ACM Press, New York, NY, USA, pp. 163–170.

Grigoriev, D., Schwarz, F., 2007. Loewy- and primary decompositions of D-modules. Adv. in Appl. Math. 38, 526–541.

Grigoriev, D.Y., 1990. Complexity of factoring and calculating the GCD of linear ordinary differential operators. J. Symbolic Comput. 10 (1), 7–37.

Guo, L., 2002. Baxter algebras and differential algebras. In: Differential Algebra and Related Topics (Newark, NJ, 2000). World Sci. Publ., River Edge, NJ, pp. 281–305.

Guo, L., Keigher, W., 2007, On differential Rota–Baxter algebras, arXiv:math/0703780v1 [math.RA].

Kamke, E., 1967. Differentialgleichungen. Lösungsmethoden und Lösungen. Teil I: Gewöhnliche Differentialgleichungen, 8th ed. In: Mathematik und ihre Anwendungen in Physik und Technik A, vol. 18. Akademische Verlagsgesellschaft, Leipzig.

Kaplansky, I., 1957. An Introduction to Differential Algebra. Actualités Sci. Ind., No. 1251 = Publ. Inst. Math. Univ. Nancago, No. 5. Hermann, Paris.

Keigher, W.F., 1997. On the ring of Hurwitz series. Comm. Algebra 25 (6), 1845–1859.

Keigher, W.F., Pritchard, F.L., 2000. Hurwitz series as formal functions. J. Pure Appl. Algebra 146 (3), 291–304.

Kolchin, E., 1973. Differential algebra and algebraic groups. In: Pure and Applied Mathematics, vol. 54. Academic Press, New York, London.

Köthe, G., 1969. Topological vector spaces (Volume I). In: Die Grundlehren der mathematischen Wissenschaften, vol. 159. Springer, New York.

Mikusiński, J., 1959. Operational calculus. In: International Series of Monographs on Pure and Applied Mathematics, vol.8. Pergamon Press, New York.

Mora, F., 1986. Groebner bases for non-commutative polynomial rings. In: AAECC-3: Proceedings of the 3rd International Conference on Algebraic Algorithms and Error-Correcting Codes. Springer-Verlag, London, UK, pp. 353–362.

Mora, T., 1994. An introduction to commutative and noncommutative Gröbner bases. Theoret. Comput. Sci. 134 (1), 131–173. Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992).

Nashed, M.Z., Votruba, G.F., 1976. A unified operator theory of generalized inverses. In: Nashed, M.Z. (Ed.), Generalized Inverses and Applications (Proc. Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1973). Academic Press, New York, pp. 1–109.

Pritchard, F.L., Sit, W.Y., 2007. On initial value problems for ordinary differential–algebraic equations. In: Rosenkranz, M., Wang, D. (Eds.), Gröbner Bases in Symbolic Analysis. Proceedings of the Special Semester on Gröbner Bases and Related Methods. In: Radon Series Comp. Appl. Math., vol. 2. Walter de Gruyter, Berlin, pp. 283–340.

Regensburger, G., Rosenkranz, M., 2007, An algebraic foundation for factoring linear boundary problems. Ann. Mat. Pura Appl. (4) (in press).

Robinson, A., 1961. Local differential algebra. Trans. Amer. Math. Soc. 97, 427–456.

Rosenkranz, M., 2005. A new symbolic method for solving linear two-point boundary value problems on the level of operators. J. Symbolic Comput. 39 (2), 171–199.

Rosenkranz, M., Buchberger, B., Engl, H.W., 2003. Solving linear boundary value problems via non-commutative Gröbner bases. Appl. Anal. 82, 655–675.

Rota, G.-C., 1969. Baxter algebras and combinatorial identities (I, II). Bull. Amer. Math. Soc. 75, 325–334.

Schwarz, F., 1989. A factorization algorithm for linear ordinary differential equations. In: ISSAC '89: Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation. ACM Press, New York, NY, USA, pp. 17–25.

Stakgold, I., 1979. Green's Functions and Boundary Value Problems. John Wiley & Sons, New York.

Tsarev, S.P., 1996. An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator. In: ISSAC '96: Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation. ACM Press, New York, NY, USA, pp. 226–231.

Tsarev, S.P., 1998. Factorization of linear partial differential operators and Darboux integrability of nonlinear PDEs. SIGSAM Bull. 32 (4), 21–28.

Ufnarovski, V., 1998. Introduction to Noncommutative Gröbner Bases Theory. pp. 259–280, In Buchberger and Winkler (1998).

van der Put, M., Singer, M.F., 2003. Galois theory of linear differential equations. In: Grundlehren der Mathematischen Wissenschaften, vol. 328. Springer, Berlin.