On the bijectivity of families of exponential/generalized polynomial maps

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Abstract

We start from a parametrized system of d generalized polynomial equations (with real exponents) for d positive variables, involving n generalized monomials with n positive parameters. Existence and uniqueness of a solution for all parameters (and for all right-hand sides) is equivalent to the bijectivity of a family of generalized polynomial/exponential maps.

We characterize the bijectivity of the family of exponential maps in terms of two linear subspaces arising from the coefficient and exponent matrices, respectively. In particular, we obtain conditions in terms of sign vectors of the two subspaces and a nondegeneracy condition involving the exponent subspace itself. Thereby, all criteria can be checked effectively.

Moreover, we characterize when the existence of a unique solution is robust with respect to small perturbations of the exponents or/and the coefficients. In particular, we obtain conditions in terms of sign vectors of the linear subspaces or, alternatively, in terms of maximal minors of the coefficient and exponent matrices.

Keywords: Birch's theorem, global invertibility, Hadamard's theorem, Descartes' rule, sign vectors, oriented matroids, perturbations, robustness

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1 Introduction

Given two matrices $W = (w^1, \ldots, w^n)$, $\tilde{W} = (\tilde{w}^1, \ldots, \tilde{w}^n) \in \mathbb{R}^{d \times n}$ with $d \leq n$ and full rank, consider the parametrized system of generalized polynomial equations

$$\sum_{j=1}^{n} w_{ij}\, c_j\, x_1^{\tilde{w}_{1j}} \cdots x_d^{\tilde{w}_{dj}} = y_i, \quad i = 1, \ldots, d$$

for d positive variables $x_i > 0$ (and right-hand sides y_i), involving the 'monomials' $c_j\, x_1^{\tilde{w}_{1j}} \cdots x_d^{\tilde{w}_{dj}} = c_j\, x^{\tilde{w}^j}$, $j = 1, \ldots, n$, in particular, the n positive pa-

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rameters $c_j > 0$. In other words, $x \in \mathbb{R}^d_{>0}$, $y \in \mathbb{R}^d$, and $c \in \mathbb{R}^n_{>0}$. As in the theory of fewnomials [15, 23], the monomials are given, however, with a positive parameter associated to every monomial.

Writing the vector of monomials as $c \circ x^{\tilde{W}} \in \mathbb{R}^n_{>0}$, thereby introducing $x^{\tilde{W}} \in \mathbb{R}^n_{>0}$ as $(x^{\tilde{W}})_j = x^{\tilde{w}^j}$ and denoting componentwise multiplication by $\circ$, yields the compact form

$$W(c \circ x^{\tilde{W}}) = y.$$

Note that, for the existence of a positive solution x, the right-hand side y must lie in the interior of $C = \operatorname{cone} W$, the polyhedral cone generated by the columns of W. The question arises whether the above equation system has a unique positive solution $x \in \mathbb{R}^d_{>0}$, for all right-hand sides $y \in C^\circ \subseteq \mathbb{R}^d$ and all positive parameters $c \in \mathbb{R}^n_{>0}$. This question is equivalent to whether the generalized polynomial map $f_c \colon \mathbb{R}^d_{>0} \to C^\circ \subseteq \mathbb{R}^d$,

$$f_c(x) = W(c \circ x^{\tilde{W}})$$

or, equivalently, the exponential map $F_c \colon \mathbb{R}^d \to C^\circ \subseteq \mathbb{R}^d$,

$$F_c(x) = W(c \circ \mathrm{e}^{\tilde{W}^\mathsf{T} x})$$

is bijective for all $c \in \mathbb{R}^n_{>0}$.

In the context of *generalized* chemical reaction networks [18, 19], the question is equivalent to whether every set of complex-balanced equilibria (an 'exponential manifold') intersects every stoichiometric compatibility class (an affine subspace) in exactly one point. For a motivation from Chemical Reaction Network Theory, see Appendix A or [9]. In the context of *classical* chemical reaction networks, the assumption of mass-action kinetics implies $W = \tilde{W}$, and in this case there is indeed exactly one complex-balanced equilibrium in every stoichiometric compatibility class.

In case $W = \tilde{W}$, the map F_c also appears in toric geometry [11], where it is related to moment maps, and in statistics [20], where it is related to log-linear models. The following result guarantees the bijectivity of F_c for all $c > 0$. It is a variant of Birch's Theorem [25, 20, 6].

Theorem 1 ([11], Section 4.2)**.** *Let $W = \tilde{W}$. Then the map F_c is a real analytic isomorphism of $\mathbb{R}^d$ onto C° for all $c > 0$.*

In this work, we characterize the bijectivity of the map F_c for all $c > 0$ (for given coefficients W and exponents $\tilde{W}$) in terms of (sign vectors of) the linear subspaces $S = \ker W \subseteq \mathbb{R}^n$ and $\tilde{S} = \ker \tilde{W} \subseteq \mathbb{R}^n$. Thereby we extend previous results, in particular, sufficient conditions for bijectivity [18, 17, 9]. Moreover, we characterize the robustness of bijectivity with respect to small perturbations of the exponents $\tilde{W}$ or/and the coefficients W, corresponding to small perturbations of the subspaces $\tilde{S}$ and S (in the Grassmannian).

Our main technical tool is Hadamard's global inversion theorem which essentially states that a C^1-map is a diffeomorphism if and only if it is locally invertible and proper. By previous results [8, 18], the map F_c is locally invertible for all $c > 0$ if and only if it is injective for all $c > 0$ which can be characterized in terms of sign vectors of the subspaces S and $\tilde{S}$. Most importantly, we show that F_c is proper if and only if it is 'proper along rays' and that properness for all $c > 0$ can be characterized in terms of sign vectors of S and $\tilde{S}$, together with a nondegeneracy condition depending on the subspace $\tilde{S}$ itself.

The crucial role of sign vectors in the characterization of existence and uniqueness of positive solutions to parametrized polynomial equations suggests a comparison with Descartes' rule of signs for univariate polynomials. Consider a univariate polynomial and order the monomials by their exponents. Now, let s be the number of sign changes in the sequence of (nonzero) coefficients, and let p be the number of positive roots (where multiple roots are counted separately). Then, Descartes' rule [24] states that $p \leq s$ and $s - p$ is even. As shown by Laguerre [16, 14] the same statement holds for generalized monomials (with real exponents). More recently it has been shown that the upper bound is sharp [1]: for given sign sequence, there exist coefficients such that $p = s$. Hence a sharp Descartes' rule states that a univariate polynomial has exactly one positive solution for all coefficients with given signs if and only if there is exactly one sign change. Indeed, this statement follows from our main result (for univariate polynomials). Hence our main result can be seen as a multivariate generalization of the sharp Descartes' rule for exactly one positive solution.

Organization of the work

In Section 2, we introduce the familiy of exponential maps F_c with $c > 0$ and discuss previous results on injectivity.

In Section 3, we present our main result, Theorem 13, characterizing the bijectivity of the family F_c, and the crucial Lemmas 11 and 16, regarding the properness of F_c. In Subsection 3.1, we discuss two extreme cases regarding the geometry of $C = \operatorname{cone} W$, the polyhedral cone generated by the columns of W. Namely, $C = \mathbb{R}^d$ or C is pointed. In the latter case, we present necessary conditions for the surjectivity of F_c. In Subsection 3.2, we show that the bijectivity of the family F_c cannot be characterized in terms of sign vectors only, cf. Example 21. Still, there are sufficient conditions for bijectivity in terms of sign vectors or in terms of faces of the Newton polytope.

In Section 4, we study the robustness of bijectivity. In Subsection 4.1, we consider perturbations of the exponents $\tilde{W}$ and show that robustness of bijectivity is equivalent to robustness of injectivity which can be characterized in terms of sign vectors, cf. Theorem 32. The criterion involves the closure of a set of sign vectors and represents another sufficient condition for bijectivity. In Subsection 4.2, we consider perturbations of the coefficients W and characterize robustness of bijectivity again in terms of sign vectors (including another closure condition), cf. Theorem 38. In particular, robustness of bijectivity implies that either $C = \mathbb{R}^d$ or C is pointed. In the latter case, the faces of C are minimally generated. Finally, in Subsection 4.3, we express the closure condition in terms of maximal minors of W and $\tilde{W}$. Further, we consider general perturbations (of both exponents and coefficients) and characterize robustness of bijectivity in terms of sign vectors and maximal minors, cf. Theorem 42.

Finally, we provide appendices on (A) a motivation from Chemical Reaction Network Theory, (B) oriented matroids, and (C) a theorem of the alternative.

Notation

We denote the positive real numbers by $\mathbb{R}_{>0}$ and the nonnegative real numbers by $\mathbb{R}_{\geq 0}$. We write $x > 0$ for $x \in \mathbb{R}^n_{>0}$ and $x \geq 0$ for $x \in \mathbb{R}^n_{\geq 0}$. For vectors $x, y \in \mathbb{R}^n$, we denote their scalar product by $x \cdot y$ and their componentwise (Hadamard) product by $x \circ y$.

For a vector $x \in \mathbb{R}^n$, we obtain the sign vector $\operatorname{sign}(x) \in \{-, 0, +\}^n$ by applying the sign function componentwise, and we write

$$\operatorname{sign}(S) = \{\operatorname{sign}(x) \mid x \in S\}$$

for a subset $S \subseteq \mathbb{R}^n$.

For a vector $x \in F^n$ with $F = \mathbb{R}$ or $F = \{-, 0, +\}$, we denote its support by $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. For a subset $X \subseteq F^n$, we say that a nonzero vector $x \in X$ has (inclusion-)minimal support, if $\operatorname{supp}(x') \subseteq \operatorname{supp}(x)$ implies $\operatorname{supp}(x') = \operatorname{supp}(x)$ for all nonzero $x' \in X$.

For a sign vector $\tau \in \{-, 0, +\}^n$, we introduce

$$\tau^- = \{i \mid \tau_i = -\}, \quad \tau^0 = \{i \mid \tau_i = 0\}, \quad \text{and} \quad \tau^+ = \{i \mid \tau_i = +\}.$$

In particular, $\operatorname{supp}(\tau) = \tau^- \cup \tau^+$. For a subset $\Sigma \subseteq \{-, 0, +\}^n$, we write

$$\Sigma_\oplus = \Sigma \cap \{0, +\}^n.$$

The inequalities $0 < -$ and $0 < +$ induce a partial order on $\{-, 0, +\}^n$: for sign vectors $\tau, \rho \in \{-, 0, +\}^n$, we write $\tau \leq \rho$ if the inequality holds componentwise. The product on $\{-, 0, +\}$ is defined in the obvious way. For $\tau, \rho \in \{-, 0, +\}^n$, we write $\tau \cdot \rho = 0$ (τ and ρ are orthogonal) if either $\tau_i \rho_i = 0$ for all i or there exist i, j with $\tau_i \rho_i = -$ and $\tau_j \rho_j = +$. For a set $\Sigma \subseteq \{-, 0, +\}^n$, we introduce the orthogonal complement

$$\Sigma^\perp = \{\tau \in \{-, 0, +\}^n \mid \tau \cdot \rho = 0 \text{ for all } \rho \in \Sigma\}.$$

Moreover, for $\tau, \rho \in \{-, 0, +\}^n$, we define the composition $\tau \circ \rho \in \{-, 0, +\}^n$ as $(\tau \circ \rho)_i = \tau_i$ if $\tau_i \neq 0$ and $(\tau \circ \rho)_i = \rho_i$ otherwise.

For a matrix $W \in \mathbb{R}^{d \times n}$, we denote its column vectors by $w^1, \ldots, w^n \in \mathbb{R}^d$. For any natural number n, we define $[n] = \{1, \ldots, n\}$. For $W \in \mathbb{R}^{d \times n}$ with $d \leq n$ and $I \subseteq [n]$ of cardinality d, we denote the square submatrix of W with column indices in I by W_I.

2 Families of exponential maps

Let $W \in \mathbb{R}^{d \times n}$, $\tilde{W} \in \mathbb{R}^{\tilde{d} \times n}$ be matrices with $d, \tilde{d} \leq n$ and full rank. Further, let

$$C = \operatorname{cone} W \subseteq \mathbb{R}^d$$

be the cone generated by the columns of W. Since W has full rank, the cone C has nonempty interior C°. Finally, let $c > 0$. We define the exponential map

$$\begin{aligned} F_c \colon \mathbb{R}^{\tilde{d}} &\to C^\circ \subseteq \mathbb{R}^d \\ x &\mapsto W(c \circ \mathrm{e}^{\tilde{W}^\mathsf{T} x}) = \sum_{i=1}^n c_i \, \mathrm{e}^{\tilde{w}^i \cdot x} \, w^i \end{aligned} \tag{1}$$

and the related subspaces

$$S = \ker W \subseteq \mathbb{R}^n \quad \text{and} \quad \tilde{S} = \ker \tilde{W} \subseteq \mathbb{R}^n. \tag{2}$$

Note that injectivity and surjectivity of F_c only depend on S and $\tilde{S}$. In fact, let $V \in \mathbb{R}^{d \times n}$, $\tilde{V} \in \mathbb{R}^{\tilde{d} \times n}$ be such that $\ker V = S$, $\ker \tilde{V} = \tilde{S}$, and let

$$G_c(x) = V(c \circ \mathrm{e}^{\tilde{V}^\mathsf{T} x})$$

be the corresponding exponential map. Then $V = UW$, $\tilde{V} = \tilde{U}\tilde{W}$ for invertible matrices $U \in \mathbb{R}^{d \times d}$, $\tilde{U} \in \mathbb{R}^{\tilde{d} \times \tilde{d}}$, and

$$G_c(x) = U F_c(\tilde{U}^\mathsf{T} x).$$

2.1 Previous results on injectivity

In the context of multiple equilibria in mass-action systems [7] and geometric modeling [8], where $d = \tilde{d}$, it was shown that the map F_c is injective for all $c > 0$ if and only if F_c is a local diffeomorphism for all $c > 0$.

Theorem 2 (Theorem 7 and Corollary 8 in [8]). *Let F_c be as in* (1) *with $d = \tilde{d}$. Then the following statements are equivalent:*

1. *F_c is injective for all $c > 0$.*
2. *$\det(\frac{\partial F_c}{\partial x}) \neq 0$ for all x and all $c > 0$.*
3. *$\det(W_I)\det(\tilde{W}_I) \geq 0$ for all subsets $I \subseteq [n]$ of cardinality d (or '≤ 0' for all I) and $\det(W_I)\det(\tilde{W}_I) \neq 0$ for some I.*

In [18], we gave an alternative proof of this result and extended it to the case $d \neq \tilde{d}$, by using the sign vectors of the subspaces S and $\tilde{S}$.

Theorem 3 (Theorem 3.6 in [18]). *Let F_c be as in* (1) *and $S, \tilde{S}$ be as in* (2). *Then the following statements are equivalent:*

1. *F_c is injective for all $c > 0$.*
2. *F_c is an immersion for all $c > 0$.*
   ($\frac{\partial F_c}{\partial x}$ is injective for all x and all $c > 0$.)
3. *$\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\}$.*

Theorems 2 and 3 characterize the injectivity of F_c with $d = \tilde{d}$ for all $c > 0$ equivalently in terms of maximal minors and sign vectors.

Corollary 4. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$ of dimension $n - d$ (with $d \leq n$). For every $W, \tilde{W} \in \mathbb{R}^{d \times n}$ (with full rank d) such that $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$, the following statements are equivalent.*

1. *$\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\}$.*

2. $\det(W_I)\det(\tilde{W}_I) \geq 0$ *for all subsets* $I \subseteq [n]$ *of cardinality* d *(or '*≤ 0*' for all* I*) and* $\det(W_I)\det(\tilde{W}_I) \neq 0$ *for some* I.

In the language of oriented matroids, Corollary 4 relates *chirotopes* (maximal minors of W and $\tilde{W}$) to *vectors* (sign vectors of $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$), see also Appendix B. Thereby, the sign vector condition is symmetric with respect to S and $\tilde{S}$.

Corollary 5 (Corollary 3.8 in [18])**.** *Let* $S, \tilde{S}$ *be subspaces of* $\mathbb{R}^n$ *of equal dimension. Then*

$$\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\} \quad \textit{if and only if} \quad \operatorname{sign}(\tilde{S}) \cap \operatorname{sign}(S^\perp) = \{0\}.$$

See also [5] for a direct proof of Corollaries 4 and 5.

3 Bijectivity

A necessary condition for the bijectivity of the map F_c is $d = \tilde{d}$. In the rest of the paper, we consider F_c as in (1) with $d = \tilde{d}$ and the related subspaces $S, \tilde{S}$ as in (2).

A first sufficient condition for the bijectivity of the map F_c for all $c > 0$ (in terms of sign vectors of S and $\tilde{S}$) was given in [18], thereby extending Theorem 1 (Birch's Theorem).

Theorem 6 (Proposition 3.9 in [18])**.** *If* $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$ *and* $(+, \dots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$*, then the map* F_c *is a real analytic isomorphism for all* $c > 0$.

As it will turn out, $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$ is sufficient for bijectivity, and the technical condition $(+, \dots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$ in [18] is not needed, cf. Corollary 15. We note that Theorems 2, 3, and 6 allowed a first multivariate generalization of Descartes' rule of signs for at most/exactly one positive solution, see [17].

In order to characterize the bijectivity of the map F_c for all $c > 0$, we start with the following observation.

Proposition 7. *The following statements are equivalent.*

1. F_c *is bijective for all* $c > 0$.
2. F_c *is a diffeomorphism for all* $c > 0$.
3. F_c *is a real analytic isomorphism for all* $c > 0$.

Proof. Let F_c be bijective for all $c > 0$. In particular, it is injective, and $\det(\frac{\partial F_c}{\partial x}) \neq 0$ for all x and $c > 0$, by Theorems 2 or 3. Hence, F_c is a local diffeomorphism for all $c > 0$. Further, F_c is real analytic and hence a local real analytic isomorphism for all $c > 0$. □

Most importantly, we will use Hadamard's global inversion theorem [13].

Theorem 8. *A* C^1*-map* $F \colon \mathbb{R}^d \to \mathbb{R}^d$ *is a diffeomorphism if and only if the Jacobian* $\det(\frac{\partial F}{\partial x}) \neq 0$ *for all* $x \in \mathbb{R}^d$ *and* $|F(x)| \to \infty$ *whenever* $|x| \to \infty$.

Obviously, we need a slightly more general version of this result that follows from the general invertibility theorem in [3], see also [12].

Theorem 9. *Let $U \subseteq \mathbb{R}^d$ be open and convex. A C^1-map $F\colon \mathbb{R}^d \to U$ is a diffeomorphism if and only if the Jacobian $\det(\frac{\partial F}{\partial x}) \neq 0$ for all $x \in \mathbb{R}^d$ and F is proper.*

Recall that a map F is *proper*, if $F^{-1}(K)$ is compact for each compact subset K of U. This is obviously necessary for the inverse F^{-1} to be continuous.

Lemma 10. *Let $U \subseteq \mathbb{R}^d$ be open. A continuous map $F\colon \mathbb{R}^d \to U$ is proper if and only if, for sequences x_n in $\mathbb{R}^d$ with $|x_n| = 1$ and $x_n \to x$ and t_n in $\mathbb{R}_{>0}$ with $t_n \to \infty$, $F(x_n t_n) \to y$ implies $y \in \partial U$.*

Proof. Suppose F is proper and $F(x_n t_n) \to y$, but $y \in U$. Take a closed ball $K \subseteq U$ around y. Then $F^{-1}(K)$ contains the unbounded sequence $x_n t_n$ and hence is not compact, a contradiction.

Conversely, let K be a compact subset of U. We need to show that every sequence X_n in $F^{-1}(K)$ has an accumulation point. Since $F^{-1}(K)$ is closed, we only need to show that X_n has a bounded subsequence. Suppose not, then $|X_n| \to \infty$. Since $F(X_n) \in K$, there is a subsequence (call it X_n again) such that $F(X_n) \to y \in K$. Now there is another subsequence (call it X_n again) such that $x_n = X_n/|X_n| \to x$, that is, the sequence x_n on the unit sphere converges. With $t_n = |X_n|$, we have $F(x_n t_n) \to y \in K \subset U$, a contradiction. □

In particular, if F is proper, then, for all nonzero $x \in \mathbb{R}^d$, $F(xt) \to y$ as $t \to \infty$ implies $y \in \partial U$. That is, if the function values converge along a ray, then the limit lies on the boundary of the range.

By Lemma 11 below, the map F_c under consideration is proper, if it is 'proper along rays'. Before we prove this result, we discuss the behaviour of F_c along a ray. For $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, we introduce

$$I_{x,\lambda} = \{i \mid \tilde{w}^i \cdot x = \lambda\}$$

and write

$$F_c(xt) = \sum_\lambda \sum_{i \in I_{x,\lambda}} c_i \, \mathrm{e}^{\lambda t} \, w^i .$$

Now, for nonzero $x \in \mathbb{R}^d$, either $|F_c(xt)| \to \infty$ as $t \to \infty$ or $F_c(xt) \to y \in C$. In the first case, there is $\lambda > 0$ such that

$$F_c(xt) \, \mathrm{e}^{-\lambda t} \to \sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$$

as $t \to \infty$. In the second case, $\sum_{i \in I_{x,\lambda}} c_i w^i = 0$ for all $\lambda > 0$ and

$$F_c(xt) \to \sum_{i \in I_{x,0}} c_i w^i \in C.$$

If $I_{x,0} = \emptyset$, then $F_c(xt) \to 0$.

Lemma 11. *The map F_c is proper, if*

$$F_c(xt) \to y \quad \text{as} \quad t \to \infty \quad \text{implies} \quad y \in \partial C \tag{$*$}$$

for all nonzero $x \in \mathbb{R}^d$.

Proof. We assume that the ray condition $(*)$ holds for all nonzero $x \in \mathbb{R}^d$.

Let $x \in \mathbb{R}^d$ with $|x| = 1$. In order to apply Lemma 10, we consider sequences x_n in $\mathbb{R}^d$ with $|x_n| = 1$ and $x_n \to x$ and t_n in $\mathbb{R}_{>0}$ with $t_n \to \infty$.

To begin with, we show that $|F_c(xt)| \to \infty$ as $t \to \infty$ implies $|F_c(x_n t_n)| \to \infty$ as $n \to \infty$. Suppose $|F_c(xt)| \to \infty$, that is, there is $\lambda > 0$ such that $F_c(xt)\,\mathrm{e}^{-\lambda t} \to \sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$ as $t \to \infty$. For x' close to x, we have the partition

$$I_{x,\lambda} = I_{x',\mu_1} \cup \dots \cup I_{x',\mu_p}$$

with μ_j close to λ and hence $\mu_j > \frac{\lambda}{2}$. Most importantly, there exists a largest μ_j such that $\sum_{i \in I_{x',\mu_j}} c_i w^i \neq 0$. Otherwise,

$$\sum_{i \in I_{x,\lambda}} c_i w^i = \sum_{i \in I_{x',\mu_1}} c_i w^i + \ldots + \sum_{i \in I_{x',\mu_p}} c_i w^i = 0.$$

Additionally, there may exist an even larger μ with $\sum_{i \in I_{x',\mu}} c_i w^i \neq 0$. In any case, there is $\lambda' > \frac{\lambda}{2}$ such that

$$F_c(x't)\,\mathrm{e}^{-\lambda' t} \to \sum_{i \in I_{x',\lambda'}} c_i w^i \neq 0$$

as $t \to \infty$ and hence $|F_c(x't)|\,\mathrm{e}^{-\frac{\lambda}{2} t} > \gamma$ with $\gamma > 0$ independent of x'; that is, $|F_c(x't)| > \gamma\,\mathrm{e}^{\frac{\lambda}{2} t}$ as $t \to \infty$. Hence $|F_c(x_n t_n)| > \gamma\,\mathrm{e}^{\frac{\lambda}{2} t_n}$ as $n \to \infty$; that is, $|F_c(x_n t_n)| \to \infty$, as claimed.

In case $C = \mathbb{R}^d$ ($\partial C = \emptyset$), the ray condition $(*)$ implies $|F_c(xt)| \to \infty$ as $t \to \infty$ and hence $|F_c(x_n t_n)| \to \infty$ as $n \to \infty$. By Lemma 10, F_c is proper.

In case $C \neq \mathbb{R}^d$, assume $F_c(x_n t_n) \to y'$ as $n \to \infty$. Then, $F_c(xt) \to y$ as $t \to \infty$, by the argument above. In particular, $\sum_{i \in I_{x,\lambda}} c_i w^i = 0$ for all $\lambda > 0$ and $y = \sum_{i \in I_{x,0}} c_i w^i$. The vectors w^i with $i \in I_{x,\lambda}$ and $\lambda > 0$ lie in the lineality space of C, and hence

$$\operatorname{cone}(w^i \mid i \in I_{x,\lambda} \text{ with } \lambda > 0) \subseteq \partial C.$$

By the ray condition $(*)$, $y \in \partial C$, and hence

$$\operatorname{cone}(w^i \mid i \in I_{x,0}) \subseteq \partial C.$$

Finally, we write

$$F_c(x_n t_n) = \sum_{i=1}^{n} c_i\,\mathrm{e}^{\tilde{w}^i \cdot x_n\, t_n}\, w^i = \sum_{\lambda} \sum_{i \in I_{x,\lambda}} c_i\,\mathrm{e}^{\tilde{w}^i \cdot x_n\, t_n}\, w^i.$$

For x_n close to x, we have $\tilde{w}^i \cdot x_n$ close to λ for $i \in I_{x,\lambda}$, in particular, $\sum_{i \in I_{x,\lambda}} c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i \to 0$ for $\lambda < 0$. The limit $F_c(x_n t_n) \to y'$ as $n \to \infty$ implies

$$\sum_{\lambda \geq 0} \sum_{i \in I_{x,\lambda}} c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i \to y',$$

and hence $y' \in \partial C$. By Lemma 10, F_c is proper. $\square$

Let $F_c(xt) \to y$ as $t \to \infty$ along the ray given by x and $F_c(x_n t_n) \to y'$ as $n \to \infty$ for a sequence $x_n t_n$ (with $x_n \to x$ and $t_n \to \infty$), approaching the ray. In the proof of Lemma 11, we have shown that, if $y = 0$, then $y' \in L$, where L is the lineality space of C. In general, if $y \in C_x = \mathrm{cone}(w^i \mid i \in I_{x,0})$, then $y' \in C_x + L$. Note that there are only finitely many index sets $I_{x,0}$ and hence finitely many limit points $y = \sum_{i \in I_{x,0}} c_i w^i$ (for fixed $c > 0$), whereas every $y' \in \partial C$ arises as a limit point (if F_c is surjective).

Using Theorem 9 (Hadamard's global inversion theorem) together with Theorems 2 or 3 and Lemma 11, we summarize our findings.

Corollary 12. *The map F_c is bijective for all $c > 0$ if and only if F_c is injective for all $c > 0$ and the ray condition $(*)$ holds for all nonzero $x \in \mathbb{R}^d$ and all $c > 0$.*

By Theorems 2 or 3, the injectivity of F_c (for all $c > 0$) can be characterized in terms of sign vectors of the subspaces S and $\tilde{S}$. By Lemma 16 below, the ray condition $(*)$ (for all nonzero $x \in \mathbb{R}^d$ and all $c > 0$) can be characterized in terms of sign vectors of S and $\tilde{S}$ together with a nondegeneracy condition depending on sign vectors of S and on the subspace $\tilde{S}$ itself.

Hence, as our main result, we characterize the bijectivity of F_c (for all $c > 0$) in terms of the subspaces S and $\tilde{S}$.

Theorem 13. *The map F_c is a diffeomorphism for all $c > 0$ if and only if*

(i) $\mathrm{sign}(S) \cap \mathrm{sign}(\tilde{S}^\perp) = \{0\}$,

(ii) for every nonzero $\tilde{\tau} \in \mathrm{sign}(\tilde{S}^\perp)_\oplus$, there is a nonzero $\tau \in \mathrm{sign}(S^\perp)_\oplus$ such that $\tau \leq \tilde{\tau}$, and

(iii) the pair $(S, \tilde{S})$ is nondegenerate.

To complete the statement, we have to define nondegeneracy.

Definition 14. Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. A vector $z \in \tilde{S}^\perp$ with a positive component is called *nondegenerate* if

- there is (a nonzero) $\tau \in \{0, +\}^n$ with $\tau^+ = \{i \mid z_i = \lambda\}$ for some $\lambda > 0$ such that $\tau \notin \mathrm{sign}(S)_\oplus$ or
- for $\tilde{\tau} = \mathrm{sign}(z) \in \mathrm{sign}(\tilde{S}^\perp)$, there is a nonzero $\tau \in \mathrm{sign}(S^\perp)_\oplus$ such that $\tilde{\tau}^0 \subseteq \tau^0$.

The pair $(S, \tilde{S})$ is called nondegenerate if every $z \in \tilde{S}^\perp$ with a positive component is nondegenerate.

First, we note that Theorem 13 immediately implies Theorems 1 and 6 (Birch's Theorem and its first extension).

Corollary 15. *The map F_c is a diffeomorphism for all $c > 0$ if* $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$.

Proof. Note that $\operatorname{sign}(S^\perp) = \operatorname{sign}(S)^\perp$, cf. [26, Prop. 6.8]. Hence, $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$ implies conditions (i) and (ii) in Theorem 13. Now, for $z \in \tilde{S}^\perp$ with a positive component $z_i = \lambda > 0$, consider $\tau \in \{0,+\}^n$ with $\tau^+ = \{i \mid z_i = \lambda\}$ and $\tilde{\tau} = \operatorname{sign}(z) \in \operatorname{sign}(\tilde{S}^\perp)$. Obviously, $\tau \cdot \tilde{\tau} \neq 0$, that is, $\tau \notin \operatorname{sign}(\tilde{S}) = \operatorname{sign}(S)$, and z is nondegenerate, as required by condition (iii). □

Second, we note that condition (i) in Theorem 13 can also be characterized in terms of maximal minors of the matrices W and $\tilde{W}$, cf. Corollary 4. Moreover, condition (ii) can be reformulated using faces of the cones $C = \operatorname{cone} W$ and $\tilde{C} = \operatorname{cone} \tilde{W}$:

(ii) for every proper face $\tilde{f}$ of $\tilde{C}$ with $\tilde{I} = \{i \mid \tilde{w}^i \in \tilde{f}\}$, there is a proper face f of C with $I = \{i \mid w^i \in f\}$ such that $\tilde{I} \subseteq I$.

Indeed, a face f of C with $I = \{i \mid w^i \in f\}$ corresponds to a supporting hyperplane with normal vector x such that $w^i \cdot x = 0$ for $i \in I$ and $w^i \cdot x > 0$ otherwise (for w^i lying on the positive side of the hyperplane). Hence f is characterized by the nonnegative sign vector $\tau = \operatorname{sign}(W^\mathsf{T} x) \in \operatorname{sign}(S^\perp)_\oplus$ with $\tau^0 = I$. Analogously, a face $\tilde{f}$ of $\tilde{C}$ with $\tilde{I} = \{i \mid \tilde{w}^i \in \tilde{f}\}$ is characterized by a nonnegative sign vector $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with $\tilde{\tau}^0 = \tilde{I}$. Clearly, $\tilde{I} \subseteq I$ is equivalent to $\tau \leq \tilde{\tau}$. (For more details on sign vectors and face lattices, see Appendix B.)

Third, before we prove Lemma 16 below, we discuss how the ray condition $(*)$ implies conditions (ii) and (iii).

Let $x \in \mathbb{R}^d$ be nonzero, and assume that the ray condition $(*)$ holds for all $c > 0$. Then, for all $c > 0$, either there is $\lambda > 0$ such that

$$F_c(xt)\, \mathrm{e}^{-\lambda t} \to \sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$$

as $t \to \infty$ or

$$F_c(xt) \to \sum_{i \in I_{x,0}} c_i w^i \in \partial C.$$

Note that the sets $I_{x,\lambda}$ are disjoint, and the sums $\sum_{i \in I_{x,\lambda}} c_i w^i$ involve different coefficients c_i for different λ. Hence,

(a) there is $\lambda > 0$ such that $\sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$ for all $c > 0$ or

(b) $\sum_{i \in I_{x,0}} c_i w^i \in \partial C$ for all $c > 0$.

To see this, assume $\neg$(a), that is, there exists $c > 0$ such that $\sum_{i \in I_{x,\lambda}} c_i w^i = 0$ for all $\lambda > 0$. Then, $\sum_{i \in I_{x,0}} c_i w^i \in \partial C$ for all $c > 0$, that is, (b).

Now, let $\lambda' = \max_i \tilde{w}^i \cdot x$.

If $\lambda' \leq 0$, then $\tilde{f} = \text{cone}(\tilde{w}^i \mid i \in I_{x,0})$ defines a proper face of $\tilde{C}$ with index set $\tilde{I} = I_{x,0}$. Indeed, $\tilde{w}^i \cdot x = 0$ for $i \in I_{x,0}$ and $\tilde{w}^i \cdot x < 0$ otherwise. Condition (b) implies that (the interior of) the $\text{cone}(w^i \mid i \in I_{x,0})$ lies in a proper face $f = \text{cone}(w^i \mid i \in I)$ of C with index set I. That is, $\tilde{I} = I_{x,0} \subseteq I$, as required by condition (ii).

If $\lambda' > 0$, let $z = \tilde{W}^\mathsf{T} x \in \tilde{S}^\perp$, having a positive component. Condition (a) implies that there is $I_{x,\lambda} = \{i \mid z_i = \lambda\}$ with $\lambda > 0$ such that $\sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$ for all $c > 0$. Equivalently, there is $\tau \in \{0, +\}^n$ with $\tau^+ = I_{x,\lambda}$ such that $\tau \notin \text{sign}(\ker W) = \text{sign}(S)$. That is, z is nondegenerate, as required by condition (iii). Moreover, let $\tilde{\tau} = \text{sign}(z) \in \text{sign}(\tilde{S}^\perp)$ and hence $\tilde{\tau}^0 = I_{x,0}$. Condition (b) implies that there is a proper face of C, characterized by a nonzero sign vector $\tau \in \text{sign}(S^\perp)_\oplus$, such that $\tilde{\tau}^0 = I_{x,0} \subseteq \tau^0$. Again, z is nondegenerate, as required by condition (iii).

Lemma 16. *The ray condition* $(*)$ *holds for all nonzero* $x \in \mathbb{R}^d$ *and for all* $c > 0$ *if and only if conditions (ii) and (iii) in Theorem 13 hold.*

Proof. To show necessity and sufficiency of (ii) and (iii), we vary over all nonzero $x \in \mathbb{R}^d$.

Let $x \in \mathbb{R}^d$ be nonzero and $\lambda' = \max_i \tilde{w}^i \cdot x$.

- If $\lambda' \leq 0$, then $\tilde{\tau} = \text{sign}(-\tilde{W}^\mathsf{T} x) \in \text{sign}(\tilde{S}^\perp)_\oplus$ defines a proper face of $\tilde{C}$ and $F_c(xt) \to \sum_{i \in \tilde{\tau}^0} c_i w^i$ as $t \to \infty$. *Necessity and sufficiency of (ii)*: The ray condition $(*)$ for all $c > 0$ is equivalent to $\sum_{i \in \tilde{\tau}^0} c_i w^i \in \partial C$ for all $c > 0$. That is, there is a proper face of C characterized by a nonzero $\tau \in \text{sign}(S^\perp)_\oplus$ such that $\tilde{\tau}^0 \subseteq \tau^0$. Equivalently, $\tau \leq \tilde{\tau}$, that is, (ii) for $\tilde{\tau}$.

  By varying over all nonzero $x \in \mathbb{R}^d$ (with $\lambda' \leq 0$), all nonzero $\tilde{\tau} \in \text{sign}(\tilde{S}^\perp)_\oplus$ are covered.

- If $\lambda' > 0$, then $z = \tilde{W}^\mathsf{T} x \in \tilde{S}^\perp$ has a positive component. *Necessity and sufficiency of (iii)*: The ray condition $(*)$ for all $c > 0$ is equivalent to

  - either there is $\lambda > 0$ such that $F_c(xt)\,\mathrm{e}^{-\lambda t} \to \sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$ as $t \to \infty$
  - or $F_c(xt) \to \sum_{i \in I_{x,0}} c_i w^i \in \partial C$,

  for all $c > 0$. That is,

  - there is $\lambda > 0$ such that $\sum_{i \in I_{x,\lambda}} c_i w^i \neq 0$ for all $c > 0$ or
  - $\sum_{i \in I_{x,0}} c_i w^i \in \partial C$ for all $c > 0$,

  thereby using that the sets $I_{x,\lambda}$ are disjoint and the sums $\sum_{i \in I_{x,\lambda}} c_i w^i$ involve different coefficients c_i for different λ. Equivalently,

  - there is $I_{x,\lambda} = \{i \mid z_i = \lambda\}$ with $\lambda > 0$ such that $c \notin \ker W = S$ for all $c \geq 0$ with $\text{supp}(c) = I_{x,\lambda}$, that is, there is $\tau \in \{0, +\}^n$ with $\tau^+ = I_{x,\lambda}$ such that $\tau \notin \text{sign}(S)$, or
  - for $\tilde{\tau} = \text{sign}(z) \in \text{sign}(\tilde{S}^\perp)$ and hence $\tilde{\tau}^0 = I_{x,0}$, there is a proper face of C, characterized by a nonzero $\tau \in \text{sign}(S^\perp)_\oplus$ such that $\tilde{\tau}^0 \subseteq \tau^0$,

that is, (iii) for z.

By varying over all nonzero $x \in \mathbb{R}^d$ (with $\lambda' > 0$), all $z \in \tilde{S}^\perp$ with a positive component are covered.

□

3.1 Special cases: $C = \mathbb{R}^d$ or C is pointed

We discuss the conditions for bijectivity in Theorem 13 for two extreme cases, regarding the geometry of the cones $C = \operatorname{cone} W$ and $\tilde{C} = \operatorname{cone} \tilde{W}$.

If $C = \mathbb{R}^d$ (that is, $\operatorname{sign}(S^\perp)_\oplus = \{0\}$), then condition (ii) is equivalent to $\tilde{C} = \mathbb{R}^d$. Hence, if $C = \mathbb{R}^d$ and F_c is bijective for all $c > 0$, then $\tilde{C} = \mathbb{R}^d$. However, the converse does not hold.

Example 17. Let F_c be given by the matrices

$$\tilde{W} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\tilde{C} = \mathbb{R}^2$ and F_c is bijective for all $c > 0$. However, $C \neq \mathbb{R}^2$.

If $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$ (that is, C is pointed and no column of W is zero), then condition (iii) holds (since $\operatorname{sign}(S)_\oplus = \{0\}$), and conditions (i) and (ii) imply $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(\tilde{S}^\perp)$ (by Proposition 19 below). Hence, if $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$ and F_c is bijective for all $c > 0$, then $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(\tilde{S}^\perp)$. However, the converse does not hold.

Example 18. Let F_c be given by the matrices

$$\tilde{W} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $(+, +, +)^\mathsf{T} \in \operatorname{sign}(\tilde{S}^\perp)$ and F_c is bijective for all $c > 0$. However, $(+, +, +)^\mathsf{T} \notin \operatorname{sign}(S^\perp)$.

If $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$, then conditions (i) and (ii) imply the surjectivity of F_c for all $c > 0$ and hence a converse of (ii).

Proposition 19. *Let $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$, and let F_c be surjective. Then, for every $\tau \in \operatorname{sign}(S^\perp)_\oplus$, there is $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with $\tilde{\tau} \geq \tau$. In particular, $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(\tilde{S}^\perp)$.*

Proof. By surjectivity, the image of F_c contains points arbitrarily close to any point y on a proper face f of C, characterized by the sign vector $\tau \in \operatorname{sign}(S^\perp)_\oplus$. Hence, there are sequences x_n in $\mathbb{R}^d$ with $|x_n| = 1$ and $x_n \to x$ and $t_n > 0$ with $t_n \to \infty$ such that $F(x_n t_n) \to y$. Explicitly,

$$F_c(x_n t_n) = \sum_{i=1}^n c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i = \sum_\lambda \sum_{i \in I_{x,\lambda}} c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i.$$

Since $(+,\dots,+)^\mathsf{T} \in \operatorname{sign}(S^\perp)$, the vectors w^i are positively independent, and $\lambda = \tilde{w}^i \cdot x > 0$ implies $\tilde{w}^i \cdot x_n > 0$ (for x_n close to x) and hence $I_{x,\lambda} = \emptyset$ for $\lambda > 0$. Moreover, $\sum_{i \in I_{x,\lambda}} c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i \to 0$ for $\lambda < 0$, and hence

$$\sum_{i \in I_{x,0}} c_i \, \mathrm{e}^{\tilde{w}^i \cdot x_n \, t_n} \, w^i \to y.$$

Let $\tilde{\tau} = \operatorname{sign}(-\tilde{W}^\mathsf{T} x) \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$, that is, $\tilde{\tau}^0 = I_{x,0}$ and $\tilde{\tau}^+ = \bigcup_{\lambda<0} I_{x,\lambda}$, characterizing a proper face $\tilde{f}$ of $\tilde{C}$. Then, $\tilde{\tau}^0 \subseteq \tau^0$, that is, $\tilde{\tau} \geq \tau$.

In particular, let $\tau = (+,\dots,+)^\mathsf{T} \in \operatorname{sign}(S^\perp)$. Then there is $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with $\tilde{\tau} \geq \tau$ and hence $\tilde{\tau} = (+,\dots,+)^\mathsf{T}$. ☐

The main conclusion of the previous result can be reformulated: for every face f of C with $I = \{i \mid w^i \in f\}$, there is a face $\tilde{f}$ of $\tilde{C}$ with $\tilde{I} = \{i \mid \tilde{w}^i \in \tilde{f}\}$ such that $I \subseteq \tilde{I}$.

Finally, the surjectivity of F_c for all $c > 0$ together with condition (ii) itself implies a converse of (ii) regarding sign vectors with minimal support.

Corollary 20. *Let $(+,\dots,+)^\mathsf{T} \in \operatorname{sign}(S^\perp)$, let F_c be surjective, and assume condition (ii) in Theorem 13. Then, for every $\tau \in \operatorname{sign}(S^\perp)_\oplus$ with minimal support, there is $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with minimal support and $\tilde{\tau} \geq \tau$.*

Proof. By surjectivity, the image of F_c contains points arbitrarily close to any point y on a maximal proper face f of C, characterized by the sign vector $\tau \in \operatorname{sign}(S^\perp)_\oplus$ with minimal support. Note that $|\tau^0| \geq d-1$, and consider a point y that is a positive linear combination of $d-1$ (but not less) linearly independent vectors w^i. By Proposition 19, there is $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with $\tilde{\tau} \geq \tau$. Now, either $\tilde{\tau}$ itself has minimal support or there is $\tilde{\rho} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with minimal support and $\tilde{\tau} \geq \tilde{\rho}$. Then, either $\tilde{\rho} \geq \tau$ or $\tilde{\rho} \not\geq \tau$ (that is, $\tilde{\rho}^0 \cap \tau^+ \neq \emptyset$). However, the latter leads to a contradiction. By (ii), there is a nonzero $\rho \in \operatorname{sign}(S^\perp)_\oplus$ with $\tilde{\rho} \geq \rho$, characterizing a proper face g of C. On the one hand, $\tilde{\tau}^0 \subseteq \tilde{\rho}^0 \subseteq \rho^0$, and g contains $d-1$ linearly independent vectors w^i of f (and hence the maximal proper face f itself). On the other hand, g contains additional vectors w^i (with $i \in \tilde{\rho}^0 \cap \tau^+$) not in f, and hence g is not a proper face. ☐

Again, the main conclusion of the previous result can be reformulated: for every maximal proper face f of C with $I = \{i \mid w^i \in f\}$, there is a maximal proper face $\tilde{f}$ of $\tilde{C}$ with $\tilde{I} = \{i \mid \tilde{w}^i \in \tilde{f}\}$ such that $I \subseteq \tilde{I}$.

3.2 Sign-vector conditions

In general, the bijectivity of F_c for all $c > 0$ cannot be *characterized* in terms of sign vectors of S and $\tilde{S}$.

Example 21. Let F_c be given by the matrices

$$\tilde{W} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & \tilde{w} \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

involving the parameter $\tilde{w} > 0$. Obviously, $\tilde{C} = C = \mathbb{R}^3$. For $\tilde{w} = 1$ or $\tilde{w} \in [2, \infty)$, the map F_c is injective for all $c > 0$, but not bijective, whereas for $\tilde{w} \in (0,1)$ or $\tilde{w} \in (1,2)$, the map F_c is bijective for all $c > 0$. Clearly, the sign vectors $\operatorname{sign}(\tilde{S}) = \operatorname{sign}(\ker \tilde{W})$ do not depend on $\tilde{w}$ and hence cannot characterize bijectivity.

As opposed to conditions (i) and (ii) in Theorem 13, the nondegeneracy condition (iii) cannot be characterized in terms of sign vectors of S and $\tilde{S}$, in general. Still,

- condition (iii) holds trivially if $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$ (and hence C is pointed), see Section 3.1,
- there is a (weakest) condition (iii') in terms of sign vectors of S and $\tilde{S}$ *sufficient* for nondegeneracy, see Proposition 22, and
- there is a *sufficient* condition for nondegeneracy using faces of the Newton polytope $\tilde{P}$, see Proposition 23. (Thereby, faces of $\tilde{P}$ correspond to nonnegative sign vectors of an affine subspace related to $\tilde{S}$.)

Proposition 22. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. The pair $(S, \tilde{S})$ is nondegenerate, if*

(iii') for all $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)$ with $\tilde{\tau}^+ \neq \emptyset$,

- *either there is no $\tau \in \operatorname{sign}(S)_\oplus$ with $\tau^+ = \tilde{\tau}^+$*
- *or there is no $\pi \in \operatorname{sign}(S)$ with $(\tilde{\tau}^+ \cup \tilde{\tau}^-) \subseteq \pi^+$.*

Proof. Let $(S, \tilde{S})$ be degenerate. In particular, let $z \in \tilde{S}^\perp$ with a positive component be degenerate, and let $\tilde{\tau} = \operatorname{sign}(z) \in \operatorname{sign}(\tilde{S}^\perp)$, where $\tilde{\tau}^+ \neq \emptyset$.

For every $\lambda > 0$ and the corresponding index set $J = \{i \mid z_i = \lambda\}$, the sign vector $\tau \in \{0, +\}^n$ with $\tau^+ = J$ satisfies $\tau \in \operatorname{sign}(S)_\oplus$. Clearly, the index sets J cover $\tilde{\tau}^+$ and, by composition, there is $\tau \in \operatorname{sign}(S)_\oplus$ with $\tau^+ = \tilde{\tau}^+$.

Further, there is no nonzero $\tau \in \operatorname{sign}(S^\perp)_\oplus$ such that $\tilde{\tau}^0 \subseteq \tau^0$. That is, there is no nonzero $\tau \in \operatorname{sign}(S^\perp)$ with $\tau_i = 0$ for $i \in \tilde{\tau}^0$ and $\tau_i \leq +$ otherwise. By Corollary 45 in Appendix C, there is $\pi \in \operatorname{sign}(S)$ with $\pi_i = +$ for $i \in (\tilde{\tau}^+ \cup \tilde{\tau}^-)$. □

Finally, we formulate a sufficient condition for nondegeneracy using faces of the Newton polytope $\tilde{P} = \operatorname{conv} \tilde{W}$. A face $\tilde{f}$ of $\tilde{P}$ with $J = \{i \mid \tilde{w}^i \in \tilde{f}\}$ corresponds to a supporting affine hyperplane with normal vector $x \in \mathbb{R}^d$ and $\lambda' \in \mathbb{R}$ such that $\tilde{w}^i \cdot x = \lambda'$ for $i \in J$ and $\tilde{w}^i \cdot x < \lambda'$ otherwise; that is, $J = I_{x,\lambda'}$. It further corresponds to $z = \tilde{W}^\mathsf{T} x \in \tilde{S}^\perp$, where $J = \{i \mid z_i = \lambda'\}$. If $\lambda' > 0$, we call the face $\tilde{f}$ of $\tilde{P}$ *positive*, and $z \in \tilde{S}^\perp$ has a positive component.

Proposition 23. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$, $\tilde{W}$ be a matrix with full rank such that $\ker \tilde{W} = \tilde{S}$, and $\tilde{P} = \operatorname{conv} \tilde{W}$ be the Newton polytope. The pair $(S, \tilde{S})$ is nondegenerate, if, for every positive face $\tilde{f}$ of $\tilde{P}$ with $J = \{i \mid \tilde{w}^i \in \tilde{f}\}$ and sign vector $\tau \in \{0, +\}^n$ with $\tau^+ = J$, it holds that $\tau \notin \operatorname{sign}(S)_\oplus$.*

Proof. Let $z \in \tilde{S}^\perp$ have a positive component, $\lambda' = \max_i z_i > 0$, and $J = \{i \mid z_i = \lambda'\}$. Then z corresponds to a positive face $\tilde{f}$ of $\tilde{P}$ with $J = \{i \mid \tilde{w}^i \in \tilde{f}\}$. If, for the sign vector $\tau \in \{0,+\}^n$ with $\tau^+ = J$, it holds that $\tau \notin \operatorname{sign}(S)_\oplus$, then z is nondegenerate, by definition. □

4 Robustness of bijectivity

We study the robustness of the bijectivity of F_c for all $c > 0$ with respect to small perturbations of the exponents $\tilde{W}$ or/and the coefficients W, corresponding to small perturbations of the subspaces $\tilde{S}$ and S (in the Grassmannian).

4.1 Perturbations of the exponents

First, we consider small perturbations of the subspace $\tilde{S}$, corresponding to the exponents $\tilde{W}$ in F_c. As it turns out, the closure of $\operatorname{sign}(\tilde{S})$ plays an important role.

Definition 24. Let $\Sigma \subseteq \{-,0,+\}^n$. We define its *closure*

$$\overline{\Sigma} = \{\tau' \in \{-,0,+\}^n \mid \tau' \leq \tau \text{ for some } \tau \in \Sigma\}.$$

Clearly, $\Sigma_1 \subseteq \overline{\Sigma_2}$ implies $\overline{\Sigma_1} \subseteq \overline{\Sigma_2}$.

Lemma 25. *Let S be a subspace of $\mathbb{R}^n$ and S_ε be a small perturbation. Then* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(S_\varepsilon)}$.

Proof. Let $\tau \in \operatorname{sign}(S)$ and a corresponding $x \in S$ with $\tau = \operatorname{sign}(x)$. Then there is $x_\varepsilon \in S_\varepsilon$ close to x. For a small enough perturbation S_ε, nonzero components remain nonzero (but zero components can become nonzero), that is, $\operatorname{sign}(x) \leq \operatorname{sign}(x_\varepsilon)$. Hence, $\tau \in \overline{\operatorname{sign}(S_\varepsilon)}$. □

We start by studying injectivity.

Lemma 26. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. If* $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ *for all small perturbations $\tilde{S}_\varepsilon$, then* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$.

Proof. Suppose $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$ does not hold. Then there is a nonzero sign vector $\tau \in \operatorname{sign}(S)$ with $\tau \notin \overline{\operatorname{sign}(\tilde{S})}$. We will find a small perturbation $\tilde{S}_\varepsilon$ such that $\tau \in \operatorname{sign}(\tilde{S}_\varepsilon^\perp)$ and hence $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ is violated.

In terms of sign vectors, there is no $\tilde{\tau} \in \operatorname{sign}(\tilde{S})$ such that $\tilde{\tau} \geq \tau$; in terms of vectors, there is no $\tilde{x} \in \tilde{S}$ such that $\tilde{x}_i > 0$ for $i \in \tau^+$ and $\tilde{x}_i < 0$ for $i \in \tau^-$. By Corollary 46 in Appendix C, there is a nonzero $x \in \tilde{S}^\perp$ such that $x_i \geq 0$ for $i \in \tau^+$, $x_i \leq 0$ for $i \in \tau^-$, and $x_i = 0$ otherwise. If $\operatorname{sign}(x) = \tau$, then $\tau \in \operatorname{sign}(\tilde{S}^\perp)$, as desired. Otherwise, we find a perturbation $x_\varepsilon = x + \varepsilon e$ with $\varepsilon > 0$ small and $e \in \mathbb{R}^n$ such that $\operatorname{sign}(x_\varepsilon) = \tau$. In particular, we choose $e_i = 1$ if $x_i = 0$ and $i \in \tau^+$, $e_i = -1$ if $x_i = 0$ and $i \in \tau^-$, and $e_i = 0$ otherwise. Then, we rescale x_ε such that $|x_\varepsilon| = |x|$. Finally, we find an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ (close to the identity) such that $Ux = x_\varepsilon$. Then $x_\varepsilon = Ux \perp U\tilde{S} = \tilde{S}_\varepsilon$, that is, $x_\varepsilon \in \tilde{S}_\varepsilon^\perp$ and $\tau \in \operatorname{sign}(\tilde{S}_\varepsilon^\perp)$, as desired. □

Lemma 27. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. If* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$, *then* $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\}$.

Proof. Assume there exists a nonzero $\tau \in \operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}^\perp)$. If $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$, then there exists $\rho \in \operatorname{sign}(\tilde{S})$ with $\tau \leq \rho$. In particular, $\tau \cdot \rho \neq 0$, thereby contradicting $\tau \in \operatorname{sign}(\tilde{S}^\perp)$ and $\rho \in \operatorname{sign}(\tilde{S})$. ◻

Proposition 28. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. Then* $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ *for all small perturbations $\tilde{S}_\varepsilon$ if and only if* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$.

Proof. ($\Rightarrow$): By Lemma 26.

($\Leftarrow$): Assume $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$. By Lemma 25, $\operatorname{sign}(\tilde{S}) \subseteq \overline{\operatorname{sign}(\tilde{S}_\varepsilon)}$ for all small perturbations $\tilde{S}_\varepsilon$ which implies $\overline{\operatorname{sign}(\tilde{S})} \subseteq \overline{\operatorname{sign}(\tilde{S}_\varepsilon)}$. Hence, $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S}_\varepsilon)}$. By Lemma 27, $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$. ◻

Corollary 29. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. Then*

$$\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})} \quad \textit{if and only if} \quad \operatorname{sign}(S^\perp) \subseteq \overline{\operatorname{sign}(\tilde{S}^\perp)}.$$

Proof. By Corollary 5, $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ is equivalent to $\operatorname{sign}(S^\perp) \cap \operatorname{sign}(\tilde{S}_\varepsilon) = \{0\}$. By Proposition 28 twice, the former statement (for all small perturbations $\tilde{S}_\varepsilon$) is equivalent to $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$ and the latter to $\operatorname{sign}(S^\perp) \subseteq \overline{\operatorname{sign}(\tilde{S}^\perp)}$. ◻

In terms of the map F_c (and the associated subspaces S and $\tilde{S}$), Proposition 28 states that

$$\begin{array}{l} F_c \text{ is injective for all } c > 0 \\ \text{and all small perturbations } \tilde{S}_\varepsilon \end{array} \quad \Leftrightarrow \quad \operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}.$$

In Proposition 30 and Theorem 32 below, we will show that

$$\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})} \quad \Rightarrow \quad F_c \text{ is bijective for all } c > 0$$

and

$$\begin{array}{l} F_c \text{ is bijective for all } c > 0 \\ \text{and all small perturbations } \tilde{S}_\varepsilon \end{array} \quad \Leftrightarrow \quad \operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}.$$

First, we prove that the closure condition

$$\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})} \tag{cc}$$

implies the bijectivity of F_c for all $c > 0$, that is, conditions (i), (ii), and (iii) in Theorem 13. For an alternative proof, see [9].

Proposition 30. *If* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$, *then the map F_c is a diffeomorphism for all $c > 0$.*

Proof. (cc) $\Rightarrow$ (i): By Lemma 27.

(cc) $\Rightarrow$ (ii):

Assume $\neg$(ii), that is, the existence of a nonzero $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)_\oplus$ with $\tau \not\leq \tilde{\tau}$ for all nonzero $\tau \in \operatorname{sign}(S^\perp)_\oplus$.

By Corollary 45 in Appendix C, the nonexistence of a nonzero $\tau \in \operatorname{sign}(S^\perp)$ with $f_i \leq +$ for $i \in \tilde{\tau}_+$ and $f_i = 0$ otherwise implies the existence of $\tau \in \operatorname{sign}(S)$ with $\tau_i = +$ for $i \in \tilde{\tau}_+$, that is, $\tilde{\tau} \leq \tau$.

Now, if $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$, then there exists $\rho \in \operatorname{sign}(\tilde{S})$ with $\tau \leq \rho$. In particular, $\tilde{\tau} \cdot \rho \neq 0$, thereby contradicting $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)$ and $\rho \in \operatorname{sign}(\tilde{S})$.

(cc) $\Rightarrow$ (iii') in Proposition 22:

Assume $\neg$(iii') and hence, by Proposition 22 (based on Corollary 45 in Appendix C), the existence of $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)$ and $\tau \in \operatorname{sign}(S)_\oplus$ with $\tilde{\tau}^+ = \tau^+ \neq \emptyset$ and further the existence of $\pi \in \operatorname{sign}(S)$ with $(\tilde{\tau}^+ \cup \tilde{\tau}^-) \subseteq \pi^+$. The sign vectors of a subspace are closed under composition, and hence $\tau' = \tau \circ (-\pi) \in \operatorname{sign}(S)$, where $\tau'_i = +$ for $i \in \tilde{\tau}^+$ and $\tau'_i = -$ for $i \in \tilde{\tau}^-$, that is, $\tilde{\tau} \leq \tau'$.

Now, if $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$, then there exists $\rho \in \operatorname{sign}(\tilde{S})$ with $\tau' \leq \rho$. In particular, $\tilde{\tau} \cdot \rho \neq 0$, thereby contradicting $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)$ and $\rho \in \operatorname{sign}(\tilde{S})$. $\square$

However, the closure condition (cc) is not necessary for bijectivity. Recall that there is a (weakest) sign-vector condition sufficient for bijectivity, involving conditions (i), (ii), and (iii'), cf. Proposition 22.

Example 31. Let F_c be given by the matrices

$$\tilde{W} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}.$$

Obviously, $\tilde{C} = C = \mathbb{R}$. Now, for $\tau = (+,+,-)^\mathsf{T} \in \operatorname{sign}(\operatorname{im} W^\mathsf{T}) = \operatorname{sign}(S^\perp)$, there is no $\tilde{\tau} \in \operatorname{sign}(\operatorname{im} \tilde{W}^\mathsf{T}) = \operatorname{sign}(\tilde{S}^\perp)$ with $\tau \leq \tilde{\tau}$. Hence, $\operatorname{sign}(S^\perp) \not\subseteq \overline{\operatorname{sign}(\tilde{S}^\perp)}$, and the closure condition (cc) does not hold. Still, there is no nonzero $\tau \in \operatorname{sign}(\ker W)_\oplus = \operatorname{sign}(S)_\oplus$, and hence condition (iii') holds. Further, conditions (i) and (ii) hold, and F_c is bijective for all $c > 0$.

In fact, the closure condition (cc) is equivalent to bijectivity for all small perturbations $\tilde{S}_\varepsilon$.

Theorem 32. *The map F_c is a diffeomorphism for all $c > 0$ and all small perturbations $\tilde{S}_\varepsilon$ if and only if* $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$.

Proof. By Lemma 25, $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$ implies $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S}_\varepsilon)}$ for all small perturbations $\tilde{S}_\varepsilon$. By Proposition 30, the latter implies the bijectivity of F_c for all $c > 0$ and all small perturbations $\tilde{S}_\varepsilon$.

Bijectivity implies injectivity, that is, $\operatorname{sign}(S) \cap \operatorname{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ for all small perturbations $\tilde{S}_\varepsilon$. By Lemma 26, the latter implies $\operatorname{sign}(S) \subseteq \overline{\operatorname{sign}(\tilde{S})}$. $\square$

4.2 Perturbations of the coefficients

Next, we consider small perturbations of the subspace S (corresponding to the coefficients W in F_c). We start by studying injectivity. By Corollary 5, the perturbed injectivity conditon $\operatorname{sign}(S_\varepsilon) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\}$ is equivalent to $\operatorname{sign}(\tilde{S}) \cap \operatorname{sign}(S_\varepsilon^\perp) = \{0\}$. By exchanging the roles of S and $\tilde{S}$ in Proposition 28, we immediately obtain the desired result.

Corollary 33. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$. Then $\operatorname{sign}(S_\varepsilon) \cap \operatorname{sign}(\tilde{S}^\perp) = \{0\}$ for all small perturbations S_ε if and only if $\operatorname{sign}(\tilde{S}) \subseteq \overline{\operatorname{sign}(S)}$.*

The closure condition

$$\operatorname{sign}(\tilde{S}) \subseteq \overline{\operatorname{sign}(S)} \tag{cc'}$$

is equivalent to $\operatorname{sign}(\tilde{S}^\perp) \subseteq \overline{\operatorname{sign}(S^\perp)}$, by Corollary 29. As opposed to (cc), it does not imply bijectivity, in fact, it implies conditions (i) and (iii) in Theorem 13, but not condition (ii).

Proposition 34. *If $\operatorname{sign}(\tilde{S}) \subseteq \overline{\operatorname{sign}(S)}$, then conditions (i) and (iii) in Theorem 13 hold.*

Proof. (cc') $\Rightarrow$ (i): By Corollary 33.

(cc') $\Rightarrow$ (iii'):

Assume $\neg$(iii') and hence, by Proposition 22, the existence of $\tilde{\tau} \in \operatorname{sign}(\tilde{S}^\perp)$ and $\tau \in \operatorname{sign}(S)_\oplus$ with $\tilde{\tau}^+ = \tau^+ \neq \emptyset$, in particular, $\tau \leq \tilde{\tau}$. Now assume (cc') and hence the existence of $\rho \in \operatorname{sign}(S^\perp)$ with $\tilde{\tau} \leq \rho$. Then $\tau \cdot \rho \neq 0$, thereby contradicting $\tau \in \operatorname{sign}(S)$ and $\rho \in \operatorname{sign}(S^\perp)$. $\square$

Example 35. Let F_c be given by the matrices

$$\tilde{W} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Obviously, $\tilde{C} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and $C = \mathbb{R}^2_{\geq 0}$. Now, $\tilde{S} = \ker \tilde{W} = \operatorname{im}(1, 0, 1)^\mathsf{T}$, $S = \ker W = \operatorname{im}(1, -1, 1)^\mathsf{T}$, and hence $\operatorname{sign}(\tilde{S}) \subseteq \overline{\operatorname{sign}(S)}$. However, $\operatorname{sign}(\tilde{S}^\perp)_\oplus = \{(0,0,0)^\mathsf{T},\ (0,+,0)^\mathsf{T}\}$, $\operatorname{sign}(S^\perp)_\oplus = \{(0,0,0)^\mathsf{T},\ (0,+,+)^\mathsf{T},\ (+,+,0)^\mathsf{T}\}$, and hence condition (ii) does not hold.

It remains to study how perturbations of the subspace S affect condition (ii).

Lemma 36. *If condition (ii) in Theorem 13 holds for all small perturbations S_ε, then either $C = \tilde{C} = \mathbb{R}^d$ or $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$.*

Proof. If neither $C = \mathbb{R}^d$ nor $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$, then C has a nontrivial lineality space. On the one hand, there is a small perturbation S_{ε_1} such that $C_{\varepsilon_1} = \mathbb{R}^d$ and hence $\tilde{C} = \mathbb{R}^d$, by (ii); on the other hand, there is a small perturbation S_{ε_2} such that $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S_{\varepsilon_2}^\perp)$ and hence $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(\tilde{S}^\perp)$, by Proposition 19; a contradiction.

If $C = \mathbb{R}^d$, then $C_\varepsilon = \mathbb{R}^d$ for all small perturbations S_ε and hence $\tilde{C} = \mathbb{R}^d$, by (ii). $\square$

That is, condition (ii) is robust only in two extreme cases regarding the geometry of the cone C. We consider the case $(+, \ldots, +)^\mathsf{T} \in \operatorname{sign}(S^\perp)$ separately.

Lemma 37. *Let* $(+,\dots,+)^\mathsf{T} \in \text{sign}(S^\perp)$*. If the map* F_c *is surjective and condition (ii) in Theorem 13 holds for all small perturbations* S_ε*, then*

(ii') every $\tau \in \text{sign}(S^\perp)_\oplus$ *has minimal support if and only if* $|\tau^0| = d-1$*, and* $\text{sign}(S^\perp)_\oplus = \text{sign}(\tilde{S}^\perp)_\oplus$.

Proof. By Proposition 19, $(+,\dots,+)^\mathsf{T} \in \text{sign}(\tilde{S}^\perp)$ (and hence $\tilde{C}$ is pointed). Let $\tau \in \text{sign}(S^\perp)_\oplus$ have minimal support, but $|\tau^0| > d-1$. By Corollary 20, there is $\tilde{\tau} \in \text{sign}(\tilde{S}^\perp)_\oplus$ with minimal support and $\tilde{\tau} \geq \tau$, that is, $\tilde{\tau}^0 \subseteq \tau^0$. Clearly, τ and $\tilde{\tau}$ characterize maximal proper faces f of C and $\tilde{f}$ of $\tilde{C}$, respectively.

Suppose that either $|\tilde{\tau}^0| > d-1$ or $\text{cone}(w^j \mid j \in \tilde{\tau}^0)$ is a proper subcone of f. Then there is $i \in \tilde{\tau}^0$ (and hence $i \in \tau^0$) such that $\text{cone}(w^j \mid j \in \tau^0 \setminus \{i\}) = f$, that is, the vector w^i is not needed to generate the face f. Now consider a small perturbation w^i_ε such that $w^i_\varepsilon \in C^\circ$ and $C_\varepsilon = C$. Then $i \notin \tau^0_\varepsilon$ for all $\tau_\varepsilon \in \text{sign}(S^\perp_\varepsilon)_\oplus$, and there is no $\tau_\varepsilon \in \text{sign}(S^\perp_\varepsilon)_\oplus$ with $\tau_\varepsilon \leq \tilde{\tau}$, contradicting (ii).

Conversely, suppose that $|\tilde{\tau}^0| = d-1$ and $\text{cone}(w^j \mid j \in \tilde{\tau}^0) = f$. Then there is $i \in \tau^0$ (but $i \notin \tilde{\tau}^0$) such that $\text{cone}(w^j \mid j \in \tau^0 \setminus \{i\}) = f$. Now consider a perturbation w^i_ε such that $w^i_\varepsilon \notin C$ and $C_\varepsilon \supset C$. Then the hyperplane containing f (generated by the vectors w^j with $j \in \tilde{\tau}^0$) is not a supporting hyperplane of C_ε, and there is no $\tau_\varepsilon \in \text{sign}(S^\perp_\varepsilon)_\oplus$ with $\tau_\varepsilon \leq \tilde{\tau}$, contradicting (ii).

As a consequence, every $\tau \in \text{sign}(S^\perp)_\oplus$ has minimal support if and only if $|\tau^0| = d-1$. By Corollary 20, the same holds for the corresponding $\tilde{\tau} \in \text{sign}(\tilde{S}^\perp)_\oplus$ with minimal support, in particular, $\tilde{\tau} = \tau$. In fact, every $\tilde{\tau} \in \text{sign}(\tilde{S}^\perp)_\oplus$ has minimal support if and only if $|\tilde{\tau}^0| = d-1$: By (ii), there is a corresponding $\tau \in \text{sign}(S^\perp)_\oplus$, necessarily with minimal support and $|\tau^0| = d-1$, in particular, $\tau = \tilde{\tau}$. That is, elements of $\text{sign}(S^\perp)_\oplus$ and $\text{sign}(\tilde{S}^\perp)_\oplus$ with minimal support are in one-to-one correspondence. By [21], every nonnegative sign vector of a subspace is the composition of nonnegative sign vectors with minimal support. Hence, $\text{sign}(S^\perp)_\oplus = \text{sign}(\tilde{S}^\perp)_\oplus$. ☐

The main conclusion of the previous result can be reformulated: every maximal proper face f of C with $I = \{i \mid w^i \in f\}$ has $d-1$ generators, that is $|I| = d-1$. Moreover, faces (and their generators) of C and $\tilde{C}$ correspond to each other.

Finally, the closure condition (cc') together with sign-vector conditions regarding the geometry of the cones C and $\tilde{C}$ is equivalent to bijectivity for all small perturbations S_ε.

Theorem 38. *The map* F_c *is a diffeomorphism for all* $c > 0$ *and all small perturbations* S_ε *if and only if* $\text{sign}(\tilde{S}) \subseteq \overline{\text{sign}(S)}$ *and*

either $C = \tilde{C} = \mathbb{R}^d$
or $(+,\dots,+)^\mathsf{T} \in \text{sign}(S^\perp) \cap \text{sign}(\tilde{S}^\perp)$ *(and hence* C *and* $\tilde{C}$ *are pointed) and condition (ii') in Lemma 37 holds.*

Proof. By Theorem 13, the bijectivity of F_c for all $c > 0$ is equivalent to conditions (i), (ii), and (iii).

By Corollary 33, condition (i), that is, $\text{sign}(S_\varepsilon) \cap \text{sign}(\tilde{S}^\perp) = \{0\}$, for all small perturbations S_ε, is equivalent to $\text{sign}(\tilde{S}) \subseteq \overline{\text{sign}(S)}$.

By Lemma 36, condition (ii) for all small perturbations S_ε implies either $C = \tilde{C} = \mathbb{R}^d$ or $(+,\dots,+)^\mathsf{T} \in \mathrm{sign}(S^\perp)$. In the latter case, Lemma 37 further implies condition (ii'), in particular, $(+,\dots,+)^\mathsf{T} \in \mathrm{sign}(\tilde{S}^\perp)$.

Conversely, $\tilde{C} = \mathbb{R}^d$ (and hence $\mathrm{sign}(\tilde{S}^\perp)_\oplus = \{0\}$) trivially implies condition (ii) for all small perturbations S_ε. By Lemma 25, $\mathrm{sign}(\tilde{S}) \subseteq \overline{\mathrm{sign}(S)}$ implies $\mathrm{sign}(\tilde{S}) \subseteq \overline{\mathrm{sign}(S_\varepsilon)}$ for all small perturbations S_ε, and by Proposition 34 (for $\tilde{S}$ and S_ε), this implies condition (iii) for all small perturbations S_ε. Finally, $(+,\dots,+)^\mathsf{T} \in \mathrm{sign}(S^\perp)$ and condition (ii') imply condition (ii) for all small perturbations S_ε: If every $\tau \in \mathrm{sign}(S^\perp)_\oplus$ has minimal support if and only if $|\tau^0| = d-1$, then $\mathrm{sign}(S_\varepsilon^\perp)_\oplus = \mathrm{sign}(S^\perp)_\oplus$ for all small perturbations S_ε. If further $\mathrm{sign}(S^\perp)_\oplus = \mathrm{sign}(\tilde{S}^\perp)_\oplus$, then $\mathrm{sign}(S_\varepsilon^\perp)_\oplus = \mathrm{sign}(\tilde{S}^\perp)_\oplus$. □

4.3 General perturbations and maximal minors

Corollary 4 relates *chirotopes* (maximal minors of W and $\tilde{W}$) to *vectors* (sign vectors of $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$). By varying over all small perturbations $\tilde{S}_\varepsilon$, we obtain the following result.

Proposition 39. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$ of dimension $n-d$ (with $d \le n$). For every $W, \tilde{W} \in \mathbb{R}^{d\times n}$ (with full rank d) such that $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$, the following statements are equivalent.*

1. $\mathrm{sign}(S) \subseteq \overline{\mathrm{sign}(\tilde{S})}$.
2. $\det(W_I) \neq 0$ *implies* $\det(W_I)\det(\tilde{W}_I) > 0$ *for all subsets* $I \subseteq [n]$ *of cardinality* d *(or '< 0' for all I).*

Proof. By Proposition 28, statement 1 is equivalent to $\mathrm{sign}(S) \cap \mathrm{sign}(\tilde{S}_\varepsilon^\perp) = \{0\}$ for all small perturbations $\tilde{S}_\varepsilon$. By Corollary 4, this is equivalent to

> $\det(W_I)\det(\tilde{W}_{\varepsilon,I}) \geq 0$ for all $I \subseteq [n]$ of cardinality d (or '≤ 0' for all I)
> and $\det(W_I)\det(\tilde{W}_{\varepsilon,I}) \neq 0$ for some I,
> for all small perturbations $\tilde{W}_\varepsilon$ of $\tilde{W}$.

This is equivalent to statement 2, thereby using that $\det(\tilde{W}_I) = 0$ implies $\det(\tilde{W}_{\varepsilon_1,I}) < 0$ and $\det(\tilde{W}_{\varepsilon_2,I}) > 0$ for some small perturbations $\tilde{W}_{\varepsilon_1}$ and $\tilde{W}_{\varepsilon_2}$. □

Now we can extend Theorem 32. In particular, we can characterize the bijectivity of F_c for all $c > 0$ and all small perturbations $\tilde{S}_\varepsilon$ not only in terms of sign vectors, but also in terms of maximal minors.

Corollary 40. *The following statements are equivalent:*

1. *F_c is a diffeomorphism for all $c > 0$ and all small perturbations $\tilde{S}_\varepsilon$.*
2. $\mathrm{sign}(S) \subseteq \overline{\mathrm{sign}(\tilde{S})}$.
3. $\det(W_I) \neq 0$ *implies* $\det(W_I)\det(\tilde{W}_I) > 0$ *for all subsets* $I \subseteq [n]$ *of cardinality* d *(or '< 0' for all I).*

Proof. (1 ⇔ 2): By Theorem 32. (2 ⇔ 3): By Proposition 39. □

The next result relates *chirotopes* to *cocircuits* (sign vectors of $S^\perp = \operatorname{im} W^\mathsf{T}$ and $\tilde{S}^\perp = \operatorname{im} \tilde{W}^\mathsf{T}$ with minimal support).

Lemma 41. *Let $S, \tilde{S}$ be subspaces of $\mathbb{R}^n$ of dimension $n - d$ (with $d \leq n$). For every $W, \tilde{W} \in \mathbb{R}^{d \times n}$ (with full rank d) such that $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$, the following statements are equivalent.*

1. $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$, *and a nonzero $\tau \in \operatorname{sign}(S^\perp)$ has minimal support if and only if $|\tau^0| = d - 1$.*
2. $\det(W_I) \det(\tilde{W}_I) > 0$ *for all subsets $I \subseteq [n]$ of cardinality d (or '< 0' for all I).*

Proof. By using the standard chirotope/cocircuit translation for subspaces of $\mathbb{R}^n$, see [10, Theorem 8.1.6]. □

Now, we can consider small perturbations of both subspaces, S and $\tilde{S}$.

Theorem 42. *The following statements are equivalent:*

1. *F_c is a diffeomorphism for all $c > 0$ and all small perturbations S_ε and $\tilde{S}_{\tilde{\varepsilon}}$.*
2. $\operatorname{sign}(S) = \operatorname{sign}(\tilde{S})$, *and a nonzero $\tau \in \operatorname{sign}(S^\perp)$ has minimal support if and only if $|\tau^0| = d - 1$.*
3. $\det(W_I) \det(\tilde{W}_I) > 0$ *for all subsets $I \subseteq [n]$ of cardinality d (or '< 0' for all I).*

Proof. (1 ⇒ 3): Statement 1 implies the injectivity of F_c for all $c > 0$, that is, $\operatorname{sign}(S_\varepsilon) \cap \operatorname{sign}(\tilde{S}_{\tilde{\varepsilon}}^\perp) = \{0\}$, for all small perturbations S_ε, $\tilde{S}_{\tilde{\varepsilon}}$. By Corollary 4, this is equivalent to

> $\det(W_{\varepsilon,I}) \det(\tilde{W}_{\tilde{\varepsilon},I}) \geq 0$ for all $I \subseteq [n]$ of cardinality d (or '≤ 0' for all I)
> and $\det(W_{\varepsilon,I}) \det(\tilde{W}_{\tilde{\varepsilon},I}) \neq 0$ for some I,
> for all small perturbations W_ε of W and $\tilde{W}_{\tilde{\varepsilon}}$ of $\tilde{W}$.

This is equivalent to statement 3.

(3 ⇒ 1): Statement 3 implies

> $\det(W_{\varepsilon,I}) \det(\tilde{W}_{\tilde{\varepsilon},I}) > 0$ for all $I \subseteq [n]$ of cardinality d (or '< 0' for all I),
> for all small perturbations W_ε, $\tilde{W}_{\tilde{\varepsilon}}$.

By Lemma 41, this implies $\operatorname{sign}(S_\varepsilon) = \operatorname{sign}(\tilde{S}_{\tilde{\varepsilon}})$ and hence $\operatorname{sign}(S_\varepsilon) \subseteq \overline{\operatorname{sign}(\tilde{S}_{\tilde{\varepsilon}})}$, for all small perturbations W_ε, $\tilde{W}_{\tilde{\varepsilon}}$. By Proposition 30, this implies statement 1.

(2 ⇔ 3): By Lemma 41. □

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Appendices

A Motivation from Chemical Reaction Networks

Let S and $\tilde{S}$ be subspaces of $\mathbb{R}^n$, representing the stoichiometric and kinetic-order subspaces of a generalized chemical reaction network [18, 19]. For a (positive) complex-balancing equilibrium $c \in \mathbb{R}^n_{>0}$, the set of all complex-balancing equilibria is given by

$$c \circ \mathrm{e}^{\tilde{S}^\perp} \subseteq \mathbb{R}^n_{>0}.$$

Further, for a (positive) species concentration $c' \in \mathbb{R}^n_{>0}$, the corresponding stoichiometric compatibility class is given by

$$(c' + S) \cap \mathbb{R}^n_{\geq 0}.$$

We want to characterize when there exists a unique complex-balancing equilibrium in every stoichiometric class, in particular, when there exists a unique intersection

$$c \circ \mathrm{e}^{\tilde{S}^\perp} \cap (c' + S)$$

for all $c, c' > 0$.

Let $u \in S$ and $v \in \tilde{S}^\perp$ such that

$$c \circ \mathrm{e}^v = c' + u.$$

Now, write $S = \ker W$ and $\tilde{S} = \ker \tilde{W}$, where $W, \tilde{W} \in \mathbb{R}^{d \times n}$ have full rank $d \leq n$, and write $v = \tilde{W}^\mathsf{T} x \in \operatorname{im} \tilde{W}^\mathsf{T} = (\ker \tilde{W})^\perp = \tilde{S}^\perp$ with $x \in \mathbb{R}^d$. Then the previous equation is equivalent to

$$W(c \circ \mathrm{e}^{\tilde{W}^\mathsf{T} x}) = W c'.$$

As a consequence, a unique intersection $c \circ \mathrm{e}^{\tilde{S}^\perp} \cap (c' + S)$ for all $c, c' > 0$ is equivalent to the bijectivity of the map

$$\begin{aligned} F_c \colon \mathbb{R}^d &\to C^\circ \subseteq \mathbb{R}^d, \\ x &\mapsto W(c \circ \mathrm{e}^{\tilde{W}^\mathsf{T} x}) = \sum_{i=1}^n c_i \, \mathrm{e}^{\tilde{w}^i \cdot x} \, w^i \end{aligned}$$

for all $c > 0$. Thereby $C = \operatorname{cone} W$ and $\tilde{w}^i$ is the i-th column of $\tilde{W}$. The bijectivity of the map (for all $c > 0$) is equivalent to a unique intersection (for all $c, c' > 0$) and hence does not depend directly on W and $\tilde{W}$, but only on $\ker W = S$ and $\ker \tilde{W} = \tilde{S}$.

B Sign vectors and face lattices

In the context of oriented matroids, we discuss the relation between sign vectors of linear subspaces and face lattices of polyhedral cones. For further details, we refer to [2, Chapter 7], [26, Chapters 2 and 6], and the encyclopedic study [4].

Let $W = (w^1, \ldots, w^n) \in \mathbb{R}^{d \times n}$ with $d \leq n$ have full rank. Then W is called a *vector configuration* (of n vectors in $\mathbb{R}^d$), and $\operatorname{im} W^\mathsf{T} \subseteq \mathbb{R}^n$ is a corresponding linear subspace. Now let $v = W^\mathsf{T} x \in \operatorname{im} W^\mathsf{T}$ with $x \in \mathbb{R}^d$. Then $v_i = w^i \cdot x$, and the sign vector $\tau = \operatorname{sign}(v) \in \operatorname{sign}(\operatorname{im} W^\mathsf{T}) \subseteq \{-, 0, +\}^n$ describes the positions of the vectors $w^1, \ldots, w^n$ relative to the hyperplane with normal vector x.

Elements of $\operatorname{sign}(\operatorname{im} W^\mathsf{T})$ are called *covectors*, and elements of $\operatorname{sign}(\operatorname{im} W^\mathsf{T})$ with minimal support are called *cocircuits*. Analogously, elements of $\operatorname{sign}(\ker W)$ are called *vectors*, and elements of $\operatorname{sign}(\ker W)$ with minimal support are called *circuits*.

The *chirotope* of the vector configuration W is the map

$$\begin{aligned} \chi_W \colon \{1, \ldots, n\}^d &\to \{-, 0, +\}\,, \\ (i_1, \ldots, i_d) &\mapsto \operatorname{sign}(\det(w^{i_1}, \ldots, w^{i_d})) \end{aligned}$$

which records for each d-tuple of vectors if it forms a positively (or negatively) oriented basis of $\mathbb{R}^d$ or it is not a basis.

The *oriented matroid* of W is a combinatorial structure that can be given by any of the above data (co/vectors, co/circuits, or chirotopes) and defined/characterized in terms of any of the corresponding axiom systems.

The *face lattice* of $C = \operatorname{cone} W \subseteq \mathbb{R}^d$, the polyhedral cone generated by the vectors $w^1, \ldots, w^n$, can be obtained from the sign vectors of the linear subspace $\operatorname{im} W^\mathsf{T}$. In fact, it is the set $\operatorname{sign}(\operatorname{im} W^\mathsf{T})_\oplus = \operatorname{sign}(\operatorname{im} W^\mathsf{T}) \cap \{0, +\}^n$ with the partial order induced by the relation $+ > 0$. A face f of C corresponds to a supporting hyperplane with normal vector x such that $w^i \cdot x = 0$ for $w^i \in f$ and $w^i \cdot x > 0$ for $w^i \notin f$ (lying on the positive side of the hyperplane). Hence f is characterized by the sign vector $\tau = \operatorname{sign}(W^\mathsf{T} x) \in \operatorname{sign}(\operatorname{im} W^\mathsf{T})_\oplus$.

The *lineality space* of a cone C is given by the set $C \cap (-C)$. It is the minimal face of C, in the sense that it is contained in all faces. The lineality space of $C = \operatorname{cone} W$ can be obtained from $\operatorname{sign}(\ker W)_\oplus$, that is, from the positive dependencies among the vectors $w^1, \ldots, w^n$.

A cone C is called *pointed* if its lineality space is $\{0\}$, that is, if it has vertex 0. Note that, if $(+, \ldots, +)^T \in \operatorname{sign}(\operatorname{im} W^\mathsf{T})_\oplus$ (that is, $\operatorname{sign}(\ker W)_\oplus = \{0\}$), then $C = \operatorname{cone} W$ is pointed.

C A general theorem of the alternative

Definition 43. Let $x \in \mathbb{R}^n$, and let $I_1, \ldots, I_n$ be intervals of $\mathbb{R}$. We define the interval

$$\begin{aligned} I(x) &\equiv x_1 I_1 + \ldots + x_n I_n \\ &= \{x_1 y_1 + \ldots + x_n y_n \in \mathbb{R} \mid y_1 \in I_1, \ldots, y_n \in I_n\} \end{aligned}$$

and write $I(x) > 0$ if $y > 0$ for all $y \in I(x)$.

Theorem 44 (Theorem 22.6 in [22])**.** *Let S be a subspace of $\mathbb{R}^n$, and let $I_1, \ldots, I_n$ be intervals of $\mathbb{R}$. Then one and only one of the following alternatives holds:*

(a) There exists a vector $x = (x_1, \ldots, x_n)^\mathsf{T} \in S$ such that

$$x_1 \in I_1, \ldots, x_n \in I_n.$$

(b) There exists a vector $x^* = (x_1^*, \ldots, x_n^*)^\mathsf{T} \in S^\perp$ such that

$$x_1^* I_1 + \ldots + x_n^* I_n > 0.$$

Corollary 45. *Let S be a subspace of $\mathbb{R}^n$, and let $J \subseteq \{1, \ldots, n\}$ be nonempty. Then either (a) there exists a vector $x \in S$ with $x_i > 0$ for $i \in J$ or (b) there exists a nonzero vector $x^* \in S^\perp$ with $x_i^* \geq 0$ for $i \in J$ and $x_i^* = 0$ otherwise.*

Proof. By Theorem 44 with $I_i = (0, +\infty)$ for $i \in J$ and $I_i = (-\infty, +\infty)$ otherwise. □

Corollary 46. *Let S be a subspace of $\mathbb{R}^n$, and let $J^+, J^- \subseteq \{1, \ldots, n\}$ with $J^+ \cap J^- = \emptyset$ and $J^+ \cup J^- \neq \emptyset$. Then either (a) there exists a vector $x \in S$ with $x_i > 0$ for $i \in J^+$ and $x_i < 0$ for $i \in J^-$ or (b) there exists a nonzero vector $x^* \in S^\perp$ with $x_i^* \geq 0$ for $i \in J^+$, $x_i^* \leq 0$ for $i \in J^-$, and $x_i^* = 0$ otherwise.*

Proof. By Theorem 44 with $I_i = (0, +\infty)$ for $i \in J^+$, $I_i = (-\infty, 0)$ for $i \in J^-$, and $I_i = (-\infty, +\infty)$ otherwise. □

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