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On integro-differential algebras

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ABSTRACT

The concept of integro-differential algebra has been introduced recently in the study of boundary problems of differential equations. We generalize this concept to that of integro-differential algebra with a weight, in analogy to the differential Rota–Baxter algebra. We construct free commutative integro-differential algebras with weight generated by a differential algebra. This gives in particular an explicit construction of the integro-differential algebra on one generator. Properties of the free objects are studied.

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1. Introduction

1.1. Motivation and goal

Differential algebra [28,33] is the study of differentiation and nonlinear differential equations by purely algebraic means, without using an underlying topology. It has been largely successful in many important areas like uncoupling of nonlinear systems, classification of singular components, and detection of hidden equations. There are various implementations that offer the main algorithms needed for such tasks, for instance the `DifferentialAlgebra` package in the Maple™ system [10].

In view of applications, there is one crucial component that does not fit well in differential algebra—the treatment of initial or boundary conditions. The problem is that the elements of a differential algebra or field are abstractions that cannot be evaluated at a specific point. For bridging this gap (first in a specific context of two-point boundary problems), a new framework was set up in [34] with the following features:

- Differential algebras are enhanced by two evaluations (multiplicative functionals to the ground field) and two integral operators (Rota–Baxter operators), leading to the notion of analytic algebra.
- The usual ring of differential operators is generalized to a ring of integro-differential operators.
- Boundary problems are formulated in terms of the operator ring (differential equations as usual, boundary conditions in terms of the evaluations).
- The Green's operator of a boundary problem is computed as an element of the operator ring.

The algebraic framework of boundary problems was subsequently refined and extended by a multiplicative structure with results on the corresponding factorizations along a given factorization of the differential operator [35,38]. The factorization approach to boundary problems was applied in [2,3] to find closed-form and asymptotic expressions for ruin probabilities and associated quantities in risk theory.

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Moreover, it was realized that the algebraic theory of boundary problems is intimately related to the theory of Rota–Baxter algebras, which can be regarded as an algebraic study of both the integral and summation operators, even though it originated from the probability study of G. Baxter [7] in 1960. Rota–Baxter algebras have found extensive applications in mathematics and physics, including quantum field theory and the classical Yang–Baxter equation [4,14,15,18,19,25]. In a nutshell, the relation with Rota–Baxter algebras is this: In the differential algebra $C^\infty(\mathbb{R})$, every point evaluation ϕ gives rise to a unique Rota–Baxter operator $(1-\phi)\circ\int$, where $\int$ is any fixed integral operator, say $f \mapsto \int_0^x f(\xi)\,d\xi$. See also Theorem 2.5 below for a more general relation between evaluations and integral operators. We refer to [5,6] for an extensive study on algebraic properties of integro-differential operators with polynomial coefficients and a single evaluation (corresponding to initial value problems).

The algebraic approach to boundary problems is currently developed for linear ordinary differential equations although some effort is under way to cover certain classes of linear partial differential equations [37]. Various parts of the theory have been implemented, first as external Mathematica®-Theorema reasoner [34], then as internal Theorema code [37,38], and recently in a Maple™ package with new features for singular boundary problems [29].

1.2. Main results and outline of the paper

Our main purpose in this paper is to explicitly construct free objects in the category of λ-integro-differential algebras, which is at the heart of the algebraic framework of boundary problems described above. The existence of such free objects is known from universal algebra via equivalence classes of terms modulo the identities they satisfy [9,12,30] and from category theory via adjoint functors and monads; see [31, Chapter VI] and the references therein. But to construct free objects explicitly in terms of normal forms is often a non-trivial task. In the case of λ-integro-differential algebras, we make use of the construction of free objects in a structure closely related to the λ-integro-differential algebra, namely the differential Rota–Baxter algebra. A Rota–Baxter algebra is an algebraic abstraction of a reformulation of the integral by parts formula where only the integral operator appears. Free commutative Rota–Baxter algebras were obtained in [21,22] in terms of shuffles and the more general mixable shuffles of tensor powers.

More recently the concept of a differential Rota–Baxter algebra was introduced [23] by putting a differential operator and a Rota–Baxter operator of the same weight together such that one is the one side inverse of the other as in the Fundamental Theorem of Calculus. One advantage of this relatively independent combination of the two operators in a differential Rota–Baxter algebra is that the free objects can be constructed quite easily by building the free Rota–Baxter algebra on top of the free differential algebra. Since the axiom of an integro-differential algebra requires more intertwined relationship between the differential and Rota–Baxter operators, a free integro-differential algebra is a quotient of a free differential Rota–Baxter algebra. With this as the starting point of our construction of free integro-differential algebras, our strategy is to find an explicitly defined linear basis for this quotient from the known basis of the free differential Rota–Baxter algebra by tensor powers. For this purpose we use regular differential algebras as our basic building block for the tensor powers.

In Section 2, we first introduce the concept of an integro-differential algebra of weight λ and study their various characterizations, especially those in connection with differential Rota–Baxter algebras. In Section 3, we start with recalling free commutative Rota–Baxter algebras of weight λ and then free commutative differential Rota–Baxter algebras of weight λ and derive the existence of free commutative integro-differential algebras. The explicit construction of free objects in the category of λ-integro-differential algebras is carried out in Section 4 (Theorem 4.6) with a preparation on regular differential algebras and a detailed discussion on the regularity of the differential algebras of differential polynomials and rational functions.

2. Integro-differential algebras of weight λ

We first introduce the concepts and basic properties related to λ-integro-differential algebras.

2.1. Definitions and preliminary examples

We recall the concepts of a derivation with weight, a Rota–Baxter operator with weight and a differential Rota–Baxter algebra with weight, before introducing our definition of an integro-differential algebra with weight.

Definition 2.1. Let $\mathbf{k}$ be a unitary commutative ring. Let $\lambda \in \mathbf{k}$ be fixed.

(a) A **differential k-algebra of weight** λ (also called a **λ-differential k-algebra**) is a unitary associative $\mathbf{k}$-algebra R together with a linear operator $d\colon R \to R$ such that

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y) \quad \text{for all } x, y \in R, \tag{1}$$

and

$$d(1) = 0. \tag{2}$$

Such an operator is called a **derivation of weight** λ or a λ-**derivation**.

(b) A **Rota–Baxter k-algebra of weight** λ is an associative **k**-algebra R together with a linear operator $P: R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy) \quad \text{for all } x, y \in R. \tag{3}$$

Such an operator is called a **Rota–Baxter operator of weight** λ or a λ-**Rota–Baxter operator**.

(c) A **differential Rota–Baxter k-algebra of weight** λ (also called a λ-**differential Rota–Baxter k-algebra**) is a differential **k**-algebra (R, d) of weight λ and a Rota–Baxter operator P of weight λ such that

$$d \circ P = \mathrm{id}_R.$$

(d) An **integro-differential k-algebra of weight** λ (also called a λ-**integro-differential k-algebra**) is a differential **k**-algebra (R, D) of weight λ with a linear operator $\Pi: R \to R$ such that

$$D \circ \Pi = \mathrm{id}_R \tag{4}$$

and

$$\Pi(D(x))\Pi(D(y)) = \Pi(D(x))y + x\Pi(D(y)) - \Pi(D(xy)) \quad \text{for all } x, y \in R. \tag{5}$$

When there is no danger of confusion, we will suppress λ and **k** from the notations. We will also denote the set of non-negative integers by $\mathbb{N}$.

Note that we require that a derivation d satisfies $d(1) = 0$. This follows from Eq. (1) automatically when $\lambda = 0$, but is a non-trivial restriction when $\lambda \neq 0$. In the next section, we give equivalent characterizations of the **hybrid Rota–Baxter axiom** (5) and discuss its relation to the **Rota–Baxter axiom** (3) as well as consequences of the **section axiom** (4). Note that the hybrid Rota–Baxter axiom does not contain a term with the weight λ.

We next give some simple examples of differential Rota–Baxter algebras and integro-differential algebras. As we shall see below (Lemma 2.3), the latter are a special case of the former. Further examples will be given in later sections. In particular, the algebras of λ-Hurwitz series are integro-differential algebras (Proposition 3.2). By Theorem 4.6, every regular differential algebra naturally gives rise to the corresponding free integro-differential algebra.

Example 2.2. (a) By the First Fundamental Theorem of Calculus

$$\frac{d}{dx}\Big(\int_a^x f(t)dt\Big) = f(x)$$

and the conventional integration-by-parts formula

$$\int_a^x f(t)g'(t)dt = f(t)g(t) - f(a)g(a) - \int_a^x f'(t)g(t)dt, \tag{6}$$

$(C^\infty(\mathbb{R}), d/dx, \int_a^x)$ is an integro-differential algebra of weight 0. As we shall see later in Theorem 2.5, integration by parts is in fact equivalent to the hybrid Rota–Baxter axiom (5).

(b) The following example from [23] of a differential Rota–Baxter algebra is also an integro-differential algebra. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let $R = C^\infty(\mathbb{R})$ denote the $\mathbb{R}$-algebra of smooth functions $f: \mathbb{R} \to \mathbb{R}$, and consider the usual "difference quotient" operator D_λ on R defined by

$$(D_\lambda(f))(x) = (f(x+\lambda) - f(x))/\lambda. \tag{7}$$

Then D_λ is a λ-derivation on R. When $\lambda = 1$, we obtain the usual difference operator on functions. Further, the usual derivation is $D_0 := \lim_{\lambda \to 0} D_\lambda$. Now let R be an $\mathbb{R}$-subalgebra of $C^\infty(\mathbb{R})$ that is closed under the operators

$$\Pi_0(f)(x) = -\int_x^\infty f(t)dt, \qquad \Pi_\lambda(f)(x) = -\lambda \sum_{n \geq 0} f(x + n\lambda).$$

For example, R can be taken to be the $\mathbb{R}$-subalgebra generated by e^{-x}: $R = \sum_{k \geq 1} \mathbb{R}e^{-kx}$. Then Π_λ is a Rota–Baxter operator of weight λ and, for the D_λ in Eq. (7),

$$D_\lambda \circ \Pi_\lambda = \mathrm{id}_R \quad \text{for all } x, y \in R, 0 \neq \lambda \in \mathbb{R},$$

reducing to the fundamental theorem $D_0 \circ \Pi_0 = \mathrm{id}_R$ when λ goes to 0. We note the close relations of $(R, D_\lambda, \Pi_\lambda)$ to the time scale calculus [1] and the quantum calculus [27].

The fact that $(R, D_\lambda, \Pi_\lambda)$ is actually an integro-differential algebra follows from Theorem 2.5(g) since the kernel of D_λ is just the constant functions (in the case $\lambda \neq 0$ one uses that $R = \sum_{k \geq 1} \mathbb{R}e^{-kx}$ does not contain periodic functions).

(c) Here is one example of a differential Rota–Baxter algebra that is not an integro-differential algebra [35, Ex. 3]. Let **k** be a field of characteristic zero, $A = \mathbf{k}[y]/(y^4)$, and $(A[x], d)$, where d is the usual derivation with $d(x^k) = k\,x^{k-1}$. We define a **k**-linear map P on $A[x]$ by

$$P(f) = \Pi(f) + f(0, 0)\, y^2,$$

where Π is the usual integral with $\Pi(x^k) = x^{k+1}/(k+1)$. Since the second term vanishes under d, we see immediately that $d \circ P = \mathrm{id}_{A[x]}$. For verifying the Rota–Baxter axiom (3) with weight zero, we compute

$$P(f)P(g) = \Pi(f)\Pi(g) + g(0,0)\,y^2\,\Pi(f) + f(0,0)\,y^2\Pi(g) + f(0,0)g(0,0)\,y^4,$$
$$P(fP(g)) = \Pi(f\,(\Pi(g) + g(0,0)\,y^2)) = \Pi(f\,\Pi(g)) + g(0,0)\,y^2\,\Pi(f),$$
$$P(P(f)g) = \Pi((\Pi(f) + f(0,0)\,y^2)\,g) = \Pi(\Pi(f)g) + f(0,0)\,y^2\,\Pi(g).$$

Since $y^4 \equiv 0$ and the usual integral Π fulfills the Rota–Baxter axiom (3), this implies immediately that P does also. However, it does not fulfill the hybrid Rota–Baxter (5) since for example

$$P(d(x))P(d(y)) = P(1)P(0) = 0$$

but we obtain

$$P(d(x))y + xP(d(y)) - P(d(xy)) = P(1)y + xP(0) - P(y) = (x + y^2)y - xy = y^3.$$

for the right-hand side.

2.2. *Basic properties of integro-differential algebras with weight*

We first show that an integro-differential algebra with weight is a differential Rota–Baxter algebra of the same weight. We then give several equivalent conditions for integro-differential algebras.

Lemma 2.3. *Let (R, D) be a differential algebra of weight λ with a linear operator $\Pi : R \to R$ such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$.*

(a) *The triple (R, D, Π) is a differential Rota–Baxter algebra of weight λ if and only if*

$$\Pi(x)\Pi(y) = J(\Pi(x)\Pi(y)) \quad \text{for all } x, y \in R, \tag{8}$$

and if and only if

$$J(x)J(y) = J(J(x)J(y)) \quad \text{for all } x, y \in R. \tag{9}$$

(b) *Every integro-differential algebra is a differential Rota–Baxter algebra.*

Note that Eq. (8) does not contain a term with λ. Also note Eq. (9) involves only the initialization J and shows in particular that $\operatorname{im} J$ is a subalgebra.

Proof. (a) Using Eq. (1), we see that

$$D(\Pi(x)\Pi(y)) = x\Pi(y) + \Pi(x)y + \lambda xy.$$

Hence the Rota–Baxter axiom

$$\Pi(x)\Pi(y) = \Pi(x\Pi(y)) + \Pi(\Pi(x)y) + \lambda\Pi(xy)$$

is equivalent to Eq. (8). Moreover, substituting $D(x)$ for x and $D(y)$ for y in Eq. (8), we get the identity (9). Since D is onto by $D \circ \Pi = \mathrm{id}_R$, we also obtain Eq. (8) from Eq. (9).
(b) Since $J \circ \Pi = \Pi \circ (D \circ \Pi) = \Pi \circ \mathrm{id}_R = \Pi$, we obtain Eq. (8) from the hybrid Rota–Baxter axiom (5) by substituting $\Pi(x)$ for x and $\Pi(y)$ for y. □

We now give several equivalent conditions for an integro-differential algebra by starting with a result on complementary projectors on algebras.

Lemma 2.4. *Let E and J be projectors on a unitary $\mathbf{k}$-algebra R such that $E + J = \mathrm{id}_R$. Then the following statements are equivalent:*

(a) *E is an algebra homomorphism,*
(b) *J is a derivation of weight -1,*
(c) *$\ker E = \operatorname{im} J$ is an ideal and $\operatorname{im} E = \ker J$ is a unitary subalgebra.*

Proof. ((a) ⇔ (b)) It can be checked directly that $E(xy) = E(x)E(y)$ if and only if $J(xy) = J(x)y + xJ(y) - J(x)J(y)$. Further it follows from $E + J = \mathrm{id}_R$ that $E(1) = 1$ if and only if $J(1) = 0$.
((a) ⇒ (c)) is clear once we see that the assumption of the lemma implies $\ker E = \operatorname{im} J$ and $\operatorname{im} E = \ker J$.
((c) ⇒ (a)) Let $x, y \in R$. Since $R = \operatorname{im} E \oplus \ker E$, we have $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1 = E(x), y_1 = E(y) \in \operatorname{im} E$ and $x_2, y_2 \in \ker E$. Then $E(x_1y_1) = x_1y_1$ since $\operatorname{im} E$ is by assumption a subalgebra. Thus

$$E(xy) = E(x_1y_1) + E(x_1y_2) + E(x_2y_1) + E(x_2y_2) = x_1y_1 = E(x)E(y),$$

where the last three summands vanish assuming that $\ker E$ is an ideal. Moreover, $1 \in \operatorname{im} E$ implies $E(1) = 1$. □

We have the following characterizations of integro-differential algebras.

Theorem 2.5. *Let (R, D) be a differential algebra of weight λ with a linear operator Π on R such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$, called the* ***initialization****, and $E = \mathrm{id}_R - J$, called the* ***evaluation****. Then the following statements are equivalent:*

(a) *(R, D, Π) is an integro-differential algebra;*
(b) *$E(xy) = E(x)E(y)$ for all $x, y \in R$;*
(c) *$\ker E = \mathrm{im}\, J$ is an ideal;*
(d) *$J(xJ(y)) = xJ(y)$ and $J(J(x)y) = J(x)y$ for all $x, y \in R$;*
(e) *$J(x\Pi(y)) = x\Pi(y)$ and $J(\Pi(x)y) = \Pi(x)y$ for all $x, y \in R$;*
(f) *$x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y))$ for all $x, y \in R$;*
(g) *(R, D, Π) is a differential Rota–Baxter algebra and $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y)$ for all $x, y \in R$;*
(h) *(R, D, Π) is a differential Rota–Baxter algebra and $J(E(x)J(y)) = E(x)J(y)$ and $J(J(x)E(y)) = J(x)E(y)$ for all $x, y \in R$.*

Remark 2.6. (I) Items (d) and (e) can be regarded as the invariance formulation of the hybrid Rota–Baxter axiom.

(II) Item (f) can be seen as a "weighted" noncommutative version of integration by parts: One obtains it in case of weight zero by substituting $\int g$ for g in the usual formula (6). This motivates also the name integro-differential algebra. Clearly, in the commutative case the respective left and right versions are equivalent.

(III) Since $\mathrm{im}\, E = \ker D$, the identities in Items (g) and (h) can be interpreted as left/right linearity of respectively Π and J over the constants of the derivation D, restricted to $\mathrm{im}\, J$ in the case of (h). Note again that (g) and (h) do not contain a term with λ.

Proof. We first note that under the assumption, we have $J^2 = \Pi \circ (D \circ \Pi) \circ D = \Pi \circ \mathrm{id}_R \circ D = J$ and so the initialization J and evaluation E are projectors. Therefore

$$\ker D = \ker J = \mathrm{im}\, E \quad \text{and} \quad \mathrm{im}\, \Pi = \mathrm{im}\, J = \ker E, \tag{10}$$

and

$$R = \ker D \oplus \mathrm{im}\, \Pi$$

is a direct sum decomposition.

((a) ⇔ (b)). It follows from Lemma 2.4 since the hybrid Rota–Baxter axiom (5) can be rewritten as

$$J(x)J(y) = J(x)y + xJ(y) - J(xy) \quad \text{for all } x, y \in R. \tag{11}$$

((b) ⇔ (c)). It follows from Lemma 2.4, since $\ker D = \ker J = \mathrm{im}\, E$ is a unitary subalgebra by Eqs. (1) and (2).

((a) ⇒ (e)). We obtain (e) by substituting in Eq. (11) respectively $\Pi(y)$ for y and $\Pi(x)$ for x.

((e) ⇔ (d)). Substituting respectively $D(y)$ for y and $D(x)$ for x in (e) gives (d). Conversely, substituting respectively $\Pi(y)$ for y and $\Pi(x)$ for x in (d) gives (e).

((e) ⇔ (f)). It follows from Eq. (1).

((a) ⇒ (g)). By Lemma 2.3, (R, D, Π) is a differential Rota–Baxter algebra. Furthermore, using Eq. (1) and $D \circ E = 0$, we see that

$$D(E(x)\Pi(y)) = E(x)y \quad \text{and} \quad D(\Pi(x)E(y)) = xE(y)$$

and so

$$J(E(x)\Pi(y)) = \Pi(E(x)y) \quad \text{and} \quad J(\Pi(x)E(y)) = \Pi(xE(y)).$$

Since we have proved (e) from (a), we can respectively substitute $E(x)$ for x and $E(y)$ for y in (e) to get (g).

((g) ⇔ (h)). Further, from $\Pi(E(x)y) = E(x)\Pi(y)$ we obtain

$$J(E(x)J(y)) = \Pi(D(E(x)J(y))) = \Pi(E(x)D(y)) = E(x)J(y),$$

Conversely, from $J(E(x)J(y)) = E(x)J(y)$ we obtain

$$\Pi(E(x)y) = \Pi(D(E(x)\Pi(y))) = J(E(x)\Pi(y)) = J(E(x)J(\Pi(y))) = E(x)\Pi(y)$$

using $\Pi = J \circ \Pi$ and $D(E(x)\Pi(y)) = E(x)y$. This proves the equivalence of the first equations in (g) and (h); the same proof gives the equivalence of the second equations.

((d) ⇒ (c)). This is clear since the identities imply that $\mathrm{im}\, J$ is an ideal.

((h) ⇒ (e)). Note that $J(E(x)J(y)) = E(x)J(y)$ gives

$$J(xJ(y)) - J(J(x)J(y)) = xJ(y) - J(x)J(y)$$

and hence $J(xJ(y)) = xJ(y)$ with the Rota–Baxter axiom in the form of Eq. (9). The identity $J(J(x)y) = J(x)y$ follows analogously. □

3. Free commutative integro-differential algebras

We first review the constructions of free commutative differential algebra with weight, free commutative Rota–Baxter algebras and free commutative differential Rota–Baxter algebras. These constructions are then applied in Section 3.3 to obtain free commutative integro-differential algebras and will be applied in Section 4 to give an explicit construction of free commutative integro-differential algebras.

3.1. Free and cofree differential algebras of weight λ

We recall the construction [23] of free commutative differential algebras of weight λ.

Theorem 3.1. *Let X be a set. Let*

$$\Delta(X) = X \times \mathbb{N} = \{x^{(n)} \,\big|\, x \in X, n \geq 0\}.$$

Let $\mathbf{k}\{X\}$ be the free commutative algebra $\mathbf{k}[\Delta X]$ on the set ΔX. Define $d_X : \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ as follows. Let $w = u_1 \cdots u_k$, $u_i \in \Delta X$, $1 \leq i \leq k$, be a commutative word from the alphabet set $\Delta(X)$. If $k = 1$, so that $w = x^{(n)} \in \Delta(X)$, define $d_X(w) = x^{(n+1)}$. If $k > 1$, recursively define

$$d_X(w) = d_X(u_1)u_2 \cdots u_k + u_1 d_X(u_2 \cdots u_k) + \lambda d_X(u_1) d_X(u_2 \cdots u_k).$$

Further define $d_X(1) = 0$ and then extend d_X to $\mathbf{k}\{X\}$ by linearity. Then $(\mathbf{k}\{X\}, d_X)$ is the free commutative differential algebra of weight λ on the set X.

The use of $\mathbf{k}\{X\}$ for free commutative differential algebras of weight λ is consistent with the notation of the usual free commutative differential algebra (when $\lambda = 0$).

We also review the following construction from [23]. For any commutative $\mathbf{k}$-algebra A, let $A^{\mathbb{N}}$ denote the $\mathbf{k}$-module of all functions $f : \mathbb{N} \to A$. We define the λ-**Hurwitz product** on $A^{\mathbb{N}}$ by defining, for any $f, g \in A^{\mathbb{N}}$, $fg \in A^{\mathbb{N}}$ by

$$(fg)(n) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k f(n-j) g(k+j).$$

We denote the $\mathbf{k}$-algebra $A^{\mathbb{N}}$ with this product by DA, and call it the $\mathbf{k}$-algebra of λ-**Hurwitz series over** A. It was shown in [23] that DA is a differential Rota–Baxter algebra of weight λ with the operators

$$\begin{aligned} &D : DA \to DA, \qquad (D(f))(n) = f(n+1),\ n \geq 0,\ f \in DA, \\ &\Pi : DA \to DA, \qquad (\Pi(f))(n) = f(n-1),\ n \geq 1,\ (\Pi(f))(0) = 0,\ f \in DA. \end{aligned}$$

In fact, DA is the cofree differential algebra of weight λ on A. We similarly have

Proposition 3.2. *The triple (DA, D, Π) is an integro-differential algebra of weight λ.*

Proof. Since (DA, D, Π) is a differential Rota–Baxter algebra, we only need to show that $\Pi(E(x)y) = E(x)\Pi(y)$ for $x, y \in DA$ by Theorem 2.5. But this is clear since $\operatorname{im} E = \ker D = A$ and Π is A-linear. □

3.2. Free commutative Rota–Baxter algebras

We briefly recall the construction of free commutative Rota–Baxter algebras. Let A be a commutative $\mathbf{k}$-algebra. Define

$$Ш(A) = \bigoplus_{k \in \mathbb{N}} A^{\otimes(k+1)} = A \oplus A^{\otimes 2} \oplus \cdots, \tag{12}$$

where and hereafter all the tensor products are taken over $\mathbf{k}$ unless otherwise stated. Let $\mathfrak{a} = a_0 \otimes \cdots \otimes a_m \in A^{\otimes(m+1)}$ and $\mathfrak{b} = b_0 \otimes \cdots \otimes b_n \in A^{\otimes(n+1)}$. If $m = 0$ or $n = 0$, define

$$\mathfrak{a} \diamond \mathfrak{b} = \begin{cases} (a_0 b_0) \otimes b_1 \otimes \cdots \otimes b_n, & m = 0, n > 0, \\ (a_0 b_0) \otimes a_1 \otimes \cdots \otimes a_m, & m > 0, n = 0, \\ a_0 b_0, & m = n = 0. \end{cases} \tag{13}$$

If $m > 0$ and $n > 0$, inductively (on $m + n$) define

$$\begin{aligned} \mathfrak{a} \diamond \mathfrak{b} \ &= (a_0 b_0) \otimes \Big((a_1 \otimes a_2 \otimes \cdots \otimes a_m) \diamond (1_A \otimes b_1 \otimes \cdots \otimes b_n) + (1_A \otimes a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots \otimes b_n) \\ &\quad + \lambda\, (a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots \otimes b_n) \Big). \end{aligned} \tag{14}$$

Extending by additivity, we obtain a $\mathbf{k}$-bilinear map

$$\diamond : Ш(A) \times Ш(A) \to Ш(A).$$

Alternatively,

$$\mathfrak{a} \diamond \mathfrak{b} = (a_0 b_0) \otimes (\bar{\mathfrak{a}} \,Ш_\lambda\, \bar{\mathfrak{b}}),$$

where $\bar{\mathfrak{a}} = a_1 \otimes \cdots \otimes a_m$, $\bar{\mathfrak{b}} = b_1 \otimes \cdots \otimes b_n$ and $Ш_\lambda$ is the mixable shuffle (quasi-shuffle) product of weight λ [19,21,26], which specializes to the shuffle product $Ш$ when $\lambda = 0$.

Define a $\mathbf{k}$-linear endomorphism P_A on $Ш(A)$ by assigning

$$P_A(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = 1_A \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

for all $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}$ and extending by additivity. Let $j_A\colon A \to Ш(A)$ be the canonical inclusion map.

Theorem 3.3 ([21,22]). *The pair $(Ш(A), P_A)$, together with the natural embedding $j_A\colon A \to Ш(A)$, is a free commutative Rota–Baxter $\mathbf{k}$-algebra on A of weight λ. In other words, for any Rota–Baxter $\mathbf{k}$-algebra (R, P) and any $\mathbf{k}$-algebra map $\varphi\colon A \to R$, there exists a unique Rota–Baxter $\mathbf{k}$-algebra homomorphism $\tilde{\varphi}\colon (Ш(A), P_A) \to (R, P)$ such that $\varphi = \tilde{\varphi} \circ j_A$ as $\mathbf{k}$-algebra homomorphisms.*

Since $\diamond$ is compatible with the multiplication in A, we will suppress the symbol $\diamond$ and simply denote xy for $x \diamond y$ in $Ш(A)$, unless there is a danger of confusion.

Let (A, d) be a commutative differential $\mathbf{k}$-algebra of weight λ. Define an operator d_A on $Ш(A)$ by assigning

$$d_A(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = d(a_0) \otimes a_1 \otimes \cdots \otimes a_n + a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n + \lambda d(a_0) a_1 \otimes a_2 \otimes \cdots \otimes a_n \tag{15}$$

for $a_0 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}$ and then extending by $\mathbf{k}$-linearity. Here we use the convention that $d_A(a_0) = d(a_0)$ when $n = 0$.

Theorem 3.4 ([23]). *Let (A, d) be a commutative differential $\mathbf{k}$-algebra of weight λ. Let $j_A\colon A \to Ш(A)$ be the $\mathbf{k}$-algebra embedding (in fact a morphism of differential $\mathbf{k}$-algebras of weight λ). The quadruple $(Ш(A), d_A, P_A, j_A)$ is a free commutative differential Rota–Baxter $\mathbf{k}$-algebra of weight λ on (A, d).*

3.3. The existence of free commutative integro-differential algebras

The free objects in the category of commutative integro-differential algebras of weight λ are defined in a similar fashion as for the category of commutative differential Rota–Baxter algebras.

Definition 3.5. Let (A, d) be a λ-differential algebra over $\mathbf{k}$. A **free integro-differential algebra of weight λ on** A is an integro-differential algebra $(\mathrm{ID}(A), D_A, \Pi_A)$ of weight λ together with a differential algebra homomorphism $i_A\colon (A, d) \to (\mathrm{ID}(A), d_A)$ such that, for any integro-differential algebra (R, D, Π) of weight λ and a differential algebra homomorphism $f\colon (A, d) \to (R, D)$, there is a unique integro-differential algebra homomorphism $\bar{f}\colon \mathrm{ID}(A) \to R$ such that $\bar{f} \circ i_A = f$.

As in Theorem 3.4, let $(Ш(A), d_A, P_A)$ be the free commutative differential Rota–Baxter algebra generated by the differential algebra (A, d). Then by Theorem 2.5, we have

Theorem 3.6. *Let (A, d) be a commutative differential $\mathbf{k}$-algebra of weight λ. Let $I_{\mathbf{ID}}$ be the differential Rota–Baxter ideal of $Ш(A)$ generated by the set*

$$\{J\big(E(x)J(y)\big) - E(x)J(y) \,\big|\, x, y \in Ш(A)\},$$

where J and E denote the projectors $P_A \circ d_A$ and $\mathrm{id}_A - P_A \circ d_A$, respectively. Let δ_A (resp. Π_A) denote d_A (resp. P_A) modulo $I_{\mathbf{ID}}$. Then the quotient differential Rota–Baxter algebra $(Ш(A)/I_{\mathbf{ID}}, \delta_A, \Pi_A)$, together with the natural map $i_A\colon A \to Ш(a) \to Ш(A)/I_{\mathbf{ID}}$, is the free integro-differential algebra of weight λ on A.

Proof. Let a λ-integro-differential algebra (R, D, Π) be given. Then by Theorem 2.5, (R, D, Π) is also a λ-differential Rota–Baxter algebra. Thus by Theorem 3.4, there is a unique homomorphism $\tilde{f}\colon Ш(A) \to R$ such that the left triangle of the following diagram commutes.

$$\begin{tikzcd}
& (Ш(A), d_A, P_A) \arrow[dd, "\tilde{f}"] \arrow[dr, "\pi"] & \\
(A, d) \arrow[ur, "j_A"] \arrow[dr, "f"'] & & (Ш(A)/I_{\mathbf{ID}}, \delta_A, \Pi_A) \arrow[dl, "\bar{f}"] \\
& (R, D, \Pi) &
\end{tikzcd}$$

Since (R, D, Π) is a λ-integro-differential algebra, $\tilde{f}$ factors through $Ш(A)/I_{\mathbf{ID}}$ and induces the λ-integro-differential algebra homomorphism $\bar{f}$ such that the right triangle commutes. Since $i_A = \pi \circ j_A$, we have $\bar{f} \circ i_A = f$ as needed.

Suppose $\bar{f}_1 : Ш(A)/I_{\mathbf{ID}} \to R$ is also a λ-integro-differential algebra homomorphism such that $\bar{f}_1 \circ i_A = f$. Define $\tilde{f}_1 = \bar{f}_1 \circ \pi$. Then $\tilde{f}_1 \circ j_A = f$. Thus by the universal property of $Ш(A)$, we have $\tilde{f}_1 = \tilde{f}$. Since π is surjective, we must have $\bar{f}_1 = \bar{f}$. This completes the proof. □

4. Construction of free commutative integro-differential algebras

As mentioned in Section 1, in integro-differential algebras the relation between d and Π is more intimate than in differential Rota–Baxter algebras. This makes the construction of their free objects more complex. Having ensured their existence in (Section 3.3), we introduce a vast class of differential algebras for which our construction applies (Section 4.1). Next we present the details of the construction and some basic properties (Section 4.2), leading on to the proof that it yields the desired free object (Section 4.3). The construction applies in particular to rings of differential polynomials $\mathbf{k}\{u\}$, yielding the free object over one generator, and to the ring of rational functions (Section 4.4).

4.1. Regular differential algebras

A free commutative integro-differential algebra can be regarded as a universal way of constructing an integro-differential algebra from a differential algebra. The easiest way of obtaining an integro-differential algebra from a differential algebra occurs when (A, d) already has an integral operator Π. This means in particular that $d \circ \Pi = \mathrm{id}_A$ so that the derivation d must be surjective. But often this will not be the case, for example when $A = \mathbf{k}\{u\}$ is the ring of differential polynomials (where u is clearly not in the image of d). But even if we cannot define an antiderivative (meaning a right inverse for d) on all of A, we may still be able to define one on $d(A)$ using an appropriate **quasi-antiderivative** Q. This means we require $d(Q(y)) = y$ for $y \in d(A)$ or equivalently $d(Q(d(x))) = d(x)$ for $x \in A$. For a general operator d, an operator Q with this property is called an inner inverse of d. It exists for many important differential algebras, in particular for differential polynomials (Proposition 4.10) and rational function (Proposition 4.12).

Before coming back to differential algebras, we recall some properties of generalized inverses for linear maps on $\mathbf{k}$-modules; for further details and references see [32, Section 8.1.].

Definition 4.1. Let $L\colon M \to N$ be a linear map between $\mathbf{k}$-modules.

(a) If a linear map $\bar{L}\colon N \to M$ satisfies $L \circ \bar{L} \circ L = L$, then $\bar{L}$ is called an **inner inverse** of L.
(b) If L has an inner inverse, then L is called **regular**.
(c) If a linear map $\bar{L}\colon N \to M$ satisfies $\bar{L} \circ L \circ \bar{L} = \bar{L}$, then $\bar{L}$ is called an **outer inverse** of L.
(d) If $\bar{L}$ is an inner inverse and outer inverse of L, then $\bar{L}$ is called a **quasi-inverse** or **generalized inverse** of L.

Proposition 4.2. *Let $L\colon M \to N$ be a linear map between $\mathbf{k}$-modules.*

(a) *If L has an inner inverse $\bar{L}\colon N \to M$, then $S = L \circ \bar{L}\colon N \to N$ is a projector onto $\operatorname{im} L$ and $E = \mathrm{id}_M - \bar{L} \circ L\colon M \to M$ is a projector onto $\ker L$.*
(b) *Given projectors $S\colon N \to N$ onto $\operatorname{im} L$ and $E\colon M \to M$ onto $\ker L$, there is a unique quasi-inverse $\bar{L}$ of L such that $\operatorname{im} \bar{L} = \ker E$ and $\ker \bar{L} = \ker S$. Thus a regular map has a quasi-inverse.*

Proof. (a) This statement is immediate.
(b) If L is regular, then by Item (a), there are submodules $\ker E \subseteq M$ and $\ker S \subseteq N$ such that

$$M = \ker L \oplus \ker E, \qquad N = \operatorname{im} L \oplus \ker S.$$

Thus L induces a bijection $L\colon \ker E \to \operatorname{im} L$. Define $\bar{L}\colon N \to M$ to be the inverse of this bijection on $\operatorname{im} L$ and to be zero on $\ker S$, then we check directly that $\bar{L}$ is a quasi-inverse of L and the unique one such that $\operatorname{im} \bar{L} = \ker E$ and $\ker \bar{L} = \ker S$. See also [32, Theorem 8.1.]. □

For a quasi-inverse $\bar{L}$ of L we note the direct sums

$$M = \operatorname{im} \bar{L} \oplus \ker L \quad \text{and} \quad N = \operatorname{im} L \oplus \ker \bar{L}.$$

Moreover, let

$$J = \mathrm{id}_M - E \quad \text{and} \quad T = \mathrm{id}_N - S,$$

then we have the relations

$$M_E := \operatorname{im} E = \ker L = \ker J, \qquad M_J := \operatorname{im} J = \operatorname{im} \bar{L} = \ker E$$
$$N_S := \operatorname{im} S = \operatorname{im} L = \ker T, \qquad N_T := \operatorname{im} T = \ker \bar{L} = \ker S$$

for the corresponding projectors.

The intuitive roles of the projectors E and J are similar as in Section 2.2, except that the "evaluation" E is not necessarily multiplicative and the image of the "initialization" J need not be an ideal. The projector S may be understood as extracting the solvable part of N, in the sense of solving $L(x) = y$ for x, as much as possible for a given $y \in N$.

Let us elaborate on this. Writing respectively $y_S = S(y)$ and $y_T = T(y)$ for the "solvable" and "transcendental" part of y, the equation $L(x) = y_S$ is clearly solved by $x^* = \bar{L}(y_S)$ while $L(x) = y_T$ is only solvable in the trivial case $y_T = 0$. So the identity $L(x^*) = y - T(y)$ may be understood in the sense that x^* solves $L(x) = y$ except for the transcendental part. We illustrate this in the following example.

Example 4.3. Consider the field $\mathbb{C}(x)$ of **complex rational functions** with its usual derivation d. We take d to be the linear map $L\colon M \to N$ where $M = N = \mathbb{C}(x)$. Any rational function can be represented by f/g with a monic denominator $g = (x-\alpha_1)^{n_1}\cdots(x-\alpha_k)^{n_k}$ having distinct roots $\alpha_i \in \mathbb{C}$. By partial fraction decomposition, it can be written uniquely as

$$r + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_i)^j},$$

where $r \in \mathbb{C}[x]$ and $\gamma_{ij} \in \mathbb{C}$. Then for the domain $\mathbb{C}(x)$ of d, we have the decomposition

$$\mathbb{C}(x) = \ker d \oplus \mathbb{C}(x)_J$$

with $\ker d = \mathbb{C}$ and

$$\mathbb{C}(x)_J = \left\{ r + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_i)^j} \,\middle|\, r \in x\,\mathbb{C}[x],\ \alpha_i \in \mathbb{C} \text{ distinct},\ \gamma_{ij} \in \mathbb{C} \right\}$$

as the initialized space. For the range $\mathbb{C}(x)$ of d, we have the decomposition

$$\mathbb{C}(x) = \operatorname{im} d \oplus \mathbb{C}(x)_T,$$

with

$$\operatorname{im} d = \left\{ r + \sum_{i=1}^{k}\sum_{j=2}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_i)^j} \,\middle|\, r \in \mathbb{C}[x],\ \alpha_i \in \mathbb{C} \text{ distinct},\ \gamma_{ij} \in \mathbb{C} \right\}$$

and

$$\mathbb{C}(x)_T = \left\{ \sum_{i=1}^{k}\frac{\gamma_i}{x-\alpha_i} \,\middle|\, \alpha_i \in \mathbb{C} \text{ distinct},\ \gamma_i \in \mathbb{C} \right\}$$

as the transcendental space.

By Proposition 4.2 there exists a unique quasi-inverse $Q : \mathbb{C}(x) \to \mathbb{C}(x)$ of d corresponding to the above decompositions, which we can describe explicitly. On $\operatorname{im} d$ we define Q by setting $Q(x^k) = x^{k+1}/(k+1)$ for $k \geq 0$ and $Q(1/(x-\alpha)^j) = 1/(1-j)(x-\alpha)^{j-1}$ for $j > 1$, and we extend it by zero on $\mathbb{C}(x)_T$. Analytically speaking, the quasi-antiderivative Q acts as $\int_0^x$ on the polynomials and as $\int_{-\infty}^x$ on the solvable rational functions: Since $\mathbb{C}(x)$ is not an integro-differential algebra, it is not possible to use a single integral operator. The associated codomain projector $S = d \circ Q$ extracts the solvable part by filtering out the residues $1/(x-\alpha)$; their antiderivatives would need logarithms, which are not available in $\mathbb{C}(x)$. The domain projector $E = \mathrm{id}_{\mathbb{C}(x)} - Q \circ d$ is almost like evaluation at 0 but is not multiplicative according to Theorem 2.5 since $\mathbb{C}(x)_J$ cannot be an ideal of the field $\mathbb{C}(x)$. In fact, one checks immediately that $E(x \cdot 1/x) = E(1) = 1$ but $E(x) \cdot E(1/x) = 0 \cdot 0 = 0$.

See Proposition 4.12 for the case when d here is replaced by the difference operator or more generally the λ-difference quotient operator d_λ with $\lambda \neq 0$ (Example 2.2). We refer to [11] for details on effectively computing the above decomposition into solvable and transcendental part of rational functions in the context of symbolic integration algorithms. See also [13] for necessary and sufficient conditions for the existence of telescopers in the differential, difference, and q-difference case in terms of (generalizations of) residues.

We can now define what makes a differential algebra such as $A = \mathbf{k}\{u\}$ and $A = \mathbb{C}(x)$ adequate for the forthcoming construction of the free integro-differential algebra.

Definition 4.4. Let (A, d) be a differential algebra of weight λ with derivation $d\colon A \to A$.

(a) If $\lambda = 0$, then (A, d) is called **regular** if its derivation d is a regular map. Then a quasi-inverse of d is called a **quasi-antiderivative**.
(b) If $\lambda \neq 0$, then (A, d) is called **regular** if its derivation d is a regular map and the kernel of one of its quasi-inverses is a nonunitary $\mathbf{k}$-subalgebra of A. Such a quasi-inverse of d is called a **quasi-antiderivative**.

We observe that the class of regular differential algebras is fairly comprehensive in the zero weight case. It includes all differential algebras over a field $\mathbf{k}$ since in that case every subspace is complemented, so all $\mathbf{k}$-linear maps are regular. In particular, all differential fields (viewed as differential algebras over their field of constants) are regular. The example $\mathbb{C}(x)$ is a case in point, but note that Example 4.3 provides an explicit quasi-antiderivative rather than plain existence.

The situation is more complex in the nonzero weight case due to the extra restriction on the derivation, which we need in our construction of free integro-differential algebras. If $\mathbf{k}$ is a field, the ring of differential polynomials $\mathbf{k}\{u\}$ is regular for any weight, and we will provide an explicit quasi-antiderivative that works also when $\mathbf{k}$ is a $\mathbb{Q}$-algebra but not a field (Proposition 4.10). Moreover, the field of complex rational functions $\mathbb{C}(x)$ with its usual difference operator is a regular differential ring of weight one, and this can be extended to arbitrary nonzero weight (Proposition 4.12).

4.2. Construction of $\mathrm{ID}(A)^*$

According to Theorem 3.6, the free integro-differential algebra $\mathrm{ID}(A)$ can be described by a suitable quotient. However, for studying this object effectively, a more explicit construction is preferable. We will achieve this, for a regular differential algebra A, by defining an integro-differential algebra $\mathrm{ID}(A)^*$, and by showing in the next subsection that it satisfies the relevant universal property. Hence we may take $\mathrm{ID}(A)^*$ to be $\mathrm{ID}(A)$.

4.2.1. Definition of $\mathrm{ID}(A)^*$ and the statement of Theorem 4.6

Let (A, d) be a regular differential algebra with a fixed quasi-antiderivative Q.

Denote

$$A_J = \operatorname{im} Q \quad \text{and} \quad A_T = \ker Q.$$

Then we have the direct sums

$$A = A_J \oplus \ker d \quad \text{and} \quad A = \operatorname{im} d \oplus A_T$$

with the corresponding projectors $E = \mathrm{id}_A - Q \circ d$ and $S = d \circ Q$, respectively. As before, we write $J = \mathrm{id}_A - E = Q \circ d$ and $T = \mathrm{id}_A - S$ for the complementary projectors. Furthermore, we use the notation $K := \ker d \supseteq \mathbf{k}$ in this subsection.

We give now an explicit construction of $\mathrm{ID}(A)^*$ via tensor products (all tensors are still over $\mathbf{k}$). First let

$$Ш_T(A) := \bigoplus_{k\geq 0} A \otimes A_T^{\otimes k} = A \oplus (A \otimes A_T) \oplus (A \otimes A_T^{\otimes 2}) + \cdots$$

be the $\mathbf{k}$-submodule of $Ш(A)$ in Eq. (12). Under our assumption that A_T is a subalgebra of A when $\lambda \neq 0$, $Ш_T(A)$ is clearly a $\mathbf{k}$-subalgebra of $Ш(A)$ under the multiplication in Eqs. (13) and (14). It is also closed under the derivation d_A defined in Eq. (15). Alternatively,

$$Ш_T(A) = A \otimes Ш^+(A_T)$$

is the tensor product algebra where $Ш^+(A_T) := \bigoplus_{n\geq 0} A_T^{\otimes n}$ is the mixable shuffle algebra [19,21,26] on the $\mathbf{k}$-algebra A_T. In the case $\lambda = 0$, this is the plain shuffle algebra, where it is sufficient for A_T to have the structure of a $\mathbf{k}$-module. So a pure tensor $\mathfrak{a}$ of $A \otimes Ш^+(A_T)$ is of the form

$$\mathfrak{a} = a \otimes \bar{\mathfrak{a}} \in A \otimes A_T^{\otimes n} \subseteq A^{\otimes(n+1)}. \tag{16}$$

We then define the **length** of $\mathfrak{a}$ to be $n + 1$.

Next let $\varepsilon \colon A \to A_\varepsilon$ be an isomorphism of K-algebras, where

$$A_\varepsilon := \{\varepsilon(a) \mid a \in A\}$$

denotes a replica of the K-algebra A, endowed with the zero derivation. We identify the image $\varepsilon(K) \subseteq A_\varepsilon$ with K so that $\varepsilon(c) = c$ for all $c \in K$. Finally let

$$\mathrm{ID}(A)^* := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K A \otimes Ш^+(A_T) \tag{17}$$

denote the tensor product differential algebra of A_ε and $Ш_T(A)$, namely the tensor product algebra where the derivation (again denoted by d_A) is defined by the Leibniz rule.

4.2.2. Definition of Π_A

We will define a linear operator Π_A on $\mathrm{ID}(A)^*$. First require that Π_A is linear over A_ε. Thus we just need to define $\Pi_A(\mathfrak{a})$ for a pure tensor $\mathfrak{a}$ in $A \otimes Ш^+(A_T)$. We will accomplish this by induction on the length n of $\mathfrak{a}$. When $n = 1$, we have $\mathfrak{a} = a \in A$. Then we have

$$a = d(Q(a)) + T(a) \quad \text{with } T(a) \in A_T \tag{18}$$

and we define

$$\Pi_A(a) := Q(a) - \varepsilon(Q(a)) + 1 \otimes T(a). \tag{19}$$

Assume $\Pi_A(\mathfrak{a})$ has been defined for $\mathfrak{a}$ of length $n \geq 1$ and consider the case when $\mathfrak{a}$ has length $n+1$. Then $\mathfrak{a} = a \otimes \bar{\mathfrak{a}}$ where $a \in A$, $\bar{\mathfrak{a}} \in A_T^{\otimes n}$ and we define

$$\Pi_A(a \otimes \bar{\mathfrak{a}}) := Q(a) \otimes \bar{\mathfrak{a}} - \Pi_A(Q(a)\bar{\mathfrak{a}}) - \lambda\, \Pi_A(d(Q(a))\,\bar{\mathfrak{a}}) + 1 \otimes T(a) \otimes \bar{\mathfrak{a}}, \tag{20}$$

where the first and last terms are manifestly in $A \otimes Ш^+(A_T)$ while the middle terms are in $\mathrm{ID}(A)^*$ by the induction hypothesis. We write $E_A = \mathrm{id}_{\mathrm{ID}(A)^*} - \Pi_A \circ d_A$ for what will turn out to be the "evaluation" corresponding to Π_A (see the discussion before Example 4.3).

We display the following relationship between Π_A, P_A and ε for later application.

Lemma 4.5. (a) *For $a \in A$, we have $E_A(a) = \varepsilon(a)$.*
(b) *For $\bar{\mathfrak{a}} \in Ш^+(A_T)$, we have $\Pi_A(\bar{\mathfrak{a}}) = P_A(\bar{\mathfrak{a}}) = 1 \otimes \bar{\mathfrak{a}}$.*

Proof. (a) Using the direct sum $A = A_J \oplus \ker d$, we distinguish two cases. If $a \in \ker d = K$, then the left-hand side is $a - \Pi_A(d_A(a)) = a - \Pi_A(0) = a$; but the right-hand is a as well since $\varepsilon\colon A \to A_\varepsilon$ is a K-algebra homomorphism. Hence assume $a \in A_J = \operatorname{im} J$. In that case $a = J(a) = Q(d(a))$ and hence $T(d(a)) = d(a) - d(Q(d(a))) = 0$. So $\Pi_A(d_A(a)) = \Pi_A(d(a)) = a - \varepsilon(a)$ by Eq. (19).
(b) This is a special case of Eqs. (18) and (20) with $Q(a) = 0$ and $T(a) = a$ since $a \in A_T$. $\square$

Theorem 4.6. *Let (A, d, Q) be a regular differential algebra of weight λ with quasi-antiderivative Q. Then the triple $(\mathrm{ID}(A)^*, d_A, \Pi_A)$, with the natural embedding*

$$i_A\colon A \to \mathrm{ID}(A)^* = A_\varepsilon \otimes_K A \otimes Ш^+(A_T)$$

to the second tensor factor, is the free commutative integro-differential algebra of weight λ generated by A.

The proof of Theorem 4.6 is given in Section 4.3.

Since $A_T \cong A/\operatorname{im} d$ as $\mathbf{k}$-modules, for different choices of Q, the corresponding A_T are isomorphic as $\mathbf{k}$-modules. Then for $\lambda = 0$ the mixable shuffle (i.e., shuffle) algebras $Ш^+(A_T)$ are isomorphic $\mathbf{k}$-algebras since in that case the algebra structure of A_T is not used; see e.g. Section 2.1 of [24]. When $\lambda \neq 0$, for A_T from different choices of Q, they are still isomorphic as $\mathbf{k}$-modules. But it is not clear that they are isomorphic as nonunitary $\mathbf{k}$-algebras. Nevertheless, the free commutative integro-differential algebras derived by Theorem 4.6 are isomorphic due to the uniqueness of the free objects. See Remark 4.13 for further discussions.

The following is a preliminary discussion on subalgebras as direct sum factors.

Lemma 4.7. *Let T and S be projectors on a unitary $\mathbf{k}$-algebra R such that $T + S = \mathrm{id}_R$. Then the following statements are equivalent:*

(a) $\operatorname{im} T = \ker S$ *is a subalgebra;*
(b) $T(T(x)T(y)) = T(x)T(y)$*;*
(c) $S(xy) = S(S(x)y + xS(y) - S(x)S(y))$.

Proof. ((a) $\Leftrightarrow$ (b)) It is clear since T is a projector.
((a) $\Rightarrow$ (c)) It follows from

$$S(T(x)T(y)) = S((x - S(x))(y - S(y)) = 0.$$

((c) $\Rightarrow$ (a)) Clearly, the identity implies that $\ker S$ is a subalgebra. $\square$

If $S = d \circ Q$ as above, we obtain from Lemma 4.7(c) an equivalent identity

$$Q(xy) = Q(d(Q(x))y + xd(Q(y)) - d(Q(x))d(Q(y)))$$

in terms of Q and d, since $Q \circ d \circ Q = Q$.

4.3. The proof of Theorem 4.6

We will verify that $(\mathrm{ID}(A)^*, d_A, \Pi_A)$ is an integro-differential algebra in Section 4.3.1 and verify its universal property in Section 4.3.2.

4.3.1. The integro-differential algebra structure on $\mathrm{ID}(A)^*$

Since d_A is clearly a derivation, by Theorem 2.5(b), we just need to check the two conditions

$$d_A \circ \Pi_A = \mathrm{id}_{\mathrm{ID}(A)^*}, \tag{21}$$

$$E_A(xy) = E_A(x)E_A(y), \quad x, y \in \mathrm{ID}(A)^*. \tag{22}$$

Since A_ε is in the kernel of d_A and in the ring of constants for Π_A, we just need to verify the equations for pure tensors $x = \mathfrak{a}, y = \mathfrak{b} \in A \otimes Ш^+(A_T)$.

We check Eq. (21) by showing $(d_A \circ \Pi_A)(\mathfrak{a}) = \mathfrak{a}$ for $\mathfrak{a} \in A \otimes Ш^+(A_T)$ by induction on the length $n \geq 1$ of $\mathfrak{a}$. When $n = 1$, we have $\mathfrak{a} = a \in A$ and obtain

$$d_A(\Pi_A(a)) = d_A(Q(a) - \varepsilon(Q(a)) + 1 \otimes T(a)) = d(Q(a)) + T(a) = a$$

by Eq. (18). Under the induction hypothesis, we consider $\mathfrak{a} = a \otimes \bar{\mathfrak{a}}$ with $\bar{\mathfrak{a}} \in A_T^{\otimes n}$, $n \geq 1$. Then we have

$$\begin{aligned} d_A(\Pi_A(a \otimes \bar{\mathfrak{a}})) &= d_A\big(Q(a) \otimes \bar{\mathfrak{a}} - \Pi_A(Q(a)\bar{\mathfrak{a}}) - \lambda\, \Pi_A(d(Q(a))\,\bar{\mathfrak{a}}) + 1 \otimes T(a) \otimes \bar{\mathfrak{a}}\big) \\ &= d(Q(a)) \otimes \bar{\mathfrak{a}} + Q(a)\bar{\mathfrak{a}} + \lambda\, d(Q(a))\bar{\mathfrak{a}} - Q(a)\bar{\mathfrak{a}} - \lambda\, d(Q(a))\bar{\mathfrak{a}} + T(a) \otimes \bar{\mathfrak{a}} \\ &= d(Q(a)) \otimes \bar{\mathfrak{a}} + T(a) \otimes \bar{\mathfrak{a}} \\ &= a \otimes \bar{\mathfrak{a}} \end{aligned}$$

by Eq. (18) again.

We next verify Eq. (22). If the length of both x and y are one, then x and y are in A. Then by Lemma 4.5(a), we have

$$E_A(xy) = \varepsilon(xy) = \varepsilon(x)\varepsilon(y) = E_A(x)E_A(y).$$

If at least one of x or y have length greater than one, then each pure tensor in the expansion of xy has length greater than one. Then the equation holds by the following lemma.

Lemma 4.8. *For any pure tensor $\mathfrak{a} = a \otimes \bar{\mathfrak{a}} \in A \otimes Ш^+(A_T)$ of length greater than one we have $E_A(\mathfrak{a}) = 0$.*

Remark 4.9. Combining Lemma 4.5(a) and Lemma 4.8 we have $\operatorname{im} E_A = A_\varepsilon$. Further, by Eq. (10), we have $\ker d_A = \operatorname{im} E_A = A_\varepsilon$.

Proof. For a given $\mathfrak{a} = a \otimes \bar{\mathfrak{a}}$ of length greater than one, we compute

$$\begin{aligned} E_A(a \otimes \bar{\mathfrak{a}}) &= a \otimes \bar{\mathfrak{a}} - \Pi_A(d_A(a \otimes \bar{\mathfrak{a}})) \quad \text{(by definition of } E_A) \\ &= a \otimes \bar{\mathfrak{a}} - \Pi_A(d(a) \otimes \bar{\mathfrak{a}}) - \Pi_A(a\bar{\mathfrak{a}}) - \Pi_A(\lambda d(a)\bar{\mathfrak{a}}) \quad \text{(by definition of } d_A) \\ &= a \otimes \bar{\mathfrak{a}} - Q(d(a)) \otimes \bar{\mathfrak{a}} + \Pi_A(Q(d(a))\bar{\mathfrak{a}}) + \lambda\, \Pi_A(d(Q(d(a)))\,\bar{\mathfrak{a}}) - 1 \otimes T(d(a)) \otimes \bar{\mathfrak{a}} \\ &\quad - \Pi_A(a\bar{\mathfrak{a}}) - \Pi_A(\lambda d(a)\bar{\mathfrak{a}}) \quad \text{(by definition of } \Pi_A) \\ &= a \otimes \bar{\mathfrak{a}} - Q(d(a)) \otimes \bar{\mathfrak{a}} + \Pi_A(Q(d(a))\bar{\mathfrak{a}}) - \Pi_A(a\bar{\mathfrak{a}}) \quad \text{(by } d \circ Q \circ d = d \text{ and } T(d(a)) = 0) \\ &= E(a) \otimes \bar{\mathfrak{a}} - \Pi_A(E(a)\bar{\mathfrak{a}}) \quad \text{(by definition of } E = \mathrm{id}_A - Q \circ d). \end{aligned}$$

Since $E(A) = K \subseteq A_\varepsilon$ and Π_A is taken to be A_ε-linear, from Lemma 4.5(b), we obtain

$$E_A(a \otimes \bar{\mathfrak{a}}) = E(a)(1_A \otimes \bar{\mathfrak{a}} - \Pi_A(\bar{\mathfrak{a}})) = 0. \quad \square$$

4.3.2. The universal property

We now verify the universal property of $(\mathrm{ID}(A)^*, d_A, \Pi_A)$ as the free integro-differential algebra on (A, d): Let $i_A : A \to \mathrm{ID}(A)^*$ be the natural embedding of A into the second tensor factor of $\mathrm{ID}(A)^* = A_\varepsilon \otimes_K A \otimes Ш^+(A_T)$. Then for any integro-differential algebra (R, D, Π) and any differential algebra homomorphism $f : (A, d) \to (R, D)$, there is a unique integro-differential algebra homomorphism $\bar{f} : (\mathrm{ID}(A)^*, d_A, \Pi_A) \to (R, D, \Pi)$ such that $\bar{f} \circ i_A = f$.

The existence of $\bar{f}$: Let a differential algebra homomorphism $f : (A, d) \to (R, D)$ be given. Note that f is in fact a K-algebra homomorphism where the K-algebra structure on R is given by $f : K \to R$. Since (R, Π) is a commutative Rota–Baxter algebra, by the universal property of $Ш(A)$ as the free commutative Rota–Baxter algebra on the commutative algebra A, there is a homomorphism $\tilde{f} : (Ш(A), P_A) \to (R, \Pi)$ of commutative Rota–Baxter algebras such that $\tilde{f} \circ j_A = f$ where $j_A : A \to Ш(A)$ is the embedding into the first tensor factor. This means that $\tilde{f}$ is an A-algebra homomorphism and, in particular, a K-algebra homomorphism. Thus $\tilde{f}$ restricts to a K-algebra homomorphism

$$\tilde{f} : A \otimes Ш^+(A_T) \to R.$$

Further, f also gives a K-algebra homomorphism

$$f_\varepsilon : A_\varepsilon \to R, \varepsilon(a) \mapsto f(a) - \Pi(D(f(a))).$$

Thus we get an algebra homomorphism on the tensor product over K:

$$\bar{f} := f_\varepsilon \otimes_K \tilde{f} : A_\varepsilon \otimes_K (A \otimes Ш^+(A_T)) \to R$$

that extends $\tilde{f}$ and f_ε. Further, we have $\bar{f} \circ j_A = f$.

It remains to check the equations

$$\bar{f} \circ d_A = D \circ \bar{f}, \qquad \bar{f} \circ \Pi_A = \Pi \circ \bar{f}. \tag{23}$$

Since A_ε is in the kernel of d_A and in the ring of constants of Π_A, we only need to verify the equations when restricted to $A\otimes Ш^+(A_T)$.

Fix $a\otimes\bar{\mathfrak{a}} = a(1\otimes\bar{\mathfrak{a}})\in A\otimes Ш^+(A_T)$. By Lemma 4.5(b), we have

$$\Pi(\bar{f}(\bar{\mathfrak{a}})) = \Pi(\tilde{f}(\bar{\mathfrak{a}})) = \tilde{f}(\Pi_A(\bar{\mathfrak{a}})) = \bar{f}(1\otimes\bar{\mathfrak{a}}).$$

Thus we obtain

$$\begin{aligned}
\bar{f}(d_A(a\otimes\bar{\mathfrak{a}})) &= \bar{f}(d(a)\otimes\bar{\mathfrak{a}}) + \bar{f}(a\bar{\mathfrak{a}}) + \bar{f}(\lambda d(a)\bar{\mathfrak{a}})\\
&= f(d(a))\bar{f}(1\otimes\bar{\mathfrak{a}}) + f(a)\bar{f}(\bar{\mathfrak{a}}) + \lambda f(d(a))\bar{f}(\bar{\mathfrak{a}})\\
&= D(f(a))\bar{f}(1\otimes\bar{\mathfrak{a}}) + f(a)D(\Pi(\bar{f}(\bar{\mathfrak{a}}))) + \lambda D(f(a))D(\Pi(\bar{f}(\bar{\mathfrak{a}})))\\
&= D(f(a))\bar{f}(1\otimes\bar{\mathfrak{a}}) + f(a)D(\bar{f}(1\otimes\bar{\mathfrak{a}})) + \lambda D(f(a))D(\bar{f}(1\otimes\bar{\mathfrak{a}}))\\
&= D(f(a)\bar{f}(1\otimes\bar{\mathfrak{a}}))\\
&= D(\bar{f}(a\otimes\bar{\mathfrak{a}})).
\end{aligned}$$

This proves the first equation in Eq. (23). We next prove the second equation by induction on the length $k \geq 1$ of $\mathfrak{a} := a\otimes\bar{\mathfrak{a}}\in A\otimes Ш^+(A_T)$. When $k=1$, we have $\mathfrak{a} = a\in A$ and

$$\begin{aligned}
\bar{f}(\Pi_A(a)) &= \bar{f}\,(Q(a) - \varepsilon(Q(a)) + 1\otimes T(a))\\
&= f(Q(a)) - f(Q(a)) + \Pi(D(f(Q(a)))) + \Pi(f(T(a)))\\
&= \Pi(f(d(Q(a)) + T(a)))\\
&= \Pi(f(a)),
\end{aligned}$$

using Lemma 4.5(a) and (b). Assume now that the claim has been proved for $k=n\geq 1$ and consider $\mathfrak{a} = a\otimes\bar{\mathfrak{a}}$ with length $n+1$. Then we have

$$\begin{aligned}
\bar{f}(\Pi_A(a\otimes\bar{\mathfrak{a}})) &= \bar{f}\,(Q(a)\otimes\bar{\mathfrak{a}} - \Pi_A(Q(a)\bar{\mathfrak{a}}) - \lambda\,\Pi_A(d(Q(a))\bar{\mathfrak{a}}) + 1\otimes T(a)\otimes\bar{\mathfrak{a}})\\
&= \bar{f}(Q(a))\bar{f}(\Pi_A(\bar{\mathfrak{a}})) - \bar{f}(\Pi_A(Q(a)\bar{\mathfrak{a}})) - \lambda\bar{f}(\Pi_A(d(Q(a))\bar{\mathfrak{a}})) + \bar{f}(P_A(T(a)\otimes\bar{\mathfrak{a}})).
\end{aligned}$$

Here we have applied Lemma 4.5(b) in the last term. Applying the induction hypothesis to the first three terms and using the fact that the restriction $\tilde{f}$ of $\bar{f}$ to $A\otimes Ш^+(A_T)$ is compatible with the Rota–Baxter operators in the last term, we obtain

$$\begin{aligned}
\bar{f}(\Pi_A(a\otimes\bar{\mathfrak{a}})) &= f(Q(a))\Pi(\bar{f}(\bar{\mathfrak{a}})) - \Pi(\bar{f}(Q(a)\bar{\mathfrak{a}})) - \lambda\Pi(\bar{f}(d(Q(a))\bar{\mathfrak{a}})) + \Pi(\bar{f}(T(a)\otimes\bar{\mathfrak{a}}))\\
&= \Pi\left(D(f(Q(a)))\Pi(\bar{f}(\bar{\mathfrak{a}}))\right) + \Pi\left(f(T(a))\bar{f}(P_A(\bar{\mathfrak{a}}))\right),
\end{aligned}$$

where we have used integration by parts in Theorem 2.5(f) in the last step. On the other hand, we have

$$\begin{aligned}
\Pi(\bar{f}(a\otimes\bar{\mathfrak{a}})) &= \Pi(f(a)\bar{f}(P_A(\bar{\mathfrak{a}})))\\
&= \Pi\left(f(d(Q(a)) + T(a))\bar{f}(P_A(\bar{\mathfrak{a}}))\right)\\
&= \Pi\left(D(f(Q(a)))\Pi(\bar{f}(\bar{\mathfrak{a}}))\right) + \Pi\left(f(T(a))\bar{f}(P_A(\bar{\mathfrak{a}}))\right).
\end{aligned}$$

Thus we have completed the proof of the existence of the integro-differential algebra homomorphism $\bar{f}$.

The uniqueness of $\bar{f}$: Suppose $\bar{f}_1 : \mathrm{ID}(A)^* \to R$ is a homomorphism of integro-differential algebras such that $\bar{f}_1\circ i_A = f$. For $1\otimes a_1\otimes\cdots\otimes a_n\in Ш^+(A_T)$, we have

$$\begin{aligned}
\bar{f}_1(1\otimes a_1\otimes\cdots\otimes a_n) &= \bar{f}_1\,(\Pi_A(a_1\Pi_A(\cdots\Pi_A(a_n)\cdots)))\\
&= \Pi(f(a_1)\Pi(\cdots\Pi(f(a_n))\cdots))\\
&= \bar{f}\,(\Pi_A(a_1\Pi_A(\cdots\Pi_A(a_n)\cdots)))\\
&= \bar{f}(1\otimes a_1\otimes\cdots\otimes a_n).
\end{aligned}$$

Thus the restrictions of $\bar{f}$ and $\bar{f}_1$ to $A\otimes Ш^+(A_T)$ are the same. Further, by Lemma 4.5(a),

$$\bar{f}_1(\varepsilon(a)) = f(a) - \bar{f}_1(\Pi_A(d_A(a))) = f(a) - \Pi(D(f(a)) = \bar{f}(\varepsilon(a)).$$

Hence the restrictions of $\bar{f}$ and $\bar{f}_1$ to A_ε are also the same. As these restrictions to $A\otimes Ш^+(A_T)$ and A_ε are K-homomorphisms, by the universal property of the tensor product over K, $\bar{f}$ and $\bar{f}_1$ agree on $\mathrm{ID}(A)^* = A_\varepsilon\otimes_K A\otimes Ш^+(A_T)$. This proves the uniqueness of $\bar{f}$ and thus completes the proof of Theorem 4.6.

4.4. Examples of regular differential algebras

In this section we show that some common examples of differential algebras, namely the algebra of differential polynomials and the algebra of rational functions, are regular where the weight can be taken arbitrary.

4.4.1. Rings of differential polynomials

Our main goal in this subsection is to prove that $(\mathbf{k}\{u\}, d)$ is a regular differential algebra for any weight, and to give an explicit quasi-antiderivative Q for d.

We start by introducing some definitions for classifying the elements of $A = \mathbf{k}\{u\}$. Let u_i, $i \geq 0$, be the i-th derivation of u. Then $\mathbf{k}\{u\}$ is the polynomial algebra on $\{u_i \mid i \geq 0\}$. For $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{N}^{k+1}$, we write $u^\alpha = u_0^{\alpha_0} \cdots u_k^{\alpha_k}$. Furthermore, we use the convention that $u^\alpha = 1$ when $\alpha \in \mathbb{N}^0$ is the degenerate tuple of length zero. Then all monomials of $\mathbf{k}\{u\}$ are of the form u^α, where α contains no trailing zero. The **order** of such a monomial $u^{(\alpha_0,\ldots,\alpha_k)} \neq 1$ is defined to be k; the order of $u^{()} = 1$ is set to -1. The order of a nonzero differential polynomial is defined as the maximum of the orders of its monomials. The following classification of monomials is crucial [17,8]: A monomial u^α of order k is called **functional** if either $k \leq 0$ or $\alpha_k > 1$. We write

$$A_T = \mathbf{k}\{u^\alpha \mid u^\alpha \text{ is functional}\}$$

for the corresponding submodule. Since the product of two functional monomials is again functional, A_T is in fact a $\mathbf{k}$-subalgebra of A. Furthermore, we write A_J for the submodule generated by all monomials $u^\alpha \neq 1$.

Proposition 4.10. *For any $\lambda \in \mathbf{k}$, the canonical derivation $d\colon A \to A$ of weight λ defined in Theorem 3.1 admits a quasi-antiderivative Q with associated direct sums $A = A_T \oplus \operatorname{im} d$ and $A = A_J \oplus \ker d$.*

Proof. The main work goes into showing the direct sum $A = A_T \oplus \operatorname{im} d$. We first show $A_T \cap \operatorname{im} d = 0$. Let $x \in A$. If x has order -1, it is an element of $\mathbf{k}$ so that $d(x) = 0$. If x has order $k \geq 0$, we distinguish the two cases of $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$, then we have $d(x) = (\partial x/\partial u_k)\, u_{k+1} + \tilde{x}$, where all terms of $\tilde{x}$ have order at most k. Hence $d(x) \notin A_T$ and therefore we have $A_T \cap \operatorname{im} d = 0$.

We now turn to the case when $\lambda \neq 0$. By Eq. (1) and an inductive argument, we find that for a product $w = \prod_{i\in I} w_i$ in A, we have

$$d(w) = \sum_{\varnothing \neq J \subseteq I} \lambda^{|J|-1} \prod_{i\in J} d(w_i) \prod_{i\notin J} w_i.$$

Then for a given monomial $u^\alpha = u^{(\alpha_0,\ldots,\alpha_k)} = \prod_{i=0}^k u_i^{\alpha_i}$ of order k we have

$$\begin{aligned} d(u^\alpha) &= \sum_{0\leq\beta_i\leq\alpha_i, \sum_{i=0}^k \beta_i \geq 1} \lambda^{\beta_0+\cdots+\beta_k-1} \prod_{i=0}^k \binom{\alpha_i}{\beta_i} u_i^{\alpha_i-\beta_i} u_{i+1}^{\beta_i} \\ &= \sum_{0\leq\beta_i\leq\alpha_i, \sum_{i=0}^k \beta_i \geq 1} \lambda^{\beta_0+\cdots+\beta_k-1} \left(\prod_{i=0}^k \binom{\alpha_i}{\beta_i} u_i^{\alpha_i-\beta_i+\beta_{i-1}} \right) u_{k+1}^{\beta_k}, \end{aligned} \tag{24}$$

with the convention $\beta_{-1} = 0$. Consider the reverse lexicographic order on monomials of order $k+1$:

$$(\beta_0, \ldots, \beta_{k+1}) < (\gamma_0, \ldots, \gamma_{k+1}) \Leftrightarrow \exists\, 0 \leq n \leq k+1 \ (\beta_i = \gamma_i \text{ for } n < i \leq k+1 \text{ and } \beta_n < \gamma_n).$$

The smallest monomial *of order* $k+1$ under this order in the sum in Eq. (24) is given by $u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} u_k^{\alpha_k-1} u_{k+1}$ when $\beta_k = 1$ and $\beta_0 = \cdots = \beta_{k-1} = 0$, coming from $u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} d(u_k^{\alpha_k})$. Thus for two monomials of order k with $u^\alpha < u^\beta$ under this order, the least monomial of order $k+1$ in $d(u^\alpha)$ is smaller than the least monomial of order $k+1$ in $d(u^\beta)$. In particular, for the least monomial u^α of order k of our given element x of order $k \geq 0$, the least monomial of order $k+1$ in $d(u^\alpha)$ is the least monomial of order $k+1$ in $d(x)$ and is given by $u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} u_k^{\alpha_k-1} u_{k+1}$. Since this monomial is not in A_T, it follows that $d(x)$ is not in A_T, showing that $A_T \cap \operatorname{im} d = 0$.

Note that the previous argument shows in particular that $d(x) \neq 0$ for $x \notin \mathbf{k}$. Thus we have

$$A = A_J \oplus \mathbf{k}.$$

We next show that every monomial u^α in $\mathbf{k}\{u\}$ is in $A_T + \operatorname{im} d$. We prove this by induction on the order of u^α. If the order is -1 or 0, then $u^\alpha \in A_T$ by definition. Assuming the claim holds for differential monomials of order less than $k > 0$, consider now a monomial u^α of order k so that $\alpha = (\alpha_0, \ldots, \alpha_k)$. If $u^\alpha \in A_T$, we are done. If not, we must have $\alpha_k = 1$. Then we distinguish the cases when $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$, then

$$\begin{aligned} u^\alpha &= u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} u_k \\ &= u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} \tfrac{1}{\alpha_{k-1}+1}\, d(u_{k-1}^{\alpha_{k-1}+1}) \\ &= d(u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} \tfrac{1}{\alpha_{k-1}+1}\, u_{k-1}^{\alpha_{k-1}+1}) - d\left(u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}}\right) \tfrac{1}{\alpha_{k-1}+1}\, u_{k-1}^{\alpha_{k-1}+1}. \end{aligned}$$

Now the first term in the result is in $\operatorname{im} d$ and the second term is in $A_T + \operatorname{im} d$ by the induction hypothesis, allowing us to complete the induction when $\lambda = 0$.

Now consider the case when $\lambda \neq 0$. Suppose the claim does not hold for some monomials $u^\alpha = u^{(\alpha_0,\dots,\alpha_{k-1},1)}$ of order k. Among these monomials, there is one such that the exponent vector $\alpha = (\alpha_0, \dots, \alpha_{k-1}, 1)$ is minimal with respect to the lexicographic order:

$$(\alpha_0, \dots, \alpha_{k-1}, 1) < (\beta_0, \dots, \beta_{k-1}, 1) \Leftrightarrow \exists\, 0 \le n \le k-1 \ (\alpha_i = \beta_i \text{ for } 1 \le i < n \text{ and } \alpha_n < \beta_n).$$

By Eq. (24), we have

$$\begin{aligned} d(u_{k-1}^{\alpha_{k-1}+1}) &= \sum_{\beta_{k-1}=1}^{\alpha_{k-1}+1} \binom{\alpha_{k-1}+1}{\beta_{k-1}} \lambda^{\beta_{k-1}-1} u_{k-1}^{\alpha_{k-1}+1-\beta_{k-1}} u_k^{\beta_{k-1}} \\ &= (\alpha_{k-1}+1) u_{k-1}^{\alpha_{k-1}} u_k + \sum_{\beta_{k-1}=2}^{\alpha_{k-1}+1} \binom{\alpha_{k-1}+1}{\beta_{k-1}} \lambda^{\beta_{k-1}-1} u_{k-1}^{\alpha_{k-1}+1-\beta_{k-1}} u_k^{\beta_{k-1}}. \end{aligned}$$

So

$$u_{k-1}^{\alpha_{k-1}} u_k = \tfrac{1}{\alpha_{k-1}+1} d(u_{k-1}^{\alpha_{k-1}+1}) - \sum_{\beta_{k-1}=2}^{\alpha_{k-1}+1} \tfrac{\lambda^{\beta_{k-1}-1}}{\alpha_{k-1}+1} \binom{\alpha_{k-1}+1}{\beta_{k-1}} u_{k-1}^{\alpha_{k-1}+1-\beta_{k-1}} u_k^{\beta_{k-1}}.$$

Thus

$$\begin{aligned} u^\alpha &= u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} u_k \\ &= u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} \tfrac{1}{\alpha_{k-1}+1} d(u_{k-1}^{\alpha_{k-1}+1}) - \sum_{\beta_{k-1}=2}^{\alpha_{k-1}+1} \tfrac{\lambda^{\beta_{k-1}-1}}{\alpha_{k-1}+1} \binom{\alpha_{k-1}+1}{\beta_{k-1}} u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} u_{k-1}^{\alpha_{k-1}+1-\beta_{k-1}} u_k^{\beta_{k-1}}. \end{aligned}$$

The monomials in the sum are in A_T. For the first term, by Eq. (1), we have

$$\begin{aligned} & u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} \tfrac{1}{\alpha_{k-1}+1} d(u_{k-1}^{\alpha_{k-1}+1}) \\ &= d\left(u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}} \tfrac{1}{\alpha_{k-1}+1} u_{k-1}^{\alpha_{k-1}+1}\right) - d(u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}}) \tfrac{1}{\alpha_{k-1}+1} u_{k-1}^{\alpha_{k-1}+1} \\ &\quad - \lambda\, d(u_0^{\alpha_0} \cdots u_{k-2}^{\alpha_{k-2}}) d\left(\tfrac{1}{\alpha_{k-1}+1} u_{k-1}^{\alpha_{k-1}+1}\right). \end{aligned}$$

As in the case of $\lambda = 0$, the first term in the result is in $\operatorname{im} d$ and the second term has the desired decomposition by the induction hypothesis. Applying Eq. (24) to both derivations in the third term, we see that the term is a linear combination of monomials of the form $u^\gamma = u^{(\gamma_0,\dots,\gamma_k)}$ where

$$\gamma = (\alpha_0 - \beta_0, \alpha_1 - \beta_1 + \beta_0, \dots, \alpha_{k-2} - \beta_{k-2} + \beta_{k-3}, \alpha_{k-1} + 1 - \beta_{k-1} + \beta_{k-2}, \beta_{k-1})$$

for some $0 \le \beta_i \le \alpha_i, 0 \le i \le k-2$ with $\sum_{i=0}^{k-2} \beta_i \ge 1$ and $\beta_{k-1} \ge 1$. If such a monomials has $\beta_{k-1} \ge 2$, then the monomial is already in A_T. If such a monomial has $\beta_{k-1} = 1$, then it has order k and has lexicographic order less than u^α since $\sum_{i=0}^{k-2} \beta_i \ge 1$. By the minimality of u^α, this monomial is in $A_T + \operatorname{im} d$. Hence u^α is in $A_T + \operatorname{im} d$. This is a contradiction, allowing us to completes the induction when $\lambda \neq 0$.

With the two direct sum decompositions, the quasi-antiderivative Q is obtained by Proposition 4.2. □

We can thus conclude that $\mathbf{k}\{u\}$ is indeed a regular differential algebra, as claimed earlier. Hence the construction $\mathrm{ID}(\mathbf{k}\{u\})^*$ developed in Section 4.2 does yield the free integro-differential algebra over the single generator u.

Proposition 4.11. *Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. Then the free integro-differential algebra $\mathrm{ID}(\mathbf{k}\{u\})$ is a polynomial algebra.*

Proof. We first take the coefficient ring to be $\mathbb{Q}$. Since $\mathrm{ID}(\mathbb{Q}\{u\})$ is isomorphic to $\mathrm{ID}(\mathbb{Q}\{u\})^*$, which is given by Eq. (17) with $A = \mathbb{Q}\{u\}$, it suffices to ensure that $Ш^+(A_T)$ is a polynomial algebra. Now observe that $A_T = \mathbb{Q}F$ is the monoid algebra generated over the set F of functional monomials. One checks immediately that the functional monomials F form a monoid under multiplication. Hence Theorem 2.3 of [24] is applicable, and we see that the mixable shuffle algebra $Ш^+(A_T) = \mathrm{MS}_{\mathbb{Q},\lambda}(F)$ is isomorphic to $\mathbb{Q}[\mathrm{Lyn}(F)]$, where $\mathrm{Lyn}(F)$ denotes the set of Lyndon words over F. This proves the proposition when $\mathbf{k} = \mathbb{Q}$. Then the conclusion follows for any commutative $\mathbb{Q}$-algebra $\mathbf{k}$ since $\mathrm{ID}(\mathbf{k}\{u\})^* \cong \mathbf{k} \otimes_{\mathbb{Q}} \mathrm{ID}(\mathbb{Q}\{u\})^*$. □

4.4.2. Rational functions

We show that the algebra of rational functions with derivation of any weight is regular.

Proposition 4.12. *Let $A = \mathbb{C}(x)$. For any $\lambda \in \mathbb{C}$ let*

$$d_\lambda : A \to A, f(x) \mapsto \begin{cases} \frac{f(x+\lambda)-f(x)}{\lambda}, & \lambda \neq 0, \\ f'(x), & \lambda = 0, \end{cases}$$

be the λ-derivation introduced in Example 2.2(b). Then d_λ is regular. In particular the difference operator on $\mathbb{C}(x)$ is a regular derivation of weight one.

Proof. We have considered the case of $\lambda = 0$ in Example 4.3. Modifying the notations there, any rational function can be uniquely expressed as

$$r + \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\gamma_{ij}}{(x-\alpha_{ij})^i}, \tag{25}$$

where $r \in \mathbb{C}[x]$, $\alpha_{ij} \in \mathbb{C}$ are distinct for any given i and $\gamma_{ij} \in \mathbb{C}$ are nonzero. Let $0 \neq \lambda \in \mathbb{C}$ be given. We have the direct sum of linear spaces

$$\mathbb{C}[x] \oplus \mathcal{R} = \mathbb{C}[x] \oplus \bigoplus_{i\geq 1} \mathcal{R}_i,$$

where $\mathcal{R}$ is the linear space from the fractions in Eq. (25), namely the linear space with basis $1/(x-\alpha)^i$, $\alpha \in \mathbb{C}$, $1 \leq i$, and $\mathcal{R}_i$, for fixed $i \geq 1$, is the linear subspace with basis $1/(x-\alpha)^i$, $\alpha \in \mathbb{C}$.

We note that the λ-**divided falling factorials**

$$\binom{x}{n}_\lambda := \frac{x(x-\lambda)(x-2\lambda)\cdots(x-(n+1)\lambda)}{n!}, \quad n \geq 0,$$

with the convention $\binom{x}{0}_\lambda = 1$, form a $\mathbb{C}$-basis of $\mathbb{C}[x]$. In fact,

$$\binom{x}{n}_\lambda = \frac{1}{n!}\sum_{k=0}^{n} s(n,k)\lambda^{n-k}x^k, \qquad x^n = n!\sum_{k=0}^{n} S(n,k)\lambda^{n-k}\binom{x}{n}_\lambda, \quad n \geq 0,$$

where $s(n,k)$ and $S(n,k)$ are Stirling numbers of the first and second kind, respectively; see [19,20] for example. By a direct computation, we have

$$d_\lambda\left(\binom{x}{n}_\lambda\right) = \frac{\binom{x+\lambda}{n}_\lambda - \binom{x}{n}_\lambda}{\lambda} = \binom{x}{n-1}_\lambda.$$

Thus $d_\lambda(\mathbb{C}[x]) = \mathbb{C}[x]$ and hence $\mathbb{C}[x] \subseteq \operatorname{im} d_\lambda$. We next note that $\mathcal{R}$, as well as $\mathcal{R}_k$, is also closed under the operator d_λ since

$$\lambda d_\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_{ij})^i}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-(\alpha_{ij}-\lambda))^i} - \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_{ij})^i}.$$

Further, for any $n \geq 0$ and $f(x) \in \mathbb{C}(x)$, we have

$$\lambda d_\lambda\left(\sum_{i=0}^{n} f(x+i\lambda)\right) = f(x+(n+1)\lambda) - f(x),$$

and similarly for $n < 0$,

$$\lambda d_\lambda\left(\sum_{i=n}^{-1} f(x+i\lambda)\right) = f(x) - f(x+n\lambda).$$

Thus for any $n \in \mathbb{Z}$, we have

$$f(x) \equiv f(x+n\lambda) \mod \operatorname{im} d_\lambda.$$

In particular,

$$1/(x-\alpha)^i \equiv 1/(x-(\alpha-n\lambda))^i \mod \operatorname{im} d_\lambda$$

and hence

$$1/(x-\alpha)^i \equiv 1/(x-\beta)^i \mod \operatorname{im} d_\lambda,$$

for some $\beta \in \mathbb{C}$ with the real part $\operatorname{Re}(\beta) \in [0, |\operatorname{Re}(\lambda)|)$. Consequently, any fraction in $\mathcal{R}$ is congruent modulo $\operatorname{im} d_\lambda$ to an element of

$$\mathbb{C}(x)_T := \left\{ \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\gamma_{ij}}{(x-\alpha_{ij})^i} \in \mathcal{R} \,\middle|\, \operatorname{Re}(\alpha_{ij}) \in [0, |\operatorname{Re}(\lambda)|) \right\}.$$

That is,

$$\mathbb{C}(x) = \operatorname{im} d_\lambda + \mathbb{C}(x)_T.$$

On the other hand, suppose there is a nonzero function

$$f(x) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\gamma_{ij}}{(x-\alpha_{ij})^i} \in \operatorname{im} d_\lambda \cap \mathbb{C}(x)_T.$$

Thus there is $g(x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{\gamma_{ij}}{(x-\beta_{ij})^i}$ such that $d_\lambda(g(x)) = f(x)$. The range of i in $f(x)$ and $g(x)$ are the same since $d_\lambda(\mathcal{R}_i) \subseteq \mathcal{R}_i$. Let $f(x) = \sum_{i=1}^{k} f_i(x)$ and $g(x) = \sum_{i=1}^{k} g_i(x)$ be the homogeneous decompositions of f and g. Then $d_\lambda(g_i(x)) = f_i(x)$, $1 \le i \le k$. Fix $1 \le i \le k$ and take $\operatorname{Re}(\lambda) > 0$ for now. List $\beta_{i,1} < \cdots < \beta_{i,m_i}$ according to their lexicographic order from the pairs $(a, b) \leftrightarrow a + i\, b \in \mathbb{C}$. Then we have

$$\lambda d_\lambda(g_i(x)) = \sum_{j=1}^{m_i} \frac{\gamma_{ij}}{(x-(\beta_{ij}-\lambda))^i} - \sum_{j=1}^{m_i} \frac{\gamma_{ij}}{(x-\beta_{ij})^i}.$$

The first fraction in the first sum, $1/(x-(\beta_{i,1}-\lambda))^i$, is not the same as any other fraction in the first sum since they are translations by λ of distinct fractions in f_i, and is not the same as any fraction in the second sum since $\operatorname{Re}(\beta_{i,1}-\lambda) < \operatorname{Re}(\beta_{i,1}) \le \operatorname{Re}(\beta_{ij})$ for $1 \le j \le m_i$. Similarly the last fraction in the second sum, $1/(x-\beta_{i,m_i})^i$, is not the same as any other terms in the sums. Thus they both have nonzero coefficients in $d_\lambda(g_i(x))$. But

$$\operatorname{Re}(\beta_{i,m_i}) - \operatorname{Re}(\beta_{i,1}-\lambda) = \operatorname{Re}(\beta_{i,m_i} - (\beta_{i,1}-\lambda)) = \operatorname{Re}(\beta_{i,m_i} - \beta_{i,1}) + \operatorname{Re}(\lambda) \ge \operatorname{Re}(\lambda).$$

Hence $\operatorname{Re}(\beta_{i,m_i})$ and $\operatorname{Re}(\beta_{i,1}-\lambda)$ cannot both be in $[0, \operatorname{Re}(\lambda))$. Thus $d_\lambda(g_i)$ and hence $d_\lambda(g)$ cannot be in $\mathbb{C}(x)_T$. This is a contradiction, showing that $\operatorname{im} d_\lambda \cap \mathbb{C}(x)_T = 0$. When $\operatorname{Re}(\lambda) < 0$, we get analogously $\operatorname{im} d_\lambda \cap \mathbb{C}(x)_T = 0$. Thus we have proved

$$\mathbb{C}(x) = \operatorname{im} d_\lambda \oplus \mathbb{C}(x)_T. \tag{26}$$

Note that $\mathbb{C}(x)_T$ is closed under multiplication, hence is a nonunitary subalgebra of $\mathbb{C}(x)$.

The above argument shows that $d_\lambda(g)$ is in $\mathbb{C}(x)_T$ for $g \in \mathcal{R}$ only when $g = 0$. Thus $\ker d_\lambda \cap \mathcal{R} = 0$. Since d_λ preserves the decomposition $\mathbb{C}(x) = \mathbb{C}[x] \oplus \mathcal{R}$, we have $\ker d_\lambda = \ker(d_\lambda)\big|_{\mathbb{C}[x]} = \mathbb{C}$. Thus we have the direct sum decomposition

$$\mathbb{C}(x) = \ker d_\lambda \oplus (x\mathbb{C}[x] \oplus \mathcal{R}),$$

and hence d_λ is injective on $x\mathbb{C}[x] \oplus \mathcal{R}$ with image $\operatorname{im} d_\lambda$. Therefore d_λ is regular with quasi-antiderivative Q defined to be the inverse of

$$d_\lambda : x\mathbb{C}[x] \oplus \mathcal{R} \to \operatorname{im} d_\lambda$$

on $\operatorname{im} d_\lambda$ and to be zero on its complement $\mathbb{C}(x)_T$; see Proposition 4.2. □

Remark 4.13. We remark that the subalgebra of $\mathbb{C}(x)$ that is a complement of $\operatorname{im} d_\lambda$ is not unique, thus giving different quasi-antiderivatives. In fact, from the proof of Proposition 4.12 it is apparent that in the decomposition (26) one can replace $\mathbb{C}(x)_T$ by

$$\mathbb{C}(x)_{T,a} = \left\{ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\gamma_{ij}}{(x-\alpha_{ij})^i} \in \mathcal{R} \,\middle|\, \operatorname{Re}(\alpha_i) \in [a, a + |\operatorname{Re}(\lambda)|) \right\},$$

for any given $a \in \mathbb{R}$. These two subalgebras are isomorphic since $\mathbb{C}(x)_{T,a}$ is isomorphic to the polynomial $\mathbb{C}$-algebra with generating set

$$\left\{ \frac{1}{x-\alpha} \,\middle|\, \alpha \in [a, a + |\operatorname{Re}(\lambda)|) \right\}.$$

Remark 4.14. In conclusion, we have given the first construction for the free integro-differential algebra $\operatorname{ID}(A)^*$ over a given regular differential algebra A. In several ways, this construction is similar to the integro-differential polynomials of [36,38]. This will be clear when one writes out the elements $a_0 \otimes a_1 \otimes a_2 \otimes \cdots$ of Eq. (16) in the form $a_0 \int a_1 \int a_2 \int \cdots$. But there are also some important differences:

(a) The integro-differential polynomials are the polynomial algebra in the variety of integro-differential algebras of weight zero, not the free algebra in this category. In fact, the polynomial algebra is always a free product of the coefficient algebra and the free algebra by Theorem 4.31 of [30].
(b) The construction of [36] uses the language of term algebras and rewrite systems whereas in this paper we use a more abstract approach through tensor products.

(c) In the integro-differential polynomials, the starting point is a given integro-differential algebra (A, D, Π) instead of a regular differential algebra as in the present paper. In the former case we can construct nested integrals over differential polynomials with coefficients in A, whereas in the latter case we can only treat differential polynomials with trivial coefficients (i.e., the derivation vanishes on them).

It would be interesting to apply the methods used in this paper to rederive and generalize the construction of the integro-differential polynomials of [36]. This would also shed some light on the constructive meaning of the free product mentioned in Item (a) above. An important step in this direction might be generalizing Section 4.4.1 to differential polynomials with nonzero derivation on the coefficient ring $\mathbf{k}$. See [16] for a construction of the free integro-differential algebra on an arbitrary set by the method of Gröbner–Shirshov bases.

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