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On the product of projectors and generalized inverses

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We consider generalized inverses of linear operators on arbitrary vector spaces and study the question when their product in reverse order is again a generalized inverse. This problem is equivalent to the question when the product of two projectors is again a projector, and we discuss necessary and sufficient conditions in terms of their kernels and images alone. We give a new representation of the product of generalized inverses that does not require explicit knowledge of the factors. Our approach is based on implicit representations of subspaces via their orthogonals in the dual space. For Fredholm operators, the corresponding computations reduce to finite-dimensional problems. We illustrate our results with examples for matrices and linear ordinary boundary problems.

Keywords: generalized inverse; projector; reverse order law; Fredholm operator; linear boundary problem; duality

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1. Introduction

Analogues of the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ for bijective operators have been studied intensively for various kinds of generalized inverses. Most articles and books are concerned with the matrix case; see for example [1–11]. For infinite-dimensional vector spaces, usually additional topological structures like Banach or Hilbert spaces are assumed; see for example [12–15]. In our approach, we systematically exploit duality results that hold in arbitrary vector spaces and a corresponding duality principle for statements about generalized inverses and projectors; see Appendix A.

The validity of the reverse order law can be reduced to the question whether the product of two projectors is a projector (Section 2). This problem is studied in [16–18] for finite-dimensional vector spaces. We discuss necessary and sufficient conditions that carry over to arbitrary vector spaces and can be expressed in terms of the kernels and images of the respective operators alone (Section 4). Applying the duality principle leads to new conditions and a characterization of the commutativity of two projectors that generalizes a result from [19].

In Section 5, we translate the results for projectors to generalized inverses and obtain necessary and sufficient conditions for the reverse order law in arbitrary vector spaces.

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Based on these conditions, we give a short proof for the characterization in Theorem 5.3 of two operators such that the reverse holds for all inner inverses (also called g-inverses or $\{1\}$-inverses). Moreover, we show that there always exist algebraic generalized inverses (also called $\{1, 2\}$-inverses) of two operators A and B such that their product in reverse order is an algebraic generalized inverse of AB.

Assuming the reverse order law to hold, Theorem 6.2 gives a representation of the product of two outer inverses ($\{2\}$-inverses) that can be computed using only kernel and image of the outer inverses of the factors. In this representation, we rely on a description of the kernel of a composition using inner inverses (Section 3) and implicit representations of subspaces via their orthogonals in the dual space. Moreover, we avoid the computation of generalized inverses by using the associated transpose map. Examples for matrices illustrating the results are given in Section 7.

An important application for our results is given by linear boundary problems (Section 9). Their solution operators (Green's operators) are generalized inverses, and it is natural to express infinite dimensional solution spaces implicitly via the (homogeneous) boundary conditions they satisfy. Green's operators for ordinary boundary problems are Fredholm operators, for which we can check the conditions for the reverse order law algorithmically and compute the implicit representation of the product (Section 8). Hence we can test if the product of two (generalized) Green's operators is again a Green's operator, and we can determine which boundary problem it solves.

2. Generalized inverses

In this section, we first recall basic properties of generalized inverses. For further details and proofs, we refer to [15,20] and the references therein. Throughout this article, U, V and W always denote vector spaces over the same field F, and we use the notation $V_1 \leq V$ for a subspace V_1 of V.

Definition 2.1 Let $T\colon V \to W$ be linear. We call a linear map $G\colon W \to V$ an *inner inverse* of T if $TGT = T$ and an *outer inverse* of T if $GTG = G$. If G is an inner and an outer inverse of T, we call G an *algebraic generalized inverse* of T.

This terminology of generalized inverses is adopted from [20]; other sources refer to inner inverses as generalized inverses or g-inverses, whereas algebraic generalized inverses are also called reflexive generalized inverses. Also the notations $\{1\}$-inverse (resp. $\{2\}$- and $\{1, 2\}$-inverse) are used, which refer to the corresponding Moore–Penrose equations the generalized inverse satisfies.

Proposition 2.2 *Let $T\colon V \to W$ and $G\colon W \to V$ be linear. The following statements are equivalent:*

(i) *G is an outer inverse of T.*
(ii) *GT is a projector and* $\operatorname{Im} GT = \operatorname{Im} G$.
(iii) *GT is a projector and* $V = \operatorname{Im} G \oplus \operatorname{Ker} GT$.
(iv) *GT is a projector and* $W = \operatorname{Im} T + \operatorname{Ker} G$.
(v) *TG is a projector and* $\operatorname{Ker} TG = \operatorname{Ker} G$.
(vi) *TG is a projector and* $W = \operatorname{Ker} G \oplus \operatorname{Im} TG$.
(vii) *TG is a projector and* $\operatorname{Im} G \cap \operatorname{Ker} T = \{0\}$.

Corresponding to (vii) and (vi), for subspaces $B \leq V$ and $E \leq W$ with

$$B \cap \operatorname{Ker} T = \{0\} \quad \text{and} \quad W = E \oplus T(B),$$

we can construct an outer inverse G of T with $\operatorname{Im} G = B$ and $\operatorname{Ker} G = E$ as follows; cf. [15, Cor. 8.2]. We consider the projector Q with

$$\operatorname{Im} Q = T(B), \quad \operatorname{Ker} Q = E. \tag{1}$$

The restriction $T|_B : B \to T(B)$ is bijective since $B \cap \operatorname{Ker} T = \{0\}$, and we can define $G = (T|_B)^{-1} Q$. One easily verifies that G is an outer inverse of T with $\operatorname{Im} G = B$ and $\operatorname{Ker} G = E$. Since by Proposition 2.2(iii) we have $V = B \oplus T^{-1}(E)$, we define the projector P in analogy to Q by

$$\operatorname{Im} P = T^{-1}(E), \quad \operatorname{Ker} P = B. \tag{2}$$

Then, by definition and by Proposition 2.2, we have

$$GTG = G, \quad TG = Q \quad \text{and} \quad GT = 1 - P,$$

and G is determined uniquely by these equations. Hence an outer inverse depends only on the choice of the defining spaces B and E. We use the notations $G = \mathrm{O}(T, B, E)$ and $G = \mathrm{O}(T, P, Q)$ for P and Q as in (2) and (1).

Obviously, G is an outer inverse of T if and only if T is an inner inverse of G. Therefore, we get a result analogous to Proposition 2.2 for inner inverses by interchanging the role of T and G. The construction of inner inverses is not completely analogous to outer inverses, see [20, Prop. 1.3]. For subspaces $B \leq V$ and $E \leq W$ such that

$$V = \operatorname{Ker} T \oplus B \quad \text{and} \quad W = \operatorname{Im} T \oplus E, \tag{3}$$

an inner inverse G of T is given on $\operatorname{Im} T$ by $(T|_B)^{-1}$ and can be chosen arbitrarily on E. For such an inner inverse with $B = \operatorname{Im} GT$ and $E = \operatorname{Ker} TG$, we write $G \in \mathrm{I}(T, B, E)$.

For constructing algebraic generalized inverses, we start with direct sums as in (3), but require $\operatorname{Ker} G = E$ and $\operatorname{Im} G = B$. We use the notation $G = \mathrm{G}(T, B, E)$.

The following result for inner inverses is well known in the matrix case [8,17,21] and its elementary proof remains valid for arbitrary vector spaces.

Proposition 2.3 *Let $T_1 : V \to W$ and $T_2 : U \to V$ be linear with outer (resp. inner) inverses G_1 and G_2. Let $P = G_1 T_1$ and $Q = T_2 G_2$. Then $G_2 G_1$ is an outer (resp. inner) inverse of $T_1 T_2$ if and only if QP (resp. PQ) is a projector.*

Proof Let $G_2 G_1$ be an outer inverse of $T_1 T_2$, that is, $G_2 G_1 = G_2 G_1 T_1 T_2 G_2 G_1$. Multiplying with T_2 from the left and with T_1 from the right yields

$$T_2 G_2 G_1 T_1 = T_2 G_2 G_1 T_1 T_2 G_2 G_1 T_1,$$

thus $QP = T_2 G_2 G_1 T_1$ is a projector. For the other direction, we multiply the previous equation with G_2 from the left and G_1 from the right and use that $G_1 T_1 G_1 = G_1$ and $G_2 T_2 G_2 = G_2$. The proof for inner inverses follows by interchanging the roles of T_i and G_i. □

3. Kernel of compositions

We now describe the inverse image of a subspace under the composition of two linear maps using inner inverses. For projectors, kernel and image of the composition can be expressed in terms of kernel and image of the corresponding factors alone. Note that a projector is an inner inverse of itself.

PROPOSITION 3.1 *Let $T_1: V \to W$ and $T_2: U \to V$ be linear and G_2 an inner inverse of T_2. For a subspace $W_1 \leq W$, we have*

$$(T_1T_2)^{-1}(W_1) = G_2(T_1^{-1}(W_1) \cap \mathrm{Im}\,T_2) \oplus \mathrm{Ker}\,T_2$$

for the inverse image of the composition. In particular,

$$\mathrm{Ker}\,T_1T_2 = G_2(\mathrm{Ker}\,T_1 \cap \mathrm{Im}\,T_2) \oplus \mathrm{Ker}\,T_2.$$

Proof Since T_2G_2 is a projector onto $\mathrm{Im}\,T_2$ by Proposition 2.2(ii) (interchanging the role of T and G), we have

$$\begin{aligned} T_1T_2(G_2(T_1^{-1}(W_1) \cap \mathrm{Im}\,T_2) + \mathrm{Ker}\,T_2) &= T_1Q_2(T_1^{-1}(W_1) \cap \mathrm{Im}\,T_2) + 0 \\ &= T_1(T_1^{-1}(W_1) \cap \mathrm{Im}\,T_2) \leq W_1 \cap \mathrm{Im}\,T_1T_2 \leq W_1. \end{aligned}$$

Conversely, let $u \in (T_1T_2)^{-1}(W_1)$. Then $T_2u = v$ with $v \in T_1^{-1}(W_1)$. Since also $v \in \mathrm{Im}\,T_2$, we have

$$T_2(u - G_2v) = T_2u - Q_2v = T_2u - v = v - v = 0,$$

that is, $u - G_2v \in \mathrm{Ker}\,T_2$. Writing $u = G_2v + u - G_2v$ yields $u \in G_2(T_1^{-1}(W_1) \cap \mathrm{Im}\,T_2) + \mathrm{Ker}\,T_2$. The sum is direct since by Proposition 2.2(vi) (interchanging the role of T and G), we have $U = \mathrm{Ker}\,T_2 \oplus \mathrm{Im}\,G_2T_2$. $\square$

COROLLARY 3.2 *Let $T: V \to W$ be linear and let $P: V \to V$ and $Q: W \to W$ be projectors. Then*

$$\mathrm{Ker}\,TQ = (\mathrm{Ker}\,T \cap \mathrm{Im}\,Q) \oplus \mathrm{Ker}\,Q \quad \textit{and} \quad \mathrm{Im}\,PT = (\mathrm{Im}\,T + \mathrm{Ker}\,P) \cap \mathrm{Im}\,P.$$

Proof Applying Proposition 3.1 yields

$$\mathrm{Ker}\,TQ = Q(\mathrm{Ker}\,T \cap \mathrm{Im}\,Q) \oplus \mathrm{Ker}\,Q = (\mathrm{Ker}\,T \cap \mathrm{Im}\,Q) \oplus \mathrm{Ker}\,Q.$$

The statement for the image follows from the duality principle A.4. $\square$

This result generalizes [17, Lemma 2.2], where the kernel and image of a product PQ of two projectors are computed as above, when PQ is again a projector.

4. Products of projectors

In view of Proposition 2.3, we study necessary and sufficient conditions for the product of two projectors to be a projector. Throughout this section let $P, Q: V \to V$ denote projectors.

The first of the following necessary and sufficient conditions for the product of P and Q to be a projector is mentioned as an exercise without proof in [22, p. 339]. In [16, Lemma 3] the same result is formulated for matrices but the proof is valid for arbitrary vector spaces. The second necessary and sufficient condition for the matrix case is given in [17, Lemma 2.2]. The simpler proof from [18] carries over to arbitrary vector spaces.

LEMMA 4.1 *The composition PQ is a projector if and only if*

$$\operatorname{Im} PQ \leq \operatorname{Im} Q \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$$

if and only if

$$\operatorname{Im} Q \leq \operatorname{Im} P \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q).$$

We obtain the following characterization of the idempotency of PQ in terms of the kernels and images of P and Q alone.

THEOREM 4.2 *The following statements are equivalent:*

(i) *The composition PQ is a projector.*
(ii) $\operatorname{Im} P \cap (\operatorname{Im} Q + \operatorname{Ker} P) \leq \operatorname{Im} Q \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$
(iii) $\operatorname{Im} Q \leq \operatorname{Im} P \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$
(iv) $\operatorname{Ker} Q \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \geq \operatorname{Ker} P \cap (\operatorname{Im} Q + \operatorname{Im} P)$
(v) $\operatorname{Ker} P \geq \operatorname{Ker} Q \cap (\operatorname{Im} Q + \operatorname{Ker} P) \cap (\operatorname{Im} Q + \operatorname{Im} P)$

Proof The equivalence of (i), (ii) and (iii) follows from the previous lemma and Corollary 3.2. By the duality principle A.4, the last two conditions are equivalent to (ii) and (iii), respectively. □

For algebraic generalized inverses, it is also interesting to have sufficient conditions for PQ as well as QP to be projectors; for example, if P and Q commute. This can again be characterized in terms of the images and kernels of P and Q alone. If $PQ = QP$, one sees with Corollary 3.2 that

$$\operatorname{Im} PQ = \operatorname{Im} P \cap \operatorname{Im} Q \quad \text{and} \quad \operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q. \tag{4}$$

In general, these conditions are necessary but not sufficient for commutativity of P and Q, see [16, Ex. 1].

Using Corollary 3.2, modularity (A1) and (A2), one obtains the following characterization of projectors with image or kernel as in (4); for further details see [23]. For the commutativity of projectors see also [22, p. 339].

PROPOSITION 4.3 *The composition PQ is a projector with*

(i) $\operatorname{Im} PQ = \operatorname{Im} P \cap \operatorname{Im} Q$ *if and only if*

$$\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q).$$

(ii) $\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q$ *if and only if*

$$\operatorname{Ker} P = (\operatorname{Ker} P \cap \operatorname{Ker} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q).$$

COROLLARY 4.4 *We have $PQ = QP$ if and only if*

$$\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Im} Q)$$

and

$$\operatorname{Ker} Q = (\operatorname{Im} P \cap \operatorname{Ker} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q).$$

In [16, Thm. 4] and [19, Thm. 3.2] different necessary and sufficient conditions for the commutativity of two projectors are given, but both require the computation of PQ as well as of QP.

5. Reverse order law for generalized inverses

Proposition 2.3 and Theorem 4.2 together give necessary and sufficient conditions for the reverse order law for outer inverses to hold, in terms of the defining spaces B_i and E_i alone.

THEOREM 5.1 *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, B_1, E_1)$ and $G_2 = \mathrm{O}(T_2, B_2, E_2)$. The following conditions are equivalent:*

(i) *$G_2 G_1$ is an outer inverse of $T_1 T_2$.*
(ii) $T_2(B_2) \cap (B_1 + E_2) \leq B_1 \oplus (E_2 \cap T_1^{-1}(E_1))$
(iii) $B_1 \leq T_2(B_2) \oplus (E_2 \cap B_1) \oplus (E_2 \cap T_1^{-1}(E_1))$
(iv) $T_1^{-1}(E_1) \oplus (E_2 \cap B_1) \geq E_2 \cap (B_1 + T_2(B_2))$
(v) $E_2 \geq T_1^{-1}(E_1) \cap (B_1 + E_2) \cap (B_1 + T_2(B_2))$

Proof Recall that $\operatorname{Im} G_i = B_i$ and $\operatorname{Ker} G_i = E_i$, and $Q = T_2 G_2$ and $P = G_1 T_1$ are projectors with

$$\operatorname{Im} P = B_1, \quad \operatorname{Ker} P = T_1^{-1}(E_1), \quad \operatorname{Im} Q = T_2(B_2) \quad \text{and} \quad \operatorname{Ker} Q = E_2.$$

By Proposition 2.3, $G_2 G_1$ is an outer inverse if and only if QP is a projector. Applying Theorem 4.2 proves the claim. □

In the following theorem, we give the analogous conditions for inner inverses, where $P = G_1 T_1$ and $Q = T_2 G_2$ are the projectors corresponding to the direct sums in (3). Note that the conditions for inner inverses only depend on the choice of B_1 and E_2, but not on B_2 and E_1.

The characterization of (iii) and the orthogonal of (v) in the following theorem generalize [17, Thm. 2.3] to arbitrary vector spaces.

THEOREM 5.2 *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with inner inverses $G_1 \in \mathrm{I}(T_1, B_1, E_1)$ and $G_2 \in \mathrm{I}(T_2, B_2, E_2)$. The following conditions are equivalent:*

(i) *$G_2 G_1$ is an inner inverse of $T_1 T_2$.*
(ii) $B_1 \cap (\operatorname{Im} T_2 + \operatorname{Ker} T_1) \leq \operatorname{Im} T_2 \oplus (\operatorname{Ker} T_1 \cap E_2)$
(iii) $\operatorname{Im} T_2 \leq B_1 \oplus (\operatorname{Ker} T_1 \cap \operatorname{Im} T_2) \oplus (\operatorname{Ker} T_1 \cap E_2)$
(iv) $E_2 \oplus (\operatorname{Ker} T_1 \cap \operatorname{Im} T_2) \geq \operatorname{Ker} T_1 \cap (\operatorname{Im} T_2 + B_1)$
(v) $\operatorname{Ker} T_1 \geq E_2 \cap (\operatorname{Im} T_2 + \operatorname{Ker} T_1) \cap (\operatorname{Im} T_2 + B_1)$

The question when the reverse order law holds for all inner inverses of T_1 and T_2 was answered for matrices in [11, Thm. 2.3], and an alternative proof was given in [24]. Using the previous characterizations, we give a short proof that generalizes the result to arbitrary vector spaces.

THEOREM 5.3 *Let $T_1\colon V \to W$ and $T_2\colon U \to V$ be linear. Then G_2G_1 is an inner inverse of T_1T_2 for all inner inverses G_1 of T_1 and G_2 of T_2 if and only if $T_1T_2 = 0$ or* $\operatorname{Ker}T_1 \leq \operatorname{Im}T_2$.

Proof If $\operatorname{Ker}T_1 \leq \operatorname{Im}T_2$ then $\operatorname{Ker}T_1 \cap \operatorname{Im}T_2 = \operatorname{Ker}T_1$ and (iii) in Theorem 5.2 the previous theorem is satisfied since $\operatorname{Ker}T_1 + B_1 = V$. The case $T_1T_2 = 0$ is trivial.

For the reverse implication, assume that $\operatorname{Im}T_2$ is not contained in $\operatorname{Ker}T_1$ and $\operatorname{Ker}T_1$ is not contained in $\operatorname{Im}T_2$. Choose $V_1, V_2 \leq V$ such that we have two direct sums $\operatorname{Ker}T_1 = (\operatorname{Im}T_2 \cap \operatorname{Ker}T_1) \oplus V_1$ and $\operatorname{Im}T_2 = (\operatorname{Im}T_2 \cap \operatorname{Ker}T_1) \oplus V_2$. Then we have

$$\operatorname{Im}T_2 + \operatorname{Ker}T_1 = (\operatorname{Im}T_2 \cap \operatorname{Ker}T_1) \oplus V_1 \oplus V_2. \tag{5}$$

By assumption, we can choose non-zero $v_1 \in V_1$ and $v_2 \in V_2$. Let $v = v_1 + v_2$. Then $v \in \operatorname{Im}T_2 + \operatorname{Ker}T_1$ and $v \notin \operatorname{Ker}T_1$, $v \notin \operatorname{Im}T_2$. Hence we can choose B_1 and E_2 such that $v \in B_1$ and $v \in E_2$ and $V = \operatorname{Ker}T_1 \oplus B_1 = \operatorname{Im}T_2 \oplus E_2$. Then

$$v \in E_2 \cap (\operatorname{Im}T_2 + \operatorname{Ker}T_1) \cap (\operatorname{Im}T_2 + B_1)$$

but $v \in \operatorname{Ker}T_1$. Hence 5.2 in the previous theorem is not satisfied for inner inverses with $\operatorname{Im}G_1 = B_1$ and $\operatorname{Ker}G_2 = E_2$. $\square$

Werner [17, Thm. 3.1] proves that for matrices, it is always possible to construct inner inverses such that the reverse order law holds. Using the necessary and sufficient condition for outer inverses above, we extend this result to algebraic generalized inverses in arbitrary vector spaces. The special case of Moore–Penrose inverses is treated in [8, Thm. 3.2], and explicit solutions are constructed in [25,26].

THEOREM 5.4 *Let $T_1\colon V \to W$ and $T_2\colon U \to V$ be linear. There always exist algebraic generalized inverses G_1 of T_1 and G_2 of T_2 such that G_2G_1 is an algebraic generalized inverse of T_1T_2.*

Proof Choose $V_1, V_2 \leq V$ as in the previous proof such that (5) holds. Moreover, choose $V_3 \leq V$ such that

$$V = (\operatorname{Im}T_2 + \operatorname{Ker}T_1) \oplus V_3 = (\operatorname{Im}T_2 \cap \operatorname{Ker}T_1) \oplus V_1 \oplus V_2 \oplus V_3.$$

Then $B_1 = V_2 \oplus V_3$ is a direct complement of $\operatorname{Ker}T_1$ and $E_2 = V_1 \oplus V_3$ is a direct complement of $\operatorname{Im}T_2$. Hence, there exist respectively an algebraic generalized inverse G_1 of T_1 with $\operatorname{Im}G_1 = B_1$ and G_2 of T_2 with $\operatorname{Ker}G_2 = E_2$. We verify that such G_1 and G_2 satisfy Theorem 5.1(iii), where $T_1^{-1}(E_1) = \operatorname{Ker}T_1$ and $T_2(B_2) = \operatorname{Im}T_2$ since G_1 and G_2 are algebraic generalized inverses:

$$\operatorname{Im}T_2 \oplus (E_2 \cap B_1) \geq \operatorname{Im}T_2 \oplus V_3 = (\operatorname{Im}T_2 \cap \operatorname{Ker}T_1) \oplus V_2 \oplus V_3 \geq B_1.$$

Similarly, we verify Theorem 5.2(iii)

$$B_1 \oplus (\mathrm{Ker}T_1 \cap \mathrm{Im}T_2) = V_2 \oplus V_3 \oplus (\mathrm{Ker}T_1 \cap \mathrm{Im}T_2) \geq V_2 \oplus (\mathrm{Ker}T_1 \cap \mathrm{Im}T_2) = \mathrm{Im}T_2.$$

Hence G_2G_1 is an algebraic generalized inverse of T_1T_2 for all $G_1 = \mathrm{G}(T_1, B_1, E_1)$ and $G_2 = \mathrm{G}(T_2, B_2, E_2)$, independent of the choice of E_1 and B_2. □

6. Representing the product of outer inverses

In this section, we assume that for two linear maps $T_1 \colon V \to W$ and $T_2 \colon U \to V$ with outer inverses G_1 and G_2, respectively, the reverse order law holds. Our goal is to find a description of the product G_2G_1 that does not require the explicit knowledge of G_1 and G_2. Using the representation via projectors, one immediately verifies that

$$\mathrm{O}(T_2, P_2, Q_2)\,\mathrm{O}(T_1, P_1, Q_1) = \mathrm{O}(T_1T_2, P_2 - G_2P_1T_2, T_1Q_2G_1)$$

but this expression involves both outer inverses G_1 and G_2. For the representation via defining spaces, we compute the kernel and the image of the product.

Lemma 6.1 *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, B_1, E_1)$ and $G_2 = \mathrm{O}(T_2, B_2, E_2)$. Then*

$$\mathrm{Ker}G_2G_1 = E_1 \oplus T_1(B_1 \cap E_2) \quad \textit{and} \quad \mathrm{Im}G_2G_1 = G_2((B_1 + E_2) \cap \mathrm{Im}T_2).$$

Proof Recall that by definition $\mathrm{Ker}G_i = E_i$ and $\mathrm{Im}G_i = B_i$. The first identity follows directly from Proposition 3.1. For the second identity, we first note that for a linear map G and subspaces V_1, V_2, we have $G(V_1 \cap V_2) = G(V_1) \cap G(V_2)$ if $\mathrm{Ker}G \leq V_1$. Hence $G_2((B_1 + E_2) \cap \mathrm{Im}T_2)$ equals

$$G_2((\mathrm{Im}G_1 + \mathrm{Ker}G_2) \cap \mathrm{Im}T_2) = G_2(\mathrm{Im}G_1) \cap G_2(\mathrm{Im}T_2) = \mathrm{Im}G_2G_1,$$

since $G_2(\mathrm{Im}T_2) = \mathrm{Im}G_2$ by Proposition 2.2(ii). □

Note that the expression for the image of the composition requires the explicit knowledge of G_2. In particular, the reverse order law takes the form

$$\mathrm{O}(T_2, B_2, E_2)\,\mathrm{O}(T_1, B_1, E_1) = \mathrm{O}(T_1T_2, G_2((B_1 + E_2) \cap \mathrm{Im}T_2), E_1 + T_1(B_1 \cap E_2)).$$

Werner [17, Thm. 2.4] gives a result in a similar spirit for inner inverses of matrices.

Using an implicit description of $\mathrm{Im}G_i$, it is possible to state the reverse order law in a form that depends on the kernels and images of the respective outer inverses alone. This approach is motivated by our application to linear boundary problems (Section 9), where it is natural to define solution spaces via the boundary conditions they satisfy.

In more detail, the Galois connection from Appendix A allows to represent a subspace B implicitly via the orthogonally closed subspace $\mathscr{B} = B^{\perp}$ of the dual space. We will therefore use the notation $G = \mathrm{O}(T, \mathscr{B}, E)$ for the outer inverse with $\mathrm{Im}G = \mathscr{B}^{\perp}$ and $\mathrm{Ker}G = E$ as well as the analogue for inner inverses.

THEOREM 6.2 *Let $T_1 : V \to W$ and $T_2 : U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, \mathscr{B}_1, E_1)$ and $G_2 = \mathrm{O}(T_2, \mathscr{B}_2, E_2)$. If G_2G_1 is an outer inverse of T_1T_2, then*

$$\mathrm{O}(T_2, \mathscr{B}_2, E_2)\,\mathrm{O}(T_1, \mathscr{B}_1, E_1) = \mathrm{O}(T_1T_2, \mathscr{B}_2 \oplus T_2^*(\mathscr{B}_1 \cap E_2^\perp), E_1 \oplus T_1(\mathscr{B}_1^\perp \cap E_2)), \quad (6)$$

where T_2^* denotes the transpose of T_2.

Proof From Lemma 6.1, we already know that $\mathrm{Ker}G_2G_1 = E_1 \oplus T_1(\mathscr{B}_1^\perp \cap E_2)$. From Proposition A.2 and 3.1, we get

$$\begin{aligned}(\mathrm{Im}G_2G_1)^\perp &= \mathrm{Ker}G_1^*G_2^* = T_2^*(\mathrm{Ker}G_1^* \cap \mathrm{Im}G_2^*) \oplus \mathrm{Ker}G_2^* \\ &= T_2^*((\mathrm{Im}G_1)^\perp \cap (\mathrm{Ker}G_2)^\perp) \oplus (\mathrm{Im}G_2)^\perp = T_2^*(\mathscr{B}_1 \cap E_2^\perp) \oplus \mathscr{B}_2,\end{aligned}$$

and thus (6) holds. □

A computational advantage of this representation is that one can determine G_2G_1 directly by computing only one outer inverse instead of computing both G_1 and G_2; see the next section for an example.

7. Examples for matrices

In this section, we illustrate our results for finite-dimensional vector spaces. In particular, we show how to compute directly the composition of two generalized inverses using the reverse order law in the form (6).

Consider the following linear maps $T_1 : \mathbb{Q}^4 \to \mathbb{Q}^3$ and $T_2 : \mathbb{Q}^3 \to \mathbb{Q}^4$ given by

$$T_1 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 3 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 2 \\ -1 & 5 & 4 \\ -1 & 5 & 4 \end{pmatrix}.$$

We first use Theorems 5.1 and 5.2 to check whether for algebraic generalized inverses $G_1 = \mathrm{G}(T_1, B_1, E_1)$ and $G_2 = \mathrm{G}(T_2, B_2, E_2)$, the composition G_2G_1 is an algebraic generalized inverse of T_1T_2.

For testing the conditions, we only need to fix $B_1 = \mathrm{Im}G_1$ and $E_2 = \mathrm{Ker}G_2$, such that $B_1 \oplus \mathrm{Ker}T_1 = \mathbb{Q}^4 = E_2 \oplus \mathrm{Im}T_2$. We have

$$\mathrm{Ker}T_1 = \mathrm{span}((0, 1, 0, 1)^T, (0, 0, 1, 1)^T), \quad \mathrm{Im}T_2 = \mathrm{span}((1, 0, -2, -2)^T, (0, 1, 1, 1)^T),$$

so we may choose for example

$$B_1 = \mathrm{span}((1, 0, 0, 0)^T, (0, 1, 0, 0)^T), \quad E_2 = \mathrm{span}((1, 0, 0, 0)^T, (0, 0, 1, 0)^T).$$

For algebraic generalized inverses, we obtain as a necessary and sufficient condition for being an outer inverse

$$B_1 \leq \mathrm{Im}T_2 \oplus (E_2 \cap B_1) \oplus (E_2 \cap \mathrm{Ker}T_1)$$

from Theorem 5.1(iii).

Since $E_2 \cap \mathrm{Ker}T_1 = \{0\}$ and $E_2 \cap B_1 = \mathrm{span}((1, 0, 0, 0)^T)$, the right hand side yields that $\mathrm{span}((1, 0, 0, 0)^T, (0, 1, 0, 0)^T, (0, 0, 1, 1)^T) \geq B_1$. Thus for all algebraic generalized inverses G_1 and G_2 with $\mathrm{Im}G_1 = B_1$ and $\mathrm{Ker}G_2 = E_2$, the product G_2G_1 is an outer inverse of T_1T_2.

The corresponding condition for inner inverses by Theorem 5.2(iii) is

$$\mathrm{Im}T_2 \leq B_1 \oplus (\mathrm{Ker}T_1 \cap \mathrm{Im}T_2) \oplus (\mathrm{Ker}T_1 \cap E_2).$$

Since $\mathrm{Ker}T_1 \cap \mathrm{Im}T_2 = \{0\}$, the right hand side yields B_1, which does not contain $\mathrm{Im}T_2$. Hence for the above choices of G_1 and G_2, the product G_2G_1 is never an inner inverse of T_1T_2.

Since G_2G_1 is an outer inverse, Theorem 6.2 allows to determine G_2G_1 directly without knowing the factors. Identifying the dual space with row vectors, the orthogonals of B_1 and E_2 are given by

$$B_1^{\perp} = \mathscr{B}_1 = \mathrm{span}((0, 0, 1, 0), (0, 0, 0, 1)), \quad E_2^{\perp} = \mathrm{span}((0, 1, 0, 0), (0, 0, 0, 1)),$$

so we have $\mathscr{B}_1^{\perp} \cap E_2 = \mathrm{span}((1, 0, 0, 0)^T)$ and $\mathscr{B}_1 \cap E_2^{\perp} = \mathrm{span}((0, 0, 0, 1))$. For explicitly computing G_2G_1, we also have to choose $B_2 = \mathrm{Im}G_2$ and $E_1 = \mathrm{Ker}G_1$. Since we have

$$\mathrm{Im}T_1 = \mathrm{span}((1, 0, 3)^T), (0, 1, 2)^T), \quad \mathrm{Ker}T_2 = \mathrm{span}((1, 1, -1)^T),$$

we may choose the complements $E_1 = \mathrm{Ker}G_1$ and $B_2 = \mathrm{Im}G_2$ as

$$E_1 = \mathrm{span}((0, 0, 1)^T) \quad \text{and} \quad B_2 = \mathrm{span}((1, 0, 0)^T, (0, 1, 0)^T).$$

Using (6), we can determine the kernel

$$E = \mathrm{Ker}G_2G_1 = E_1 \oplus T_1(\mathscr{B}_1^{\perp} \cap E_2) = \mathrm{span}((1, 0, 0)^T, (0, 0, 1)^T).$$

The image of G_2G_1 is by (6) given via the orthogonal

$$(\mathrm{Im}G_2G_1)^{\perp} = \mathscr{B}_2 \oplus T_2^*(\mathscr{B}_1 \cap E_2^{\perp}) = \mathrm{span}((0, 0, 1), (-1, 5, 4)),$$

which means that $B = \mathrm{Im}G_2G_1 = \mathrm{span}((5, 1, 0)^T)$. Therefore, we can directly determine G as the unique outer inverse

$$G = \mathrm{O}(T_1T_2, B, E) = \begin{pmatrix} 0 & \frac{5}{12} & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One easily checks that G is an outer inverse of T.

8. Fredholm operators

We now turn to algorithmic aspects of the previous results. As already emphasized, for arbitrary vector spaces we can express conditions for the reverse order law in terms of the defining spaces alone. Nevertheless, in general it will not be possible to compute sums and intersections of infinite-dimensional subspaces. For algorithmically checking the conditions of Theorem 5.1 or 5.2 and for computing the reverse order law in the form (6), we consider finite (co)dimensional spaces and Fredholm operators.

Recall that a linear map T between vector spaces is called *Fredholm* operator if $\dim \mathrm{Ker}T < \infty$ and $\mathrm{codim}\, \mathrm{Im}T < \infty$. Moreover, for finite codimensional subspaces $V_1 \leq V$, we have $\mathrm{codim}\, V_1 = \dim V_1^{\perp}$. In this case, V_1 can be implicitly represented by the finite-dimensional subspace $V_1^\perp \leq V^*$. For an application to linear ordinary boundary problems, see the next section.

We assume that for finite-dimensional subspaces, we can compute sums and intersections and check inclusions, both in vector spaces and in their duals. With the following lemma, the intersection of a finite-dimensional subspace with a finite codimensional subspace is reduced to computing kernels of matrices.

Definition 8.1 Let $u = (u_1, \ldots, u_m)^T \in V^m$ and $\beta = (\beta_1, \ldots, \beta_n)^T \in (V^*)^n$. We call

$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \ldots & \beta_1(u_m) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \ldots & \beta_n(u_m) \end{pmatrix} \in F^{n\times m}$$

the *evaluation matrix* of β and u.

Lemma 8.2 *Let $U \leq V$ and $\mathscr{B} \leq V^*$ be generated respectively by $u = (u_1, \ldots, u_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Let $k^1, \ldots, k^r \in F^m$ be a basis of* $\operatorname{Ker}\beta(u)$*, and $\kappa^1, \ldots, \kappa^s \in F^n$ a basis of* $\operatorname{Ker}(\beta(u))^T$*. Then*

(i) *$U \cap \mathscr{B}^\perp$ is generated by $\sum_{i=1}^m k_i^1 u_i, \ldots, \sum_{i=1}^m k_i^r u_i$ and*
(ii) *$U^\perp \cap \mathscr{B}$ is generated by $\sum_{i=1}^n \kappa_i^1 \beta_i, \ldots, \sum_{i=1}^n \kappa_i^s \beta_i$.*

Proof A linear combination $v = \sum_{\ell=1}^m c_\ell u_\ell$ is in $\mathscr{B}^\perp$ if and only if $\beta_i(v) = 0$ for $1 \leq i \leq n$, that is, $\sum_{\ell=1}^m c_\ell \beta_i(u_\ell) = 0$ for $1 \leq i \leq n$. Hence $\beta(u) \cdot (c_1, \ldots, c_m)^T = 0$. Analogously, one sees that the coefficients of linear combination in $U^\perp \cap \mathscr{B}$ are in the kernel of $(\beta(u))^T$. □

We reformulate the conditions of Theorem 5.1 such that for Fredholm operators they only involve operations on finite-dimensional subspaces and intersections like in the previous lemma. Similarly, one can rewrite the conditions of Theorem 5.2.

Corollary 8.3 *Let $T_1 : V \to W$ and $T_2 : U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, \mathscr{B}_1, E_1)$ and $G_2 = \mathrm{O}(T_2, \mathscr{B}_2, E_2)$. Let $\mathscr{C}_2 = T_2(\mathscr{B}_2^\perp)^\perp$ and $K_1 = T_1^{-1}(E_1)$. The following conditions are equivalent:*

(i) *$G_2 G_1$ is an outer inverse of $T_1 T_2$.*
(ii) $\mathscr{C}_2 + (\mathscr{B}_1 \cap E_2^\perp) \geq \mathscr{B}_1 \cap (E_2 \cap K_1)^\perp$
(iii) $\mathscr{B}_1 \geq \mathscr{C}_2 \cap (E_2 \cap \mathscr{B}_1^\perp)^\perp \cap (E_2 \cap K_1)^\perp$
(iv) $K_1 \oplus (E_2 \cap \mathscr{B}_1^\perp) \geq E_2 \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$
(v) $E_2 \geq K_1 \cap (\mathscr{B}_1 \cap E_2^\perp)^\perp \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$

Proof Taking the orthogonal of both sides of 5.1 (ii) and (iii), respectively, and applying Proposition A.1 we get (ii) and (iii). For (iv) and (v), we can apply Proposition A.1 directly to the corresponding conditions of Theorem 5.1. □

We note that using Lemma 8.2, it also possible to determine constructively the implicit representation (6) of a product of generalized inverses; see the next section.

9. Examples for linear ordinary boundary problems

As an example involving infinite dimensional spaces and Fredholm operators, we consider solution (Green's) operators for linear ordinary boundary problems. Algebraically, linear boundary problems can be represented as a pair $(T, \mathscr{B})$, where $T: V \to W$ is a surjective linear map and $\mathscr{B} \leq V^*$ is an orthogonally closed subspace of (homogeneous) boundary conditions. We say that $v \in V$ is a solution of $(T, \mathscr{B})$ for a given $w \in W$ if $Tv = w$ and $v \in \mathscr{B}^\perp$.

For a regular boundary problem (having a unique solution for every right-hand side), the Green's operator is defined as the unique right inverse G of T with $\operatorname{Im} G = \mathscr{B}^\perp$; see [28] for further details. The product $G_2 G_1$ of the Green's operators of two boundary problems $(T_1, \mathscr{B}_1)$ and $(T_2, \mathscr{B}_2)$ is then the Green's operator of the regular boundary problem $(T_1 T_2, \mathscr{B}_2 \oplus T_2^*(\mathscr{B}_1))$, see also Theorem 6.2.

For boundary problems having at most one solution, that is $\mathscr{B}^\perp \cap \operatorname{Ker} T = \{0\}$, the linear algebraic setting has been extended in [23] by defining generalized Green's operators as outer inverses. More specifically, one first has to project an arbitrary right-hand side $w \in W$ onto $T(\mathscr{B}^\perp)$, the image of the 'functions' satisfying the boundary conditions, along a complement E of $T(\mathscr{B}^\perp)$. The corresponding generalized Green's operator is defined as the outer inverse $G = \mathrm{O}(T, \mathscr{B}, E)$, and we refer to $E \leq W$ as an *exceptional space* for the boundary problem $(T, \mathscr{B})$.

The question when the product of two outer inverses is again an outer inverse, is the basis for factoring boundary problems into lower order problems; see [28,29] for the case of regular boundary problems. This, in turn, provides a method to factor certain integral operators.

As an example, let us consider the boundary problem

$$\boxed{\begin{array}{l} u'' = f \\ u'(0) = u'(1) = u(1) = 0. \end{array}} \tag{7}$$

In the above setting, this means we consider the pair $(T_1, \mathscr{B}_1)$ with $T_1 = \mathrm{D}^2$ and $\mathscr{B}_1 = \operatorname{span}(\mathrm{E}_0\,\mathrm{D}, \mathrm{E}_1\,\mathrm{D}, \mathrm{E}_1)$, where D denotes the usual derivation on smooth functions and E_c the evaluation at $c \in \mathbb{R}$. The boundary problem is only solvable for *forcing functions* f satisfying the *compatibility condition* $\int_0^1 f(\xi)\,\mathrm{d}\xi = 0$; more abstractly, we have $T_1(\mathscr{B}_1^\perp) = \mathscr{C}_1^\perp$ with $\mathscr{C}_1 = \operatorname{span}(\int_0^1)$, where $\int_0^1$ denotes the functional $f \mapsto \int_0^1 f(\xi)\,\mathrm{d}\xi$. For computing a generalized Green's operator of $(T_1, \mathscr{B}_1, E_1)$, we have to project f onto $\mathscr{C}_1^\perp$ along a fixed complement E_1. In [30], we computed the generalized Green's operator

$$G_1(f) = x \int_0^x f(\xi)\,\mathrm{d}\xi - \int_0^x \xi f(\xi)\,\mathrm{d}\xi - \frac{1}{2}(x^2+1) \int_0^1 f(\xi)\,\mathrm{d}\xi + \int_0^1 \xi f(\xi)\,\mathrm{d}\xi$$

of (7) for $E_1 = \mathbb{R}$ being the constant functions. It is easy to see that in this case we have $T_1^{-1}(E_1) = \operatorname{span}(1, x, x^2)$.

As a second boundary problem, we consider

$$\boxed{\begin{array}{l} u'' - u = f \\ u'(0) = u'(1) = u(1) = 0, \end{array}}$$

or $(T_2, \mathscr{B}_2)$ with $T_2 = \mathrm{D}^2 - 1$ and $\mathscr{B}_2 = \operatorname{span}(\mathrm{E}_0\,\mathrm{D}, \mathrm{E}_1\,\mathrm{D}, \mathrm{E}_1)$. For the corresponding generalized Green's operator G_2 with exceptional space $E_2 = \operatorname{span}(x)$, we will now check if the products G_1G_2 and G_2G_1 are again generalized Green's operators of $T_1T_2 = T_2T_1 = \mathrm{D}^4 - \mathrm{D}^2$, using condition (ii) of Corollary 8.3.

We use the algorithm from [30], implemented in the package `IntDiffOp` for the computer algebra system MAPLE, to compute the compatibility conditions. The algorithm is based on the identity

$$T(\mathscr{B}^{\perp})^{\perp} = G^{*}(\mathscr{B} \cap (\mathrm{Ker}\,T)^{\perp}),$$

for any right inverse G of T, which follows from Propositions A.2 and 3.1. Moreover, a right inverse of the differential operator can be computed by the variation of constants and the intersection $\mathscr{B} \cap (\mathrm{Ker}\,T)^{\perp}$ using Lemma 8.2. Thus, we obtain $\mathscr{C}_2 = \mathrm{span}(\int_0^1(\exp(-x)+\exp(x)))$, where $\int_0^1(\exp(-x)+\exp(x))$ denotes the functional $f \mapsto \int_0^1(\exp(-\xi)+\exp(\xi))\, f(\xi)\,\mathrm{d}\xi$.

The space $T_2^{-1}(E_2) = \mathrm{span}(x, \exp(x), \exp(-x))$ can be computed using Proposition 3.1 and a right inverse of the differential operator; this is also implemented in the `IntDiffOp` package. Hence, we have $E_1 \cap T_2^{-1}(E_2) = \{0\}$ and therefore $\mathscr{B}_2 \cap (E_1 \cap T_2^{-1}(E_2))^{\perp} = \mathscr{B}_2$. Computing $\mathscr{B}_2 \cap E_1^{\perp}$ with Lemma 8.2 yields $\mathscr{B}_2 \cap E_1^{\perp} = \mathrm{span}(\mathrm{E}_0\,\mathrm{D}, \mathrm{E}_1\,\mathrm{D})$; thus G_1G_2 is not an outer inverse of the product $T_2T_1 = \mathrm{D}^4 - \mathrm{D}^2$ by Corollary 8.3(ii).

On the other hand, we have $E_2 \cap T_1^{-1}(E_1) = \mathrm{span}(x) = E_2$, hence we know by Corollary 8.3(ii) that G_2G_1 is an outer inverse of $T_1T_2 = \mathrm{D}^4 - \mathrm{D}^2$. Furthermore, by Theorem 6.2 we can determine which boundary problem it solves without computing G_1 and G_2. With Lemma 8.2, we obtain $\mathscr{B}_1^{\perp} \cap E_2 = \{0\}$ and $\mathscr{B}_1 \cap E_2^{\perp} = \mathrm{span}(\mathrm{E}_0\,\mathrm{D} - \mathrm{E}_1, \mathrm{E}_1\,\mathrm{D} - \mathrm{E}_1)$. Since applying the transpose T_2^{*} to $\mathrm{E}_0\,\mathrm{D} - \mathrm{E}_1$ and $\mathrm{E}_1\,\mathrm{D} - \mathrm{E}_1$ corresponds to multiplying $T_2 = \mathrm{D}^2 - 1$ from the right, G_2G_1 is the generalized Green's operator of

$$(\mathrm{D}^4 - \mathrm{D}^2, \mathrm{span}(\mathrm{E}_0\,\mathrm{D}, \mathrm{E}_1\,\mathrm{D}, \mathrm{E}_1, \mathrm{E}_0\,\mathrm{D}^3 - \mathrm{E}_1\,\mathrm{D}^2, \mathrm{E}_1\,\mathrm{D}^3 - \mathrm{E}_1\,\mathrm{D}^2), \mathbb{R})$$

by (6); or, in traditional notation, it solves the boundary problem

$$\boxed{\begin{array}{l} u'''' - u'' = f \\ u'(0) = u'(1) = u(1) = u'''(0) - u''(1) = u'''(1) - u''(1) = 0, \end{array}}$$

with exceptional space $\mathbb{R}$.

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Appendix A. Duality

In the appendix, we summarize duality results for arbitrary vector spaces and their duals that generalize the standard duality for finite-dimensional vector spaces but do not require any topological assumptions; see [30, Sections 9.2 and 9.3] and [28] for further details. The notation should also remind of the analogous and well-known results for Hilbert spaces.

Let V and W be vector spaces over a field F and $\langle , \rangle : V \times W \to F$ be a bilinear map. For $V_1 \leq V$, we define the orthogonal

$$V_1^{\perp} = \{w \in W \mid \langle v, w\rangle = 0 \text{ for all } v \in V_1\} \leq W.$$

The orthogonal $W_1^{\perp}$ for $W_1 \leq W$ is defined analogously. A subspace U is called orthogonally closed if $U = U^{\perp\perp}$. It follows directly from the definition that for all subsets $X_1, X_2 \subseteq V$, we have $X_1 \subseteq X_2 \Rightarrow X_1^{\perp} \supseteq X_2^{\perp}$ and $X_1 \subseteq X_1^{\perp\perp}$; and the same holds for subsets of W. Let $\mathbb{P}(V)$ denote the projective geometry of V, that is, the partially ordered set (poset) of all subspaces ordered by inclusion. Then we have an order-reversing Galois connection between $\mathbb{P}(V)$ and $\mathbb{P}(W)$ defined by $U \mapsto U^{\perp}$.

We now consider the canonical bilinear form $V \times V^* \to F$ of a vector space V and its dual V^* defined by $\langle v, \beta\rangle \mapsto \beta(v)$. Then every subspace $V_1 \leq V$ is orthogonally closed with respect to the canonical bilinear form, and every finite-dimensional subspace $\mathscr{B} \leq V^*$ is orthogonally closed. The Galois connection gives an order-reversing bijection between $\mathbb{P}(V)$ and the poset of all orthogonally closed subspaces of V^*. So we can describe any subspace $V_1 \leq V$ implicitly by the corresponding orthogonally closed subspace $V_1^{\perp}$. We denote the poset of all orthogonally closed subspaces of V^* with $\overline{\mathbb{P}}(V^*)$.

The projective geometry $\mathbb{P}(V)$ is a modular lattice, where join and meet are defined as the sum and intersection of subspaces. Modularity means that for all $V_1, V_2, V_3 \in \mathbb{P}(V)$ with $V_1 \leq V_3$ we have

$$V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap V_3. \tag{A1}$$

Moreover, for spaces $V_1 \leq V_3$ and $V_2 \leq V_4$, we have

$$V = V_1 + V_2 = V_3 \oplus V_4 \;\Rightarrow\; V_1 = V_3 \text{ and } V_2 = V_4, \tag{A2}$$

since $V_3 \cap V_4 = \{0\}$ implies $V_3 = (V_1 \oplus V_2) \cap V_3 = V_1$ and $V_4 = (V_1 \oplus V_2) \cap V_4 = V_2$.

One can also show that $\overline{\mathbb{P}}(V^*)$ is a modular lattice, where the meet is the intersection and the join is the orthogonal closure of the sum of subspaces. Using this fact, one can prove in particular that the sum of two orthogonally closed subspaces is orthogonally closed. The following theorem summarizes Section 9.3 of [30].

Proposition A.1 *The map $V_1 \mapsto V_1^{\perp}$ gives an order-reversing lattice isomorphism with inverse $\mathscr{B}_1 \mapsto \mathscr{B}_1^{\perp}$ between the complemented modular lattices $\mathbb{P}(V)$ and $\overline{\mathbb{P}}(V^*)$. In particular, the intersection of orthogonally closed subspaces in V^* is orthogonally closed and*

$$(V_1 + V_2)^{\perp} = V_1^{\perp} \cap V_2^{\perp} \quad \textit{and} \quad (\mathscr{B}_1 \cap \mathscr{B}_2)^{\perp} = \mathscr{B}_1^{\perp} + \mathscr{B}_2^{\perp}.$$

The sum of two orthogonally closed subspaces in V^* is orthogonally closed and

$$(V_1 \cap V_2)^{\perp} = V_1^{\perp} + V_2^{\perp} \quad \textit{and} \quad (\mathscr{B}_1 + \mathscr{B}_2)^{\perp} = \mathscr{B}_1^{\perp} \cap \mathscr{B}_2^{\perp}.$$

Furthermore, orthogonality preserves direct sums, such that

$$V = V_1 \oplus V_2 \;\Rightarrow\; V^* = V_1^{\perp} \oplus V_2^{\perp} \quad \textit{and} \quad V^* = \mathscr{B}_1 \oplus \mathscr{B}_2 \;\Rightarrow\; V = \mathscr{B}_1^{\perp} \oplus \mathscr{B}_2^{\perp}.$$

For a linear map $A : V \to W$ between vector spaces, the *transpose* $A^* : W^* \to V^*$ is defined by $\gamma \mapsto \gamma \circ A$. The transposition map $A \mapsto A^*$ from $L(V, W)$ to $L(W^*, V^*)$ is linear, and it is injective since for all $w \neq 0$ there exists a linear map $h \in W^*$ with $h(w) \neq 0$. Moreover, the transpose of a composition is given by $(A_1 A_2)^* = A_2^* A_1^*$.

The image of an orthogonally closed space under the transpose map is orthogonally closed, and we have following identities, see, for example, [28, Prop. A.6].

PROPOSITION A.2 *Let V and W be vector spaces and $A\colon V \to W$ be linear. Then*

$$A(V_1)^{\perp} = (A^*)^{-1}(V_1^{\perp}), \qquad A(\mathscr{B}_1^{\perp}) = (A^*)^{-1}(\mathscr{B}_1)^{\perp},$$
$$A^*(\mathscr{C}_1)^{\perp} = A^{-1}(\mathscr{C}_1^{\perp}), \qquad A^*(W_1^{\perp}) = A^{-1}(W_1)^{\perp},$$

for subspaces $V_1 \leq V$, $W_1 \leq W$, $\mathscr{C}_1 \leq W^*$ and orthogonally closed subspaces $\mathscr{B}_1 \leq V^*$. In particular,

$$(\operatorname{Im} A)^{\perp} = \operatorname{Ker} A^*, \quad \operatorname{Im} A = (\operatorname{Ker} A^*)^{\perp}, \quad (\operatorname{Im} A^*)^{\perp} = \operatorname{Ker} A, \quad \operatorname{Im} A^* = (\operatorname{Ker} A)^{\perp},$$

for the image and kernel of A and A^*.

The property of being a projector, outer/inner/algebraic generalized inverse carries over to the transpose.

PROPOSITION A.3 *A linear map $P\colon V \to V$ is a projector if and only if its transpose P^* is a projector. A linear map $G\colon W \to V$ is an outer/inner/algebraic generalized inverse of $T\colon V \to W$ if and only if G^* is an outer/inner/algebraic generalized inverse of T^*.*

Proof This follows from the defining equations for these properties. For example, if G is an outer inverse of T, we have $G^*T^*G^* = (GTG)^* = G^*$, and the reverse implication follows from the injectivity of the transposition map. $\square$

With the results of this section, we obtain the following duality principle for generalized inverses.

Remark A.4 Given a valid statement for linear maps on arbitrary vector spaces V over a common field involving inclusions, $\{0\}$ and V, sums and intersections, direct sums, kernels and images, projectors, and outer/inner/algebraic generalized inverses, we obtain a valid dual statement by

- reversing the order of the linear maps and the corresponding domains and codomains,
- reversing inclusions and interchanging V and $\{0\}$,
- interchanging sums and intersections,
- interchanging kernels and images.

For example, one easily checks that in Proposition 2.2, the statements (v)–(vii) are the duals of (ii)–(iv) in this sense.