AAECC (2007) 18:583–601
DOI 10.1007/s00200-007-0054-9

Parametrizing compactly supported orthonormal wavelets by discrete moments

Georg Regensburger

Received: 30 December 2005 / Revised: 19 March 2007 / Published online: 16 August 2007

Abstract We discuss parametrizations of filter coefficients of scaling functions and compactly supported orthonormal wavelets with several vanishing moments. We introduce the first discrete moments of the filter coefficients as parameters. The discrete moments can be expressed in terms of the continuous moments of the related scaling function. To solve the resulting polynomial equations we use symbolic computation and in particular Gröbner bases. The cases of four to ten filter coefficients are discussed and explicit parametrizations are given.

Keywords Orthonormal wavelets · Parametrization · Filter coefficients · Moments · Gröbner bases

1 Introduction

Over the last two decades wavelets have become a fundamental tool in many areas of applied mathematics and engineering ranging from signal and image processing to numerical analysis, see for example Daubechies [13], Mallat [26], and Strang and Nguyen [35]. A function $\psi \in L^2(\mathbb{R})$ is an *orthonormal wavelet* if the family

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad \text{for } j, k \in \mathbb{Z},$$

This work was supported by the Austrian Science Fund (FWF) under the SFB grant F1322.

G. Regensburger (✉)
Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences, Altenbergerstraße 69, 4040 Linz, Austria
e-mail: georg.regensburger@oeaw.ac.at

is an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. The first known example is the Haar wavelet [16]

$$\psi(x) = \begin{cases} 1, & \text{for } 0 \leq x < \frac{1}{2}, \\ -1, & \text{for } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Daubechies [12] introduced a general method to construct compactly supported wavelets. It is based on *scaling functions* which satisfy a *dilation equation*

$$\phi(x) = \sum_{k=0}^{N} h_k \phi(2x - k) \tag{1}$$

given by a linear combination of real *filter coefficients* h_k and dilated and translated versions of the scaling function. We outline her construction in Sect. 2. The corresponding scaling function for the Haar wavelet is the box function

$$\phi(x) = \begin{cases} 1, & \text{for } 0 \leq x < 1, \\ 0, & \text{otherwise} \end{cases}$$

with the filter coefficients $h_0 = h_1 = 1$. In general, there is no closed analytic form for the scaling function, and for computations with wavelets only the filter coefficients are used.

Conditions on the scaling function imply, using the dilation equation (1), constraints on the filter coefficients. Orthonormality gives quadratic equations and vanishing moments of the associated wavelet and normalization linear constraints. For the existence of a wavelet at least one vanishing moment is necessary. Daubechies wavelets [12] have the maximal number of vanishing moments for a fixed number of filter coefficients and so there are only finitely many solutions. See Sect. 2 for details.

Parametrizing all possible filter coefficients that correspond to compactly supported orthonormal wavelets has been studied by several authors [20,25,29,31,34,37–39]. For a discussion and illustrations of scaling functions with six filter coefficients depending on two parameters see also [3] and [18]. Applications of parametrized wavelets to compression are for example discussed in [17] and [30]. In all parametrizations the filter coefficients are expressed in terms of trigonometric functions and there is no natural interpretation of the angular parameters for the resulting scaling function. Furthermore, one has to solve transcendental constraints for the parameters to find wavelets with more than one vanishing moment.

In the proposed parametrization we introduce the first discrete moments of the filter coefficients as parameters. The discrete moments can be expressed in terms of the continuous moments of the scaling function, see Sect. 3. Moreover, we do not want to parametrize all possible filter coefficients but only such with a high number of vanishing moments. More precisely, we omit one vanishing moment condition from the construction of Daubechies wavelets. We also use the fact that the even discrete moments are determined by the odd up to the number of vanishing moments, see Sect. 3. We discussed a first parametrization using the same approach in [30]. In this paper, we present new simplified parametrizations, discuss all computational aspects and different cases in detail, and give a parametrization for ten filter coefficients and at least four vanishing moments.

We solve the resulting parametrized polynomial equations for the filter coefficients using symbolic computation and for the more involved equations in particular Gröbner bases. Gröbner bases were introduced by Buchberger in [4], see also [5]. For further details on Gröbner bases we refer to [1,6,11]. Applications of Gröbner bases to the design of wavelets and filter coefficients are for example discussed in [8,9,15,23,24, 27,28,32]. The idea of using the first discrete moment as a parameter to simplify the Gröbner basis computations was also used in Selesnick and Burrus [32] and Lebrun and Selesnick [23].

In Sects. 4–7 we describe in detail the cases of four to ten filter coefficients. We give explicit parametrizations and discuss several special parameter values, for example, for the Daubechies wavelets. The corresponding Maple worksheet with all computations, several MATLAB functions and a GUI to compute with and illustrate parametrized wavelets are available on request from the author.

2 Equations for the filter coefficients

We outline the construction of orthonormal wavelets based on scaling functions and recall the polynomial equations for the filter coefficients, see for example Daubechies [13] or Strang and Nguyen [35].

Orthonormality of the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, that is,

$$\int \phi(x)\phi(x-l)dx = \delta_{0,l}$$

implies, using the dilation equation (1), the quadratic equations

$$\sum_{k\in\mathbb{Z}} h_k h_{k-2l} = 2\delta_{0,l}, \quad \text{for } l \in \mathbb{Z}, \tag{2}$$

where we set $h_k = 0$ for $k < 0$ and $k > N$. We can assume that $h_0 h_N \neq 0$. Then with Eq. (2) we see that N must be odd and the number of filter coefficients even. We have one nonhomogeneous equation

$$\sum_{k=0}^{N} h_k^2 = 2 \tag{3}$$

and the homogeneous equations

$$\sum_{k=0}^{N} h_k h_{k-2l} = 0, \quad \text{for } l = 1, \ldots, (N-1)/2. \tag{4}$$

If the filter coefficients satisfy the necessary conditions for orthogonality (2) and the normalization

$$\sum_{k=0}^{N} h_k = 2, \tag{5}$$

there exists a unique solution of the dilation equation (1) in $L^2(\mathbb{R})$ with support $[0, N]$ and for which $\int \phi = 1$, see Lawton [21]. For almost all such scaling functions the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ are orthogonal, and then

$$\psi(x) = \sum_{k=0}^{N} (-1)^k h_{N-k} \phi(2x-k) \tag{6}$$

is an orthonormal wavelet. Necessary and sufficient conditions for orthonormality were given by Cohen [10] and Lawton [22], see also Daubechies [13, Chap. 6.3.]. The only example with four filter coefficients that satisfies the Eqs. (2) and (5) and where the integer translates of the corresponding scaling are not orthogonal is $h_0 = h_3 = 1$ and $h_1 = h_2 = 0$ with the scaling function

$$\phi(x) = \begin{cases} 1/3, & \text{for } 0 \le x < 3, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Vanishing moments of the associated wavelet are related to several properties of the scaling function and wavelet. For example, to the smoothness, the polynomial reproduction and the approximation order of the scaling function, and the decay of the wavelet coefficients for smooth functions, see Strang and Nguyen [35] and the survey by Unser and Blu [36] for details. The condition that the first p moments of the wavelet ψ vanish, that is,

$$\int x^l \psi(x)\, dx = 0, \quad \text{for } l = 0, \ldots, p-1$$

is using Eq. (6) equivalent to the *sum rules*

$$\sum_{k=0}^{N} (-1)^k k^l h_k = 0, \quad \text{for } l = 0, \ldots, p-1. \tag{8}$$

We say that ψ has p vanishing moments. Since the vector space of all polynomials with degree less then p is invariant under translation and dilation, we can equivalently require vanishing moments of $\psi(x+n-1)$ with $N = 2n-1$. This corresponds to Daubechies choice [12,13] where the wavelet has support $[1-n, n]$. For the computations we use the resulting linear equations

$$\sum_{k=0}^{2n-1} (-1)^{n-k} h_k (n-k)^l = 0, \quad \text{for } l = 0, \ldots, p-1$$

since they have smaller coefficients. Note that the normalization of the filter coefficients (5) and the first sum rule

$$\sum_{k=0}^{N}(-1)^k h_k = 0 \tag{9}$$

are equivalent to

$$\sum_{\substack{k=0\\k \text{ even}}}^{N} h_k = \sum_{\substack{k=0\\k \text{ odd}}}^{N} h_k = 1. \tag{10}$$

The following proposition is a consequence of the first Newton identities, which give a relation between power sums and elementary symmetric functions, see Bourbaki [2, A.IV. 70] and Knuth [19, p. 497].

Proposition 1 *Let $x_0, \ldots, x_n$ be variables of a polynomial ring over a commutative ring. Then*

$$\left(\sum_{k=0}^{n} x_k^2\right) = \left(\sum_{k=0}^{n} x_k\right)^2 - 2\left(\sum_{\substack{0\le i<j\le n\\ j-i \text{ even}}} x_i x_j\right) - 2\left(\sum_{\substack{k=0\\k \text{ even}}}^{n} x_k\right)\left(\sum_{\substack{k=0\\k \text{ odd}}}^{n} x_k\right). \tag{11}$$

Proof The Newton identities tell us in particular that

$$\left(\sum_{k=0}^{n} x_k^2\right) = \left(\sum_{k=0}^{n} x_k\right)^2 - 2\left(\sum_{0\le i<j\le n} x_i x_j\right).$$

The last sum in this equation is

$$\left(\sum_{0\le i<j\le n} x_i x_j\right) = \left(\sum_{\substack{0\le i<j\le n\\ j-i \text{ even}}} x_i x_j\right) + \left(\sum_{\substack{0\le i<j\le n\\ j-i \text{ odd}}} x_i x_j\right)$$

and the proposition follows by observing that

$$\left(\sum_{\substack{0\le i<j\le n\\ j-i \text{ odd}}} x_i x_j\right) = \left(\sum_{\substack{k=0\\k \text{ even}}}^{n} x_k\right)\left(\sum_{\substack{k=0\\k \text{ odd}}}^{n} x_k\right).$$

$\square$

If the filter coefficients satisfy the homogeneous equations (4) from the orthonormality conditions then

$$\sum_{\substack{0\leq i<j\leq n\\ j-i \text{ even}}} h_i h_j = 0.$$

Therefore we see with the identity (11) that the normalization and the first sum rule, see Eqs. (5), (9) and (10) together with (4) imply the nonhomogeneous equation (3). So we can replace the quadratic equation (3) by the linear equation (9), which simplifies the computations.

3 Discrete and continuous moments

In this section, we discuss relations between the *discrete moments*

$$m_n = \sum_{k=0}^{N} h_k k^n$$

of the filter coefficients and the *continuous moments* of the scaling function

$$M_n = \int x^n \phi(x)\, dx.$$

We first recall a well-known recursive relation between discrete and continuous moments, see for example Strang and Nguyen [35, p. 396].

Let ϕ be a scaling function satisfying $M_0 = \int \phi = 1$. Then $m_0 = 2$ and

$$M_n = \frac{1}{2^{n+1}-2} \sum_{i=1}^{n} \binom{n}{i} m_i M_{n-i},$$

$$m_n = \left(2^{n+1}-2\right) M_n - \sum_{i=1}^{n-1} \binom{n}{i} m_i M_{n-i}, \quad \text{for } n > 0.$$

Using the recursion we obtain for the first moments

$$M_1 = 1/2\, m_1$$
$$M_2 = 1/6\, m_1^2 + 1/6\, m_2$$
$$M_3 = 1/28\, m_1^3 + 1/7\, m_1 m_2 + 1/14\, m_3$$

and

$$
\begin{aligned}
m_1 &= 2\,M_1 \\
m_2 &= -4\,M_1^2 + 6\,M_2 \\
m_3 &= 12\,M_1^3 - 24\,M_1 M_2 + 14\,M_3.
\end{aligned}
$$

Explicit formulas expressing the discrete moments in terms of the continuous and vice versa are given in [30].

For the parametrization of the filter coefficients we use the fact that the even moments are determined by the odd moments up to the number of vanishing moments, see [30]. In more detail, if the first two moments of the associated wavelet vanish, then

$$m_2 = m_1^2/2, \tag{12}$$

and if the first four moments vanish, we additionally have

$$m_4 = -1/2\,m_1^4 + 2\,m_1^2 m_2 + 2\,m_1 m_3 - 7/2\,m_2^2 = -3/8\,m_1^4 + 2\,m_1 m_3. \tag{13}$$

4 Four filter coefficients

In the case of four filter coefficients, we have the following system equations (normalization, first sum rule, parameter $m = m_1$, and orthogonality):

$$
\begin{aligned}
h_0 + h_1 + h_2 + h_3 &= 2 \\
h_0 - h_1 + h_2 - h_3 &= 0 \\
h_1 + 2\,h_2 + 3\,h_3 &= m \\
h_0 h_2 + h_1 h_3 &= 0.
\end{aligned}
$$

We solve the three linear equations for h_0, substitute the solution into the quadratic equation, and obtain

$$-2\,{h_0}^2 + (5 - m)h_0 - 1/4\,m^2 + 2\,m - 15/4 = 0. \tag{14}$$

We first consider the solution

$$h_0 = 5/4 - 1/4\,m - 1/4\sqrt{-m^2 + 6\,m - 5}.$$

Since

$$-m^2 + 6\,m - 5 = -(m-1)(m-5), \tag{15}$$

we can choose $m \in [1, 5]$ to get real filter coefficients. We set $m = a + 3$ to obtain parameter values symmetrically around zero. This correspond to a Tschirnhaus transformation for the polynomial (15) and simplifies the expression for the filter coefficients. Substituting the solution for h_0 into the solution for the linear equations we get:

$$\begin{aligned} h_0 &= 1/2 - 1/4\,a - 1/4\,w \\ h_1 &= 1/2 - 1/4\,a + 1/4\,w \\ h_2 &= 1/2 + 1/4\,a + 1/4\,w \\ h_3 &= 1/2 + 1/4\,a - 1/4\,w \end{aligned} \tag{16}$$

with $w = \sqrt{4 - a^2}$ and $a = m - 3 \in [-2, 2]$.

Notice that for $a = -a$ we obtain the flipped filter coefficients.

4.1 Special parameter values

For $a = 0$ we get the filter coefficients $(0, 1, 1, 0)$, which correspond to a translated Haar scaling function and wavelet. The parameter values $a = -2, 2$ give also Haar scaling functions with the filter coefficients $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$.

The Daubechies wavelet has two vanishing moments, so we have one more sum rule

$$2\,h_0 - h_1 + h_3 = 0.$$

Substituting the parametrized filter coefficients into this equations and solving for a, we get the two solutions $a = -\sqrt{3}, \sqrt{3}$ with the first discrete moments $m = 3 - \sqrt{3}, 3 + \sqrt{3}$. The first solution gives the famous Daubechies filters [12]

$$1/4\,(1 + \sqrt{3}, 3 + \sqrt{3}, 3 - \sqrt{3}, 1 - \sqrt{3}) \tag{17}$$

and the second the flipped version.

For $a = -8/5$ we get the rational filters $(3/5, 6/5, 2/5, -1/5)$. These rational filter coefficients give the smoothest scaling function with respect to the Hölder continuity, see Daubechies [13, p. 242].

4.2 Second root

If we choose the second root

$$h_0 = 5/4 - 1/4\,m + 1/4\,\sqrt{-m^2 + 6\,m - 5}$$

for the quadratic equation (14) and apply again the Tschirnhaus transformation $m = a + 3$, we obtain the parametrized filter coefficients:

$$\begin{aligned}
h_0 &= 1/2 - 1/4\,a + 1/4\,w \\
h_1 &= 1/2 - 1/4\,a - 1/4\,w \\
h_2 &= 1/2 + 1/4\,a - 1/4\,w \\
h_3 &= 1/2 + 1/4\,a + 1/4\,w
\end{aligned}$$

with $w = \sqrt{4 - a^2}$ and $a = m - 3 \in [-2, 2]$.

Comparing this solution with the parametrized filter coefficients (16), we see that w is replaced by $-w$ and so the two first and the two last filter coefficients are swapped. Notice that again for $a = -a$ we obtain the flipped filters.

For $a = 0$ we now get the filter coefficients $(1, 0, 0, 1)$, which give the scaling function (7) where the integer translates of the scaling function are not orthogonal. The parameter values $a = -2, 2$ also give Haar scaling functions with the filter coefficients $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. This parametrization does not contain filter coefficients with a second vanishing moment. The corresponding scaling functions are, compared to the parametrization (16), irregular.

5 Six filter coefficients

For six filter coefficients we have two vanishing moments, and we can use the relation $m_2 = m_1^2/2$, see Eq. (12). This gives an additional linear constraint, and we have the following linear equations with $m = m_1$:

$$\begin{aligned}
h_0 + h_1 + h_2 + h_3 + h_5 + h_4 &= 2 \\
-h_0 + h_1 - h_2 + h_3 - h_4 + h_5 &= 0 \\
-3\,h_0 + 2\,h_1 - h_2 + h_4 - 2\,h_5 &= 0 \\
h_1 + 2\,h_2 + 3\,h_3 + 4\,h_4 + 5\,h_5 &= m \\
h_1 + 4\,h_2 + 9\,h_3 + 16\,h_4 + 25\,h_5 &= m^2/2
\end{aligned}$$

and the quadratic equations

$$\begin{aligned}
h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 &= 0 \\
h_0h_4 + h_1h_5 &= 0.
\end{aligned}$$

We solve the linear equations for h_0, substitute the solution into the quadratic equations and obtain:

$$\begin{aligned}
&-8\,{h_0}^2 + (1/2\,m^2 - 7\,m + 21)h_0 - \frac{1}{64}\,m^4 + \frac{3}{8}\,m^3 - \frac{13}{4}\,m^2 + 12\,m - \frac{253}{16} = 0 \\
&2\,{h_0}^2 + \left(-1/8\,m^2 + \frac{7}{4}\,m - \frac{21}{4}\right)h_0 + \frac{1}{256}\,m^4 - \frac{3}{32}\,m^3 + \frac{13}{16}\,m^2 - 3\,m + \frac{253}{64} = 0.
\end{aligned} \tag{18}$$

Since the first equation is minus four times the second equation, we have, as in the case of four filter coefficients, only one quadratic equation to solve. We first consider the solution

$$h_0 = \frac{21}{16} - \frac{7}{16}m + \frac{1}{32}m^2 - \frac{1}{32}\sqrt{-m^4 + 20\,m^3 - 136\,m^2 + 360\,m - 260}.$$

The Tschirnhaus transformation $m = a + 5$ for the polynomial

$$-m^4 + 20\,m^3 - 136\,m^2 + 360\,m - 260$$

yields

$$-a^4 + 14\,a^2 + 15 = -\left(a^2 - 15\right)\left(a^2 + 1\right).$$

So we get real filter coefficients for $a \in [-\sqrt{15}, \sqrt{15}]$ or the first discrete moment $m \in [5 - \sqrt{15}, 5 + \sqrt{15}]$. Substituting the solution for h_0 into the solution for the linear equations, we get the following parametrized filter coefficients with at least two vanishing moments:

$$\begin{aligned}
h_0 &= -3/32 - 1/8\,a + 1/32\,a^2 - 1/32\,w \\
h_1 &= 5/32 - 1/8\,a + 1/32\,a^2 + 1/32\,w \\
h_2 &= 15/16 - 1/16\,a^2 + 1/16\,w \\
h_3 &= 15/16 - 1/16\,a^2 - 1/16\,w \\
h_4 &= 5/32 + 1/8\,a + 1/32\,a^2 - 1/32\,w \\
h_5 &= -3/32 + 1/8\,a + 1/32\,a^2 + 1/32\,w
\end{aligned} \tag{19}$$

with $w = \sqrt{-a^4 + 14\,a^2 + 15}$ and $a = m - 5 \in [-\sqrt{15}, \sqrt{15}]$.

5.1 Special parameter values

The Daubechies wavelet has one more vanishing moment, that is, it satisfies the sum rule

$$-9\,h_0 + 4\,h_1 - h_2 - h_4 + 4\,h_5 = 0.$$

Substituting the parametrized filter coefficients into this equations and solving for a, we get one real solution $a = -\sqrt{5 + 2\sqrt{10}}$, which gives the filter coefficients

$$\begin{aligned}
1/16\,(1 + \sqrt{10} + w,\, 5 + \sqrt{10} + 3\,w,\, 10 - 2\sqrt{10} + 2\,w,\\
10 - 2\sqrt{10} - 2\,w,\, 5+\sqrt{10} - 3\,w,\, 1+\sqrt{10}-w)
\end{aligned} \tag{20}$$

with $w = \sqrt{5 + 2\sqrt{10}}$.

The Daubechies filters with four nonzero filter coefficients (17) satisfy two sum rules and are therefore contained in this parametrization. Their first discrete moment is $m = 3 - \sqrt{3}$. So here the corresponding parameter is $a = -2 - \sqrt{3}$. We get a translated version for $a = -\sqrt{3}$.

For $a = -\sqrt{15}$ we obtain

$$1/8\,(3 + \sqrt{15}, 5 + \sqrt{15}, 0, 0, 5 - \sqrt{15}, 3 - \sqrt{15}).$$

The parameter $a = -1$ gives the first coiflet

$$1/16\,(1 - \sqrt{7}, 5 + \sqrt{7}, 14 + 2\sqrt{7}, 14 - 2\sqrt{7}, 1 - \sqrt{7}, -3 + \sqrt{7}),$$

see Daubechies [14] and [13, Chap. 8.2.]. For $a = 0$ we get

$$1/32\,(-3 - \sqrt{15}, 5 + \sqrt{15}, 30 + 2\sqrt{15}, 30 - 2\sqrt{15}, 5 - \sqrt{15}, -3 + \sqrt{15}).$$

The corresponding scaling functions and wavelets for $a > 0$ become increasingly irregular.

5.2 Second root

If we choose the second solution for the quadratic equation (18) and apply the Tschirnhaus transformation $m = a + 5$, we obtain:

$$\begin{aligned}
h_0 &= -3/32 - 1/8\,a + 1/32\,a^2 + 1/32\,w \\
h_1 &= 5/32 - 1/8\,a + 1/32\,a^2 - 1/32\,w \\
h_2 &= 15/16 - 1/16\,a^2 - 1/16\,w \\
h_3 &= 15/16 - 1/16\,a^2 + 1/16\,w \\
h_4 &= 5/32 + 1/8\,a + 1/32\,a^2 + 1/32\,w \\
h_5 &= -3/32 + 1/8\,a + 1/32\,a^2 - 1/32\,w
\end{aligned}$$

with $w = \sqrt{-a^4 + 14\,a^2 + 15}$ and $a = m - 5 \in [-\sqrt{15}, \sqrt{15}]$.

Notice that substituting $a = -a$ gives the flipped filter coefficients from the parametrization (19).

6 Eight filter coefficients

For eight filter coefficients we have three vanishing moments, and we can use as in the previous section the relation $m_2 = 1/2\,m_1^2$, see Eq. (12). We have the following six linear equations with $m = m_1$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 & -1 & 2 & -3 & 4 \\ -9 & 4 & -1 & 0 & -1 & 4 & -9 & 16 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 49 & 36 & 25 & 16 & 9 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} h_7 \\ h_6 \\ h_5 \\ h_4 \\ h_3 \\ h_2 \\ h_1 \\ h_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ m \\ 1/2\, m^2 \end{pmatrix} \tag{21}$$

and the quadratic equations

$$\begin{aligned} h_0h_2 + h_1h_3 + h_3h_5 + h_2h_4 + h_4h_6 + h_5h_7 &= 0 \\ h_0h_4 + h_1h_5 + h_3h_7 + h_2h_6 &= 0 \\ h_0h_6 + h_1h_7 &= 0. \end{aligned}$$

We solve the linear equations for h_0 and h_1 and substitute the solutions into the quadratic equations. Then we compute a Gröbner basis with respect to the lexicographic order with $h_1 >_{\text{lex}} h_0$ treating m as a parameter, that is, we compute a Gröbner basis in $\mathbb{Q}(m)[h_1, h_0]$.

The Gröbner basis has two elements. The first element is a quadratic polynomial in h_0 and the second linear in h_1 and h_0. We consider the following solution for the quadratic equation from the Gröbner basis

$$h_0 = -\frac{1}{512}\,\frac{m^5 - 42\,m^4 + 684\,m^3 - 5416\,m^2 + 20840\,m - 31088 + w}{m^2 - 14\,m + 50}$$

with $w =$

$$\sqrt{-(m^8 - 56\,m^7 + 1336\,m^6 - 17696\,m^5 + 141792\,m^4 - 699328\,m^3 + 2049600\,m^2 - 3186176\,m + 1891904)(m-8)^2}.$$

We set $m = a + 7$, which corresponds to a Tschirnhaus transformation for the first factor of the polynomial under the square root in w, and obtain

$$h_0 = -\frac{1}{512}\,\frac{a^5 - 7\,a^4 - 2\,a^3 + 30\,a^2 - 55\,a - 15 + w}{a^2 + 1}$$

with

$$w = \sqrt{-(a^8 - 36\,a^6 + 182\,a^4 - 1540\,a^2 + 945)(a-1)^2}. \tag{22}$$

To get real filter coefficients, we can choose a in

$$[-\sqrt{\beta}, -\sqrt{\alpha}] \quad \text{or} \quad [\sqrt{\alpha}, \sqrt{\beta}], \tag{23}$$

where α denotes the smaller and β the larger real root of

$$x^4 - 36\,x^3 + 182\,x^2 - 1540\,x + 945,$$

with numerical approximations

$$\sqrt{\alpha} = 0.8113601077\ldots \quad \text{and} \quad \sqrt{\beta} = 5.636256558\ldots$$

We substitute the solution for h_0 into the linear equation from the Gröbner basis, solve for h_1 and obtain with w as in (22)

$$h_1 = -\frac{1}{512}\,\frac{a^6 - 10\,a^5 + 39\,a^4 - 28\,a^3 - 25\,a^2 + 86\,a - 63 - (1+a)w}{a^3 - a^2 + a - 1}.$$

The denominator

$$a^3 - a^2 + a - 1 = (a-1)(a^2+1)$$

is zero for $a = 1$. We first assume $a < 1$. Then we can also simplify the root (22) and obtain with the solution for the linear equations (21) the following parametrized filter coefficients with at least three vanishing moments:

$$\begin{aligned}
h_0 &= -\frac{1}{512}\,\frac{a^5 - 7\,a^4 - 2\,a^3 + 30\,a^2 - 55\,a - 15 + (1-a)w}{a^2+1}\\
h_1 &= -\frac{1}{512}\,\frac{a^5 - 9\,a^4 + 30\,a^3 + 2\,a^2 - 23\,a + 63 + (1+a)w}{a^2+1}\\
h_2 &= \frac{1}{512}\,\frac{3\,a^5 - 5\,a^4 - 102\,a^3 + 186\,a^2 - 261\,a + 35 + 3(1-a)w}{a^2+1}\\
h_3 &= \frac{1}{512}\,\frac{3\,a^5 - 11\,a^4 - 70\,a^3 + 358\,a^2 - 229\,a + 525 + 3(1+a)w}{a^2+1}\\
h_4 &= -\frac{1}{512}\,\frac{3\,a^5 + 11\,a^4 - 70\,a^3 - 358\,a^2 - 229\,a - 525 + 3(1-a)w}{a^2+1}\\
h_5 &= -\frac{1}{512}\,\frac{3\,a^5 + 5\,a^4 - 102\,a^3 - 186\,a^2 - 261\,a - 35 + 3(1+a)w}{a^2+1}\\
h_6 &= \frac{1}{512}\,\frac{a^5 + 9\,a^4 + 30\,a^3 - 2\,a^2 - 23\,a - 63 + (1-a)w}{a^2+1}\\
h_7 &= \frac{1}{512}\,\frac{a^5 + 7\,a^4 - 2\,a^3 - 30\,a^2 - 55\,a + 15 + (1+a)w}{a^2+1}
\end{aligned} \tag{24}$$

with

$$w = \sqrt{-a^8 + 36\,a^6 - 182\,a^4 + 1540\,a^2 - 945},$$

$a = m - 7 < 1$ and a in the intervals (23).

If we choose the second root for the quadratic equation from the Gröbner basis and perform the same computations as before with the assumption $a < 1$, we obtain the filter coefficients (24) with w replaced by $-w$.

6.1 Different order on the variables

We now compute a Gröbner basis with respect to the lexicographic order with $h_0 >_{\mathrm{lex}} h_1$. The Gröbner basis has again two elements. The first element is a quadratic polynomial in h_1 and the second linear in h_0 and h_1.

We consider the following solution for the quadratic equation from the Gröbner basis

$$h_1 = -\frac{1}{512}\,\frac{m^5 - 44\,m^4 + 772\,m^3 - 6704\,m^2 + 28712\,m - 48384 - w}{m^2 - 14\,m + 50}$$

with $w =$

$$\sqrt{-(m^8 - 56\,m^7 + 1336\,m^6 - 17696\,m^5 + 141792\,m^4 - 699328\,m^3 + 2049600\,m^2 - 3186176\,m + 1891904)(m-6)^2}.$$

We set again $a = m + 7$ and obtain

$$h_1 = -\frac{1}{512}\,\frac{a^5 - 9\,a^4 + 30\,a^3 + 2\,a^2 - 23\,a + 63 - w}{a^2 + 1}$$

with

$$w = \sqrt{-(a^8 - 36\,a^6 + 182\,a^4 - 1540\,a^2 + 945)(a+1)^2}. \tag{25}$$

We get real filter coefficients for a in the same intervals (23) as in the previous section. We substitute the solution for h_1 into the linear equation from the second Gröbner basis, solve for h_0 and obtain with w as in (25)

$$h_0 = -\frac{1}{512}\,\frac{a^6 - 6\,a^5 - 9\,a^4 + 28\,a^3 - 25\,a^2 - 70\,a - 15 + (a-1)w}{a^3 + a^2 + a + 1}.$$

The denominator

$$a^3 + a^2 + a + 1 = (a+1)(a^2+1)$$

is zero for $a = -1$. We assume $a > -1$. Then we can also simplify the root (25) and obtain with the solution for the linear equations (21) the filter coefficients from Eq. (24) with w replaced by $-w$. From the previous section we know that this parametrization is also valid for $a < 1$ and hence for a in the intervals (23). Notice that substituting $a = -a$ in this parametrization gives the flipped filter coefficients from Eq. (24).

If we choose the second root for the quadratic equation from the Gröbner basis and perform the same computations as before with the assumption $a > -1$, we obtain the filter coefficients (24). Therefore the parametrization (24) is also valid for a in the intervals (23).

6.2 Special parameter values

The Daubechies wavelet satisfies one more sum rule

$$64\,h_0 - 27\,h_1 + 8\,h_2 - h_3 + h_5 - 8\,h_6 + 27\,h_7 = 0.$$

Substituting the parametrized filter coefficients (24) into this equations and solving for a, we get two real solution $a = -\sqrt{\beta}, -\sqrt{\alpha}$, where α denotes the smaller and β the larger real root of

$$x^4 - 28\,x^3 + 126\,x^2 - 1260\,x + 1225$$

or numerically

$$a = -4.989213573\ldots, -1.029063869\ldots$$

The first parameter gives the Daubechies wavelet with extremal phase [13, p. 195] and the second the "least asymmetric" [13, p. 198]. Notice that the symbolic expression for the parameter a with the parametrization (24) give us a closed form representation of the filter coefficients of the Daubechies wavelet. Compare this with the results obtained by Chyzak et al. [9], where also Gröbner bases are used, and the different approach by Shann and Yen [33].

The Daubechies wavelet with six nonzero filter coefficients (20) has the first discrete moment $m = 5 - \sqrt{5 + 2\sqrt{10}}$, so the corresponding parameter value for the parametrization (24) is $a = -2 - \sqrt{5 + 2\sqrt{10}}$.

7 Ten filter coefficients

For ten filter coefficients we require four vanishing moments. We can therefore use the two relations $m_2 = 1/2\,m_1^2$ and $m_4 = -3/8\,m_1^4 + 2\,m_1 m_3$, see Eqs. (12) and (13). This gives two additional linear constraints and we have the following linear equations with the two parameters $a = m_1$ and $c = m_3$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -4 & 3 & -2 & 1 & 0 & -1 & 2 & -3 & 4 & -5 \\ 16 & -9 & 4 & -1 & 0 & -1 & 4 & -9 & 16 & -25 \\ -64 & 27 & -8 & 1 & 0 & -1 & 8 & -27 & 64 & -125 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 81 & 64 & 49 & 36 & 25 & 16 & 9 & 4 & 1 & 0 \\ 729 & 512 & 343 & 216 & 125 & 64 & 27 & 8 & 1 & 0 \\ 6561 & 4096 & 2401 & 1296 & 625 & 256 & 81 & 16 & 1 & 0 \end{pmatrix} \begin{pmatrix} h_9 \\ h_8 \\ h_7 \\ h_6 \\ h_5 \\ h_4 \\ h_3 \\ h_2 \\ h_1 \\ h_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ 1/2\,a^2 \\ c \\ -\frac{3}{8}\,a^4 + 2\,ac \end{pmatrix} \tag{26}$$

Table 1 Number of real solutions for f from (27)

Parameter a	# Real solutions for c
(1.6417, 7.6167]	Two
(7.6167, 9)	Four
9	Two, singular point
(9, 10.3832]	Four
(10.3832, 16.3583)	Two

and the quadratic equations

$$\begin{aligned} h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 + h_4h_6 + h_5h_7 + h_6h_8 + h_7h_9 &= 0 \\ h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 + h_4h_8 + h_5h_9 &= 0 \\ h_0h_6 + h_1h_7 + h_2h_8 + h_3h_9 &= 0 \\ h_0h_8 + h_1h_9 &= 0. \end{aligned}$$

We solve the linear equations for h_0 and substitute the solutions into the quadratic equations. We compute a Gröbner basis with respect to the lexicographic order with $h_0 >_{\text{lex}} c$ treating a as a parameter, that is, a Gröbner basis in $\mathbb{Q}(a)[h_0, c]$. The Gröbner basis consists of two elements. The first is the polynomial

$$\begin{aligned} f = {} & 81\,a^{12} - 2916\,a^{11} + 40716\,a^{10} - 864\,a^9c - 155520\,a^9 + 31104\,a^8c - 2354328\,a^8 \\ & - 496512\,a^7c + 2880\,a^6c^2 + 31658688\,a^7 + 3768768\,a^6c - 93312\,a^5c^2 \\ & - 102669504\,a^6 - 4056192\,a^5c + 1540224\,a^4c^2 - 3072\,a^3c^3 - 590398848\,a^5 \\ & - 176214528a^4c - 15303168a^3c^2 + 55296a^2c^3 + 6210049216a^4 + 1512364544a^3c \\ & + 97677312\,a^2c^2 - 489472\,ac^3 + 1024\,c^4 - 22429995264\,a^3 - 5357366784\,a^2c \\ & - 358511616\,ac^2 + 1419264\,c^3 + 41210318592\,a^2 + 8252955648\,ac \\ & + 548785152\,c^2 - 39607335936\,a - 4229148672\,c + 16394918400 \end{aligned} \tag{27}$$

in the two parameters a, c and has $\deg_a(f) = 12$ and $\deg_c(f) = 4$. All possible parameters must lie on the real algebraic curve defined by the polynomial f. This curve has genus eleven and two finite singular points with multiplicity two and coordinates

$$a = 9, \;\; c = 729/4 \pm 3/8\,\sqrt{210}. \tag{28}$$

We compute the discriminant f with respect to c. Approximating its zeros, we see that we have real solutions for c if the first discrete moment

$$a \in [1.641693500\ldots, 16.35830649\ldots].$$

The number of real solutions for c is given in Table 1.

The second element in the Gröbner basis is linear in h_0. We solve this polynomial for h_0 and obtain with the solution for the linear equations (26) the following parametrized filter coefficients with at least four vanishing moments:

$$
\begin{aligned}
h_0&=\frac{1}{36864}\,\frac{9a^6-180a^5+948a^4-48a^3c+9840a^3+960a^2c-116824a^2-9568ac+32c^2+384480a+31680c-482976}{a-9}\\
h_1&=-\frac{1}{36864}\,\frac{9a^6-144a^5+624a^4-48a^3c+1536a^3+768a^2c+12824a^2-5728ac+32c^2-237312a+12672c+665280}{a-9}\\
h_2&=-\frac{1}{9216}\,\frac{9a^6-180a^5+948a^4-48a^3c+8976a^3+960a^2c-99064a^2-9472ac+32c^2+257760a+30816c-151200}{a-9}\\
h_3&=\frac{1}{9216}\,\frac{9a^6-144a^5+624a^4-48a^3c+2544a^3+768a^2c-9976a^2-5824ac+32c^2-53280a+13536c+120960}{a-9}\\
h_4&=\frac{1}{6144}\,\frac{9a^6-180a^5+948a^4-48a^3c+8304a^3+960a^2c-88408a^2-9376ac+32c^2+216288a+29952c-151200}{a-9}\\
h_5&=-\frac{1}{6144}\,\frac{9a^6-144a^5+624a^4-48a^3c+3360a^3+768a^2c-24904a^2-5920ac+32c^2+27072a+14400c+12096}{a-9}\\
h_6&=-\frac{1}{9216}\,\frac{9a^6-180a^5+948a^4-48a^3c+7824a^3+960a^2c-82552a^2-9280ac+32c^2+202464a+29088c-151200}{a-9}\\
h_7&=\frac{1}{9216}\,\frac{9a^6-144a^5+624a^4-48a^3c+3984a^3+768a^2c-34264a^2-6016ac+32c^2+65952a+15264c-34560}{a-9}\\
h_8&=\frac{1}{36864}\,\frac{9a^6-180a^5+948a^4-48a^3c+7536a^3+960a^2c-79192a^2-9184ac+32c^2+195552a+28224c-151200}{a-9}\\
h_9&=-\frac{1}{36864}\,\frac{9a^6-144a^5+624a^4-48a^3c+4416a^3+768a^2c-40360a^2-6112ac+32c^2+88704a+16128c-60480}{a-9}
\end{aligned}
\tag{29}
$$

with $a \neq 9$, $c \in \mathbb{R}$ such that $f(a, c) = 0$ with f from (27).

7.1 Special parameter values

For the Daubechies wavelet we have an additional sum rule which we add to the linear equations (26). We solve the linear equations, substitute the solution into the quadratic equations and obtain four polynomials in the two parameters a and c. We compute a Gröbner basis with respect to the lexicographic order with $c >_{\text{lex}} a$. It consists of two polynomials. The first is a univariate polynomial of degree 16 in a. Solving for a, we obtain four real solutions $a = 9 \pm \sqrt{\alpha}, 9 \pm \sqrt{\beta}$, where α denotes the smaller and β the larger positive real root of

$$x^8 - 72x^7 + 1692x^6 - 20472x^5 - 3258\,x^4 + 1386504\,x^3 - 8218980\,x^2 - 1640520\,x + 16769025$$

or numerically

$$a = 2.387816036\ldots, 7.767314070\ldots, 10.23268592\ldots, 15.61218396\ldots$$

The second polynomial in the Gröbner basis has degree 15 in a but depends only linearly on c. So we can express the corresponding values for the parameter c in terms of the first discrete moment a and obtain the numerical approximations

$$c = 1.701845088\ldots, 109.6494477\ldots, 275.3639993\ldots, 953.0313413\ldots$$

The first choice for a and the corresponding c gives the Daubechies wavelet with extremal phase [13, p. 195] and the second the "least asymmetric" [13, p. 198]. The two other choices give the flipped versions. We have again a closed form of the filter coefficients of the Daubechies wavelet with the symbolic expression for the parameters a and c and the parametrization (29), compare with [9] and [33].

To compute the filter coefficients for $a = 9$, we solve the linear equations (26) with the parameter values (28) for h_0 and substitute the solution into the quadratic equations. Then we solve the four univariate polynomials and obtain two solutions for h_0 which give two different filter coefficients. The second choice for c from (28) gives the flipped filter coefficients.

Acknowledgments I would like to thank Otmar Scherzer for raising my interest in wavelets, Josef Schicho for interesting and helpful discussions, and the reviewers for useful remarks.

References

1. Becker, T., Weispfenning, V.: Gröbner Bases, Graduate Texts in Mathematics, vol. 141. Springer, New York (1993)
2. Bourbaki, N.: Algebra. II. Chap. 4–7. Elements of Mathematics (Berlin). Springer, Berlin (1990)
3. Bratteli, O., Jorgensen, P.: Wavelets through a Looking Glass. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston (2002)
4. Buchberger, B.: An algorithm for finding the bases elements of the residue class ring modulo a zero dimensional polynomial ideal (German). Ph.D. Thesis, University of Innsbruck (1965) (English translation published in [7])
5. Buchberger, B.: Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. Aequationes Math. **4**, 374–383 (1970)
6. Buchberger, B: Introduction to Gröbner bases. In: Buchberger, B., Winkler, F. (eds.) Gröbner Bases and Applications (Linz, 1998), Lond Math. Soc. Lect Note Ser., vol. 251, pp. 3–31. Cambridge University Press, Cambridge (1998)
7. Buchberger, B.: Bruno Buchberger's Ph.D. thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. J. Symb. Comput. **41**(3–4): 475–511 (2006) (Translated from the 1965 German original by Michael P. Abramson)
8. Charoenlarpnopparut, C., Bose, N.: Multidimensional FIR filter bank design using Gröbner bases. IEEE Trans. Circuits Syst., II, Analog Digit. Signal Process. **46**(12), 1475–1486 (1999)
9. Chyzak, F., Paule, P., Scherzer, O., Schoisswohl, A., Zimmermann, B.: The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval. Exp. Math. **10**(1), 67–86 (2001)
10. Cohen, A.: Ondelettes, analyses multirésolutions et filtres miroirs en quadrature. Ann. Inst. H. Poincaré Anal. Non Linéaire **7**(5), 439–459 (1990)
11. Cox, D., Little, J., O'Shea, D.: Ideals, Varieties, and Algorithms, 2nd edn. Undergraduate Texts in Mathematics. Springer, New York (1997). An introduction to computational algebraic geometry and commutative algebra
12. Daubechies, I.: Orthonormal bases of compactly supported wavelets. Comm. Pure Appl. Math. **41**(7), 909–996 (1988)
13. Daubechies, I.: Ten Lectures on Wavelets. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1992)
14. Daubechies, I.: Orthonormal bases of compactly supported wavelets. II. Variations on a theme. SIAM J. Math. Anal. **24**(2), 499–519 (1993)
15. Faugère, J.C., Moreau de Saint-Martin, F., Rouillier, F.: Design of regular nonseparable bidimensional wavelets using Gröbner basis techniques. IEEE Trans. Signal Process. **46**(4), 845–856 (1998)
16. Haar, A.: Zur Theorie der orthogonalen Funktionensysteme. (Erste Mitteilung.). Math. Ann. **69**, 331–371 (1910)
17. Hereford, J.M., Roach, D.W., Pigford, R.: Image compression using parameterized wavelets with feedback. pp. 267–277. SPIE (2003)
18. Jorgensen, P.E.T.: Matrix factorizations, algorithms, wavelets. Notices Am. Math. Soc. **50**(8), 880–894 (2003)
19. Knuth, D.E.: The art of computer programming. Vol. 1: Fundamental algorithms, 3rd edn. Addison-Wesley, Reading. xx, p. 650 (1997)

20. Lai, M.J., Roach, D.W.: Parameterizations of univariate orthogonal wavelets with short support. In: Approximation Theory, X, St Louis, MO, 2001, Innov. Appl. Math., pp. 369–384. Vanderbilt University Press, Nashville, TN (2002)
21. Lawton, W.M.: Tight frames of compactly supported affine wavelets. J. Math. Phys. **31**(8), 1898–1901 (1990)
22. Lawton, W.M.: Necessary and sufficient conditions for constructing orthonormal wavelet bases. J. Math. Phys. **32**(1), 57–61 (1991)
23. Lebrun, J., Selesnick, I.: Gröbner bases and wavelet design. J. Symb. Comput. **37**(2), 227–259 (2004)
24. Lebrun, J., Vetterli, M.: High-order balanced multiwavelets: Theory, factorization, and design. IEEE Trans. Signal Process. **49**(9), 1918–1930 (2001)
25. Lina, J.M., Mayrand, M.: Parametrizations for Daubechies wavelets. Phys. Rev. E (3) **48**(6), R4160–R4163 (1993)
26. Mallat, S.: A Wavelet Tour of Signal Processing. Academic, San Diego (1998)
27. Park, H.: Symbolic computation and signal processing. J. Symb. Comput. **37**(2), 209–226 (2004)
28. Park, H., Kalker, T., Vetterli, M.: Gröbner bases and multidimensional FIR multirate systems. Multidimensional Syst. Signal Process. **8**(1–2), 11–30 (1997)
29. Pollen, D.: $\mathrm{SU}_I(2, F[z, 1/z])$ for F a subfield of **C**. J. Am. Math. Soc. **3**(3), 611–624 (1990)
30. Regensburger, G., Scherzer, O.: Symbolic computation for moments and filter coefficients of scaling functions. Ann. Comb. **9**(2), 223–243 (2005)
31. Schneid, J., Pittner, S.: On the parametrization of the coefficients of dilation equations for compactly supported wavelets. Computing **51**(2), 165–173 (1993)
32. Selesnick, I.W., Burrus, C.S.: Maximally flat low-pass FIR filters with reduced delay. IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process. **45**(1), 53–68 (1998)
33. Shann, W.C., Yen, C.C.: On the exact values of orthonormal scaling coefficients of lengths 8 and 10. Appl. Comput. Harmon. Anal. **6**(1), 109–112 (1999)
34. Sherlock, B.G., Monro, D.M.: On the space of orthonormal wavelets. IEEE Trans. Signal Process. **46**(6), 1716–1720 (1998)
35. Strang, G., Nguyen, T.: Wavelets and filter banks. Wellesley–Cambridge Press, Wellesley (1996)
36. Unser, M., Blu, T.: Wavelet theory demystified. IEEE Trans. Signal Process. **51**(2), 470–483 (2003)
37. Wang, S.H., Tewfik, A.H., Zou, H.: Correction to 'parametrization of compactly supported orthonormal wavelets'. IEEE Trans. Signal Process. **42**(1), 208–209 (1994)
38. Wells, R.O. Jr.: Parametrizing smooth compactly supported wavelets. Trans. Am. Math. Soc. **338**(2), 919–931 (1993)
39. Zou, H., Tewfik, A.H.: Parametrization of compactly supported orthonormal wavelets. IEEE Trans. Signal Process. **41**(3), 1428–1431 (1993)