A Skew Polynomial Approach to Integro-Differential Operators

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ABSTRACT

We construct the algebra of integro-differential operators over an ordinary integro-differential algebra directly in terms of normal forms. In the case of polynomial coefficients, we use skew polynomials for defining the integro-differential Weyl algebra as a natural extension of the classical Weyl algebra in one variable. Its normal forms, algebraic properties and its relation to the localization of differential operators are studied. Fixing the integration constant, we regain the integro-differential operators with polynomial coefficients.

Categories and Subject Descriptors

I.1.1 [**Symbolic and Algebraic Manipulation**]: Expressions and Their Representation—*simplification of expressions*; I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*algebraic algorithms*

General Terms

Theory, Algorithms

Keywords

Integro-differential operators, skew polynomials, Weyl algebra, integro-differential algebra, Baxter algebra.

1. INTRODUCTION

Skew polynomials provide a powerful framework for studying linear differential operators from an algebraic and algorithmic perspective [24, 12, 10]. In this paper, we develop a related approach for ordinary *integro-differential operators*, complementing the development reported in [27].

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We have introduced the algebra of integro-differential operators in [28] for a symbolic treatment of linear boundary problems following [26]. It is based on integro-differential algebras (Section 2), which bring together the usual derivation structure with a suitable notion of indefinite integration and evaluation. Integro-differential operators are constructed as the corresponding operator algebra. They can be applied for solving boundary problems and for factoring them along a given factorization of the underlying differential equation. A prototype implementation of integro-differential operators in Theorema is presented in [7].

In contrast to our earlier construction, the present treatment of integro-differential operators is directly based on normal forms (Section 3). This is useful for analyzing the algebraic structure and developing algorithms. In this context, polynomial coefficients are of particular interest.

We construct an integro-differential analog of the classical Weyl algebra in one variable—henceforth called the *differential Weyl algebra*—as a skew polynomial ring (Section 4). The integro-differential Weyl algebra has a natural decomposition into the differential Weyl algebra, the *integro Weyl algebra* (Section 5), and the two-sided *evaluation ideal*. Unlike its differential part, the integro-differential Weyl algebra has zero divisors and is neither simple nor Noetherian.

The integro Weyl algebra forms a curious counterpart to the differential Weyl algebra. Following an analogous construction as a skew polynomial ring, the resulting algebra is also a Noetherian integral domain, but otherwise exhibits some striking differences: It is not a simple ring and it lacks a canonical action on the polynomials but it has a natural grading.

Compared to the algebra of integro-differential operators, the integro-differential Weyl algebra has a finer structure, which can be specialized naturally in two different ways, either discarding or fixing the evaluation (Section 6). Factoring out the evaluation ideal leads to a *localization*, where the "integral" is added as a two-sided inverse of the derivation. Factoring out a suitable relation *choosing the integration constant*, we obtain the algebra of integro-differential operators.

Some *notational conventions*: We fix a ground field K of characteristic 0. The inner direct sum of modules is written as $M = M_1 \dotplus M_2$. We use the symbol $\leq$ for algebraic substructures. Unless specified otherwise the variables i, j, k, m, n range over the nonnegative integers.

2. INTEGRO-DIFFERENTIAL ALGEBRAS

In this section, we summarize basic properties of integro-differential algebras from [28]. We recall that $(\mathcal{F}, \partial)$ is a differential K-algebra if $\partial\colon \mathcal{F} \to \mathcal{F}$ is K-linear map satisfying the *Leibniz rule*

$$\partial(fg) = \partial(f)\, g + f\, \partial(g). \tag{1}$$

For convenience, we assume $K \leq \mathcal{F}$ and write f' for $\partial(f)$.

Definition 1. *Let $\mathcal{F}$ be a commutative algebra over a field K. We call $(\mathcal{F}, \partial, \smallint)$ an* integro-differential algebra *if $(\mathcal{F}, \partial)$ is a differential algebra, $\smallint\colon \mathcal{F} \to \mathcal{F}$ is a K-linear section of ∂, that is,*

$$\partial \smallint = 1, \tag{2}$$

and the differential Baxter axiom

$$(\smallint f')(\smallint g') = (\smallint f')g + f(\smallint g') - \smallint (fg)' \tag{3}$$

holds. Then we call $\smallint$ an integral operator *for ∂.*

We refer to the elements of $\mathcal{I} = \mathrm{Im}(\smallint)$ as *initialized*, while those of $\mathcal{C} = \mathrm{Ker}(\partial)$ are usually called *constants*. Since $\smallint$ is a section of ∂, we have projectors $\smallint \partial$ and

$$P = 1 - \smallint \partial, \tag{4}$$

and a direct sum $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$ with $\mathcal{C} = \mathrm{Im}(P)$ and $\mathcal{I} = \mathrm{Ker}(P)$. Conversely, for every projector P onto a complement of $\mathcal{C}$ there exists a unique section of ∂ such that (4) holds; see for example [22, p. 17] or [25].

The standard example $\mathcal{F} = C^\infty(\mathbb{R})$ comes from analysis, where ∂ is the usual derivation and $\smallint$ the integral operator

$$\smallint_c^x\colon f \mapsto \int_c^x f(\xi)\, d\xi.$$

for $c \in \mathbb{R}$. Here (2) is the Fundamental Theorem, while (3) can be verified either directly or by using the characterization of integral operators below. The projector $P\colon f \mapsto f(c)$ corresponds to a point evaluation. For an algorithmic approach to constant coefficient ODE, the subalgebra of exponential polynomials is important.

The polynomial ring $K[x]$ with the usual derivation is similarly seen to form an integro-differential algebra, with integral operator $\smallint_c^x\colon x^n \mapsto (x^{n+1} - c^{n+1})/(n+1)$ for $c \in K$. The corresponding projector is the evaluation homomorphism determined by $x \mapsto c$; we call c the *constant of integration*.

Substituting respectively $\smallint f$ for f and $\smallint g$ for g in (3) and using (1), (2) gives the plain *Baxter axiom* (of weight zero)

$$\smallint f \cdot \smallint g = \smallint (f \smallint g) + \smallint (g \smallint f), \tag{5}$$

which is obviously an algebraic version of integration by parts (corresponding to the rewrite rule for $\smallint f \smallint$ in Table 1). A *Baxter algebra* $(\mathcal{F}, \smallint)$ is then a K-algebra $\mathcal{F}$ with a K-linear operation $\smallint$ fulfilling the Baxter axiom (5); we refer to [14, 2, 29] for more details.

Substituting $\smallint g$ for g in (3), one obtains with (1), (2) the following one-sided variant of the differential Baxter axiom

$$\smallint fg = f \smallint g - \smallint (f' \smallint g), \tag{6}$$

which we used in [28] for the definition of integro-differential algebras. In the commutative case, both versions of the Baxter axiom are equivalent, but (3) has the advantage that it generalizes to noncommutative algebras over rings and Baxter operators with nonzero weight. Compare to the setting in [15], where a similar structure was introduced independently under the name of differential Rota-Baxter algebras. They only require (2) and the Baxter axiom (5) rather than its differential variant (3).

One can characterize what makes (3) stronger than (5). A section $\smallint$ of ∂ is an integral operator if and only if it is also $\mathcal{C}$-linear. Moreover, we can characterize the integral operators among sections by requiring the projector in (4) to be multiplicative. Another equivalent formulation of the differential Baxter axiom (corresponding to the usual integration by parts and the identity for $\smallint f \partial$ in Table 1) is

$$\smallint f g' = fg - \smallint f' g - P(f)\, P(g), \tag{7}$$

following from $\mathcal{C}$-linearity of $\smallint$ and multiplicativity of P.

In the rest of the paper, we focus on ordinary differential equations. Thus we call an (integro-)differential algebra *ordinary* if $\dim_K \mathrm{Ker}(\partial) = 1$. Note that this terminology deviates from [17, p. 58], where it only refers to having a single derivation. In an ordinary differential algebra $\mathcal{F}$, we clearly have $K = \mathcal{C}$, so $\mathcal{F}$ is an algebra over its field of constants. A section is then automatically $\mathcal{C}$-linear, so the pure Baxter axiom (5) and its differential version (3) are equivalent.

In this case, the corresponding projector is a character

$$\mathtt{E} = 1 - \smallint \partial \tag{8}$$

since it is multiplicative (by the above characterization of integral operators) and its image is $\mathcal{C} = K$. We write $\mathcal{M}(\mathcal{F})$ for the set of all characters on $(\mathcal{F}, \partial, \smallint)$, including in particular the *evaluation* $\mathtt{E}$.

3. THE ALGEBRA OF INTEGRO-DIFFERENTIAL OPERATORS

In analogy to differential operators over a differential algebra, it is natural to consider the algebra of linear operators over an integro-differential algebra. In [28] we defined the algebra of integro-differential operators as the quotient of the free algebra in the corresponding operators modulo the parametrized equations in Table 1. We showed that they form an infinite two-sided noncommutative Gröbner basis (or a Noetherian and confluent rewrite system [1]) and determined the corresponding normal forms. (See also [27] for a summary.) For the theory of Gröbner bases, we refer to [5, 6], for its noncommutative extension to [3, 21]. In this section, we want to define the algebra of integro-differential operators directly in terms of their normal forms.

Let $\mathcal{F}$ be an ordinary integro-differential algebra over K. In the following, the variables f, g are used for elements of $\mathcal{F}$ and φ, ψ for characters in $\mathcal{M}(\mathcal{F})$. Moreover, we use $U \bullet f$ for the action of U on f, where U is a combination of $\partial, \smallint$, functions in $\mathcal{F}$ and characters in $\mathcal{M}(\mathcal{F})$. In particular, we have $\partial \bullet f$ for the derivation, $\smallint \bullet f$ for the integral operator and $\varphi \bullet f$ for the application of characters, while $g \bullet f$ denotes the product in $\mathcal{F}$.

We remark that Table 1 is to be understood as including implicit equations for $\smallint \smallint$, $\smallint \partial$ and $\smallint \varphi$ by substituting $f = 1$ in the equations for $\smallint f \smallint$, $\smallint f \partial$ and $\smallint f \varphi$, respectively. Moreover, one obtains the equation $\mathtt{E} \smallint = 0$ from the definition of the evaluation $\mathtt{E}$.

For defining the algebra of integro-differential operators in terms of normal forms, we use the fact [28, Prop. 17] that every integro-differential operator can be uniquely written as a sum of a differential, an integral, and a so-called boundary

gf	$=$	$g \bullet f$	∂f	$=$	$f\partial + \partial \bullet f$
$\varphi\psi$	$=$	ψ	$\partial\varphi$	$=$	0
φf	$=$	$(\varphi \bullet f)\,\varphi$	$\partial{\textstyle\int}$	$=$	1
${\textstyle\int} f {\textstyle\int}$	$=$	$({\textstyle\int} \bullet f)\,{\textstyle\int} - {\textstyle\int}\,({\textstyle\int} \bullet f)$			
${\textstyle\int} f\partial$	$=$	$f - {\textstyle\int}(\partial \bullet f) - (\mathtt{E} \bullet f)\,\mathtt{E}$			
${\textstyle\int} f\varphi$	$=$	$({\textstyle\int} \bullet f)\,\varphi$			

Table 1: Relations for Integro-differential Operators

operator. Since all these operators form subalgebras, we first describe them separately, and then the interaction between them. It is clear that the normal forms constitute an algebra isomorphic to the algebra of integro-differential operators in the sense of [28].

Moreover, for simplicity we take the evaluation $\mathtt{E}$ as the only character. For $\mathcal{F} = C^\infty[a,b]$, this amounts to considering only initial conditions, but the approach can be extended by using the normal forms for Stieltjes boundary conditions [28, Def. 14].

We first recall the well-known algebra of *differential operators* $\mathcal{F}[\partial]$ over $\mathcal{F}$. It is defined as sums of terms of the form $f\partial^i$ with the usual addition or, more abstractly, as the free left $\mathcal{F}$-module generated by the ∂^i. The multiplication is determined by viewing $\mathcal{F}$ as a subalgebra of $\mathcal{F}[\partial]$ and by using the equation

$$\partial \cdot f = f\partial + \partial \bullet f \tag{9}$$

coming from the Leibniz rule (1).

Clearly, sums of terms of the form $f{\textstyle\int} g$ represent linear integral operators. But they cannot be normal forms since, by linearity, $f{\textstyle\int}\lambda g$ and $\lambda f{\textstyle\int} g$ with $\lambda \in K$ represent the same operator. This can be solved by choosing a K-basis $\mathcal{B}$ for $\mathcal{F}$. We additionally require $1 \in \mathcal{B}$ so that we can represent integral operators of the form $f{\textstyle\int}$. Moreover, we use in the following the convention that $f{\textstyle\int} g$ is to be understood as an abbreviation for the corresponding basis expansion if g is not a basis element.

We define the algebra of *integral operators* $\mathcal{F}[{\textstyle\int}]$ over $\mathcal{F}$ as sums of terms of the form $f{\textstyle\int} b$ with $b \in \mathcal{B}$ (or as the free left $\mathcal{F}$-module generated by the ${\textstyle\int} b$). The multiplication is based on the equation

$${\textstyle\int} b \cdot {\textstyle\int} = ({\textstyle\int} \bullet b){\textstyle\int} - {\textstyle\int}\,({\textstyle\int} \bullet b) \tag{10}$$

corresponding to the Baxter axiom (5). Note that $\mathcal{F}[{\textstyle\int}]$ does not contain $\mathcal{F}$; it is an algebra without unit element.

We define the algebra of *boundary operators* $\mathcal{F}[\mathtt{E}]$ as sums of terms of the form $f\mathtt{E}\partial^i$. Their product is determined by

$$\mathtt{E}\partial^i \cdot f\mathtt{E}\partial^j = (\mathtt{E}\partial^i \bullet f)\,\mathtt{E}\partial^j, \tag{11}$$

which is a result of the Leibniz rule and the equations $\partial\mathtt{E} = 0$, $\mathtt{E}f = (\mathtt{E} \bullet f)\mathtt{E}$, $\mathtt{E}^2 = \mathtt{E}$. Also $\mathcal{F}[\mathtt{E}]$ does not contain $\mathcal{F}$.

The additive structure on integro-differential operators is then constructed as the direct sum

$$\mathcal{F}[\partial, {\textstyle\int}] = \mathcal{F}[\partial] \oplus \mathcal{F}[{\textstyle\int}] \oplus \mathcal{F}[\mathtt{E}].$$

We regard the summands as being embedded in $\mathcal{F}[\partial, {\textstyle\int}]$.

The multiplication within the summands is given by (9), (10), and (11). It remains to define the multiplication between different summands. To start with, multiplying a differential operator with an integral operator is given by

$$\partial \cdot f{\textstyle\int} b = f \bullet b + (\partial \bullet f){\textstyle\int} b,$$

corresponding to (1) and (2). So we have $\mathcal{F}[\partial]\,\mathcal{F}[{\textstyle\int}] \subset \mathcal{F}[\partial] + \mathcal{F}[{\textstyle\int}]$. The multiplication in the reverse order is based on

$${\textstyle\int} b \cdot f\partial = b \bullet f - {\textstyle\int}(\partial b \bullet f) - (\mathtt{E}b \bullet f)\,\mathtt{E},$$

corresponding to the variant of the Baxter axiom (7), so that $\mathcal{F}[{\textstyle\int}]\,\mathcal{F}[\partial] \subset \mathcal{F}[\partial] + \mathcal{F}[{\textstyle\int}] + \mathcal{F}[\mathtt{E}]$.

The equations for multiplying a boundary operator from either side with a differential or integral operator are

$$\partial^i \cdot f\mathtt{E}\partial^j = (\partial^i \bullet f)\mathtt{E}\partial^j,$$

$$\mathtt{E}\partial^i \cdot f\partial^j = \sum_{k=0}^{i} (\mathtt{E} \bullet f_k)\mathtt{E}\partial^{j+k},$$

$${\textstyle\int} b \cdot f\mathtt{E}\partial^i = ({\textstyle\int} b \bullet f)\mathtt{E}\partial^i,$$

$$\mathtt{E}\partial^i \cdot f{\textstyle\int} b = \sum_{l=0}^{i-1} (\mathtt{E} \bullet g_l)\mathtt{E}\partial^l,$$

where $\partial^i f = \sum_{k=0}^{i} f_k \partial^k$ and $\sum_{k=1}^{i} f_k \partial^{k-1} b = \sum_{l=0}^{i-1} g_l \partial^l$ as differential operators in $\mathcal{F}[\partial]$. Besides the rules used for (11), this involves the rule ${\textstyle\int} f\mathtt{E} = ({\textstyle\int} \bullet f)\mathtt{E}$. So we have $\mathcal{F}[\partial]\,\mathcal{F}[\mathtt{E}]$, $\mathcal{F}[\mathtt{E}]\,\mathcal{F}[\partial] \subset \mathcal{F}[\mathtt{E}]$ as well as $\mathcal{F}[{\textstyle\int}]\,\mathcal{F}[\mathtt{E}]$, $\mathcal{F}[\mathtt{E}]\,\mathcal{F}[{\textstyle\int}] \subset \mathcal{F}[\mathtt{E}]$ in these cases.

Since by the above definitions multiplying a boundary operator with any integro-differential operator gives a boundary operator, we see that $\mathcal{F}[\mathtt{E}]$ is the ideal in $\mathcal{F}[\partial, {\textstyle\int}]$ generated by the evaluation $\mathtt{E}$, which we call the *evaluation ideal* of $\mathcal{F}[\partial, {\textstyle\int}]$. Here and in the following an ideal always means a two-sided ideal. So we have

$$\mathcal{F}[\partial, {\textstyle\int}] = \mathcal{F}[\partial] \dotplus \mathcal{F}[{\textstyle\int}] \dotplus (\mathtt{E}) \tag{12}$$

as a direct sum of $\mathcal{F}$-modules or K-vector spaces.

In the rest of this paper we will deal with the important special case $\mathcal{F} = K[x]$ from a skew polynomial perspective. Using the natural K-basis (x^k) yields a natural K-basis for all normal forms. In this case the above construction can be simplified substantially. We know from the Weyl algebra that the Leibniz rule (9) reduces to $\partial \cdot x = x\partial + 1$ and one can verify (compare Lemma 11) that ${\textstyle\int} \cdot {\textstyle\int} = x{\textstyle\int} - {\textstyle\int} x$ suffices to derive (10) for all polynomials. This is the basis for the skew polynomial construction in the following section.

4. THE INTEGRO-DIFFERENTIAL WEYL ALGEBRA

For analyzing rings of formal differential operators it is convenient to view them as skew polynomial rings. Specializing the coefficients to $K[x]$, one is led to the corresponding Weyl algebra. Our goal is to gain a skew polynomial perspective on the above ring $\mathcal{F}[\partial, {\textstyle\int}]$ for $\mathcal{F} = K[x]$. In this context, we write ℓ instead of ${\textstyle\int}$ to avoid confusion between iterated integrals ℓ^m and integrals with upper bounds ${\textstyle\int}^m$.

We recall the construction of skew polynomials [24] [12, p. 276] [10]. Let A be a (noncommutative) ring without zero divisors, ξ an indeterminate, $\sigma\colon A \to A$ an injective endomorphism (also known as "twist") and $\delta\colon A \to A$ a σ-derivation. The skew polynomial ring $A[\xi; \sigma, \delta]$ consists of the elements $a_0 + a_1\xi + \cdots + a_n\xi^n$ with $a_0, \ldots, a_n \in A$.

While the addition is defined termwise, the multiplication is determined by the rule

$$\xi a = \sigma(a)\,\xi + \delta(a).$$

It is well-known that $A[\xi;\sigma,\delta]$ is an integral domain since the usual degree equality $\deg fg = \deg f + \deg g$ is valid. We write $A[\xi;\delta]$ for $A[\xi;1,\delta]$.

We concentrate for a moment on the integral operators. One is tempted to take $A = K[x]$ and $\xi = \ell$. But the Baxter axiom requires $\ell x = x\ell - \ell^2$, in violation of the degree requirement. The way out is to reverse the adjunction of x and ℓ, thus picking $A = K[\ell]$ for the coefficient ring and $\xi = x$ for the indeterminate. (In the case of the differential Weyl algebra, the order of adjunction does not matter: This is the point of the well-known automorphism $x \leftrightarrow -\partial$, which does not carry over to its integro counterpart)

We choose a coefficient ring A that includes both ∂ and ℓ so that $A[x;\delta]$ yields in one stroke integro-differential operators that are "almost" isomorphic to $K[x][\partial,\int]$. It turns out that $A[x;\delta]$ has a finer structure than $K[x][\partial,\int]$; their relations will be studied in Section 6.

The coefficient ring A should contain all K-linear combinations of ∂ and ℓ, taking into account that $\partial\ell = 1$. Its derivation δ is set up so as to ensure the relations $\partial x - x\partial = 1$ and $x\ell - \ell x = \ell^2$ when $A[x;\delta]$ is introduced.

Definition 2. *The algebra $K\langle\partial,\ell\rangle$ is the quotient of the free algebra $K\langle D,L\rangle$ modulo the ideal $(DL-1)$. We write ∂ and ℓ for the corresponding residue classes. We define a derivation δ on $K\langle\partial,\ell\rangle$ by $\delta(\partial) = -1$ and $\delta(\ell) = \ell^2$.*

Note that δ is well-defined: Defining first a derivation on the free algebra by $\delta(D) = -1$ and $\delta(L) = L^2$, one sees immediately that $\delta(DL-1) = (DL-1)\,L$, so the passage to the quotient is legitimate.

The algebra $K\langle\partial,\ell\rangle$ is studied by N. Jacobson [16] from the general perspective of one-sided inverses in rings. His results imply immediately that $K\langle\partial,\ell\rangle$ is neither (left or right) Artinian nor (left or right) Noetherian. Extending this approach, L. Gerritzen [13] describes the right modules and derivations on $K\langle\partial,\ell\rangle$; using his classification [13, Prop. 7.1], we have $\delta = -\partial_0$. Some of the following results (without the differential structure—see below) can be found in their papers. Their approach is based on representation theory, while our treatment is based on a more algorithmic normal form perspective.

We shall now establish a decomposition of $K\langle\partial,\ell\rangle$ that is akin to (12). For this goal, observe that the monomials $\ell^i\partial^j$ form a K-basis of $K\langle\partial,\ell\rangle$ since they are normal forms with respect to the Gröbner basis $DL-1$.

In analogy to Equation (8) and [16], we define

$$\mathtt{E} = 1 - \ell\partial \qquad \text{and} \qquad e_{ij} = \ell^i\mathtt{E}\partial^j.$$

The e_{ij} satisfy the multiplication table for matrix units; see for example [16] and [18, Ex. 21.26]. The e_{ij} together with the pure ∂ and ℓ monomials form another K-basis. Indeed, iterating $\ell^{i+1}\partial^{j+1} = -e_{ij} + \ell^i\partial^j$, we obtain

$$\ell^{i+1}\partial^{j+1} = \begin{cases} \ell^{i-j} - \sum_{k=0}^{j} e_{ij} & \text{for } i > j, \\ \partial^{j-i} - \sum_{k=0}^{i} e_{ij} & \text{for } i \le j. \end{cases}$$

Hence ∂^j, ℓ^i, and e_{ij} generate $K\langle\partial,\ell\rangle$ over K. Using the relation $e_{ij} = \ell^i\partial^j - \ell^{i+1}\partial^{j+1}$, one sees that they are also linearly independent because the $\ell^i\partial^j$ are.

We note also that the K-vector space generated by the e_{ij} is the ideal $(\mathtt{E})$ since $\ell e_{ij} = e_{i+1,j}$ and $\partial e_{ij} = e_{i-1,j}$ for $i > 0$ and $\partial e_{0j} = 0$; analogously for multiplication on the right. Confer also [16, 13]. In analogy to $\mathcal{F}[\partial,\int]$, we refer to $(\mathtt{E})$ as the *evaluation ideal* of $K\langle\partial,\ell\rangle$.

Proposition 3. *We have*

$$K\langle\partial,\ell\rangle = K[\partial] \dotplus K[\ell]\ell \dotplus (\mathtt{E})$$

as a direct sum of K-vector spaces, where $K[\partial]$ is a differential subring of $K\langle\partial,\ell\rangle$ while $K[\ell]\ell$ is a differential subring without unit and $(\mathtt{E})$ is a δ-ideal.

Proof. We have already seen the decomposition part. All three summands are obviously closed under addition, multiplication and the first one also under derivations. For the second note that $\delta(q) = \frac{dq}{d\ell}\,\ell^2 \in K[\ell]\ell$ for all $q \in K[\ell]\ell$. The third summand is closed under δ since

$$\delta(\mathtt{E}) = -\delta(\ell)\partial - \ell\delta(\partial) = -\ell^2\partial + \ell = \ell\mathtt{E} \in (\mathtt{E}).$$

This completes the proof since δ is a derivation. $\square$

Since $\partial\mathtt{E} = \partial - \partial = 0$ and $\mathtt{E}\ell = \ell - \ell = 0$, we obtain also $\partial^{i+1}e_{ij} = 0$ and $e_{ij}\ell^{j+1} = 0$, so every element of $(\mathtt{E})$ is both a left and and a right zero-divisor. The following minimality property of the evaluation ideal was also noted in [16].

Lemma 4. *Every nonzero ideal in $K\langle\partial,\ell\rangle$ contains $(\mathtt{E})$.*

Proof. Assume I is an ideal and $0 \neq f \in I$. Write now $f = p + q + e$ where $p \in K[\partial]$, $q \in K[\ell]\ell$ and $e \in (\mathtt{E})$ as in Propostion 3. Assume first $p \neq 0$. For a sufficiently high $k \geq 0$ we may assume that $\partial^k f \in K[\partial]$ since $\partial^k e_{ij} = 0$ for $k > i$ while q just gets "shifted" into $K[\partial]$. We may assume $\partial^k f$ is monic. Now let $\mathtt{E}\partial^m$ be the term with highest ∂-power in $\mathtt{E}\partial^k f \in I$. Then $\mathtt{E}\partial^k f\ell^m = \mathtt{E} \in I$ since $\mathtt{E}\partial^n\ell^m = 0$ for $m > n$. If $p = 0$ but $q \neq 0$ we may reason analogously by first looking at $f\ell^k$ for a suitable k.

Therefore, assume now $p = q = 0$ and $e \neq 0$. Let k be maximal such that e_{kj} occurs in e. Then we have $\partial^k f = \partial^k e \in \mathtt{E}K[\partial]\backslash\{0\}$ since all terms e_{ij} with $i < k$ vanish but the terms e_{kj} do not. By the same argument as above $\mathtt{E} \in I$. $\square$

The lattice of differential ideals turns out to be particularly simple.

Proposition 5. *The only proper δ-ideal of $K\langle\partial,\ell\rangle$ is $(\mathtt{E})$.*

Proof. We have already seen in Proposition 3 that $(\mathtt{E})$ is a δ-ideal. Suppose that $I \neq 0$ is another δ-ideal. By Lemma 4 we have $(\mathtt{E}) \subseteq I$. Assume there exists $f = p + q + e \in I\backslash(\mathtt{E})$ where $p \in K[\partial]$, $q \in K[\ell]\ell$ and $e \in (\mathtt{E})$ as above, but with either p or q unequal to zero. Using the same trick as before, we can find $k \geq 0$ such that $\partial^k f \in K[\partial]\backslash\{0\}$. Now, if ∂^m is the leading term of $\partial^k f$, we have $\delta^m(\partial^k f) \in K$ since K has characteristic 0. Hence $I = K\langle\partial,\ell\rangle$. $\square$

We consider for a moment $K[\partial]$, the subring of polynomials in ∂. The derivation δ extends uniquely to the Laurent polynomials $K[\partial,\partial^{-1}]$ if we view them as the localization of $K[\partial]$ by ∂. Intuitively, another way of getting the Laurent polynomials is making ℓ also a left inverse of ∂ in $K\langle\partial,\ell\rangle$. That would mean to set $\mathtt{E} = 1 - \ell\partial = 0$. It turns out that the intuition is right in this case; compare [13, Prop. 2.6] for the algebraic part.

PROPOSITION 6. *The map*

$$K\langle\partial,\ell\rangle/(\mathtt{E}) \xrightarrow{\sim} K[\partial,\partial^{-1}]$$

defined by $\partial + (\mathtt{E}) \mapsto \partial$ *and* $\ell + (\mathtt{E}) \mapsto \partial^{-1}$ *is a differential isomorphism.*

PROOF. The map φ given by $\ell^i\partial^j \mapsto \partial^{j-i}$ is a well-defined K-vector space homomorphism from $K\langle\partial,\ell\rangle$ to $K[\partial,\partial^{-1}]$. We claim that it is also a differential ring homomorphism. Since it is additive, we need to check this just for basis elements of $K\langle\partial,\ell\rangle$: We have $\varphi(\ell^i\partial^j \cdot \ell^k\partial^m) = \varphi(\ell^{i+k-j}\partial^m) = \partial^{j+m-i-k} = \varphi(\ell^i\partial^j)\,\varphi(\ell^k\partial^m)$, assuming $k \geq j$. The computation for $j > k$ is almost the same. We have furthermore $\varphi(\delta(\ell^i\partial^j)) = \varphi(i\ell^{i+1}\partial^j - j\ell^i\partial^{j-1}) = (i-j)\,\partial^{j-i-1} = \delta(\partial^{j-i}) = \delta(\varphi(\ell^i\partial^j))$.

We compute the kernel of φ by considering the basis that corresponds to the decomposition in Proposition 3. The basis vectors ℓ^i and ∂^j are sent to nonzero elements (even basis elements) in $K[\partial,\partial^{-1}]$. On the other hand, we have $\varphi(e_{ij}) = \varphi(\ell^i\partial^j - \ell^{i+1}\partial^{j+1}) = 0$ for all i, j. Hence we conclude $\ker\varphi = (\mathtt{E})$, and by the First Isomorphism Theorem the claim follows. $\square$

Using Proposition 6 and Lemma 4 together with the Third Isomorphism Theorem—see for instance [11, Thm. 1.23]—we see that the ideals of $K\langle\partial,\ell\rangle$ are completely described by the ideals in $K[\partial,\partial^{-1}]$. This is a principal ideal domain by [4, Th. 2.18].

The main purpose of this section is to define the integro-differential analog of the differential Weyl algebra. As noted before Lemma 4, $K\langle\partial,\ell\rangle$ is not an integral domain. One can nevertheless introduce the skew polynomials as before (even with non-injective twists); see [11, Sec. 5.2], [20, Sec. 1.1.2], [18, Ex. 1.9]. Consequently the skew polynomials have zero divisors, and the degree equality must be replaced by the inequality $\deg fg \leq \deg f + \deg g$. The crucial point is that the normal forms are unique as before.

DEFINITION 7. *The* integro-differential Weyl algebra *is given by the skew polynomial ring* $K\langle\partial,\ell\rangle[x;\delta]$ *and is denoted by* $\mathrm{A}_1(\partial,\ell)$.

Any infinite ascending chain $I_1 < I_2 < \cdots$ of left ideals in A yields the infinite ascending chain $RI_1 < RI_2 < \cdots$ of left ideals in $R = A[\xi;\delta]$; similarly for right ideals and for descending chains. Consequently $\mathrm{A}_1(\partial,\ell)$ is also neither (left or right) Artinian nor (left or right) Noetherian. The latter is in stark contrast to the differential Weyl algebra, as the following proposition is.

Over a $\mathbb{Q}$-algebra A, simplicity of skew polynomial rings can be decided by the following practical characterization from [18, Th. 3.15]. The ring $A[\xi;\delta]$ is simple if and only if A has no nontrivial δ-ideals and δ is not an inner derivation. Otherwise, the skew polynomials with coefficients in a δ-ideal of A form an ideal in $A[\xi;\delta]$. Since we have seen in Proposition 3 that $(\mathtt{E})$ is a nontrivial δ-ideal in $K\langle\partial,\ell\rangle$, we can use this criterion to see that the integro-differential Weyl algebra—unlike its differential companion—is not simple.

PROPOSITION 8. *The ring* $\mathrm{A}_1(\partial,\ell)$ *is not simple.*

PROOF. It remains to prove that δ is not an inner derivation. For assume $\delta = [p,\cdot]$ for some $p \in K\langle\partial,\ell\rangle$. Application to ∂ yields $-1 = [p,\partial]$. But $K\langle\partial,\ell\rangle/(\mathtt{E})$ being a commutative ring, every commutator of $K\langle\partial,\ell\rangle$ lies in the ideal $(\mathtt{E})$. Thus we obtain $-1 \in (\mathtt{E})$, in contradiction to Proposition 3. $\square$

5. THE INTEGRO WEYL ALGEBRA

For comparing $\mathrm{A}_1(\partial,\ell)$ with the construction in Section 3, it is useful to investigate the subring of integral operators.

DEFINITION 9. *The subring of* $\mathrm{A}_1(\partial,\ell)$ *consisting of skew polynomials with coefficients in* $K[\ell]$ *is called the* integro Weyl algebra *and denoted by* $\mathrm{A}_1(\ell)$.

Obviously we have $\mathrm{A}_1(\ell) = K[\ell][x;\delta]$, with the derivation δ restricted to $K[\ell]$. In the same fashion, the differential Weyl algebra $\mathrm{A}_1(\partial) = K[\partial][x;\delta]$ is the subring of $\mathrm{A}_1(\partial,\ell)$ consisting of skew polynomials with coefficients in $K[\partial]$.

Note that—unlike its integro-differential companion—the integro Weyl algebra is indeed an integral domain since $K[\ell]$ is. It provides an interesting and natural example of an Ore algebra, which to our knowledge has not been studied in the literature [10, 19].

At first sight, $\mathrm{A}_1(\ell)$ seems to be very similar to $\mathrm{A}_1(\partial)$, but we shall soon realize that appearances are deceptive. To start with, recall that $\mathrm{A}_1(\partial)$ has a canonical action on $K[x]$ in the following sense: If $x \in \mathrm{A}_1(\partial)$ acts by multiplication and $\partial \in \mathrm{A}_1(\partial)$ as a derivation, then $\partial \bullet f = f'$ yields the usual differentiation. The corresponding statement for $\mathrm{A}_1(\ell)$ would require $x \in \mathrm{A}_1(\ell)$ to act by multiplication and $\ell \in \mathrm{A}_1(\ell)$ as a Baxter operator. But this admits any integrals $\ell \bullet f = \int_c^x f$ with arbitrary $c \in K$. We will come back to this in Section 6. Another important difference to the differential case is that $\mathrm{A}_1(\ell)$ comes with a natural grading (by total degree in x and ℓ).

For comparing $\mathrm{A}_1(\ell) \leq \mathrm{A}_1(\partial,\ell)$ with the corresponding summand $K[\int] \leq \mathcal{F}[\partial,\int]$, it is necessary to consider different K-bases for $\mathrm{A}_1(\ell)$. The construction of skew polynomials comes with the basis $(\ell^i x^j)$, which we shall call the *left basis* (since the coefficients appear to the left of the indeterminate). It is an easy exercise to determine the transition to the corresponding *right basis* $(x^j\ell^i)$.

LEMMA 10. *We have the identities*

$$x^n\ell^m = \sum_{k=0}^{n} \frac{(-m)^{\underline{k}}\,n^{\underline{k}}}{k!}\,(-1)^k\,\ell^{m+k}x^{n-k}, \tag{13}$$

$$\ell^m x^n = \sum_{k=0}^{n} \frac{(-m)^{\underline{k}}\,n^{\underline{k}}}{k!}\,x^{n-k}\ell^{m+k}, \tag{14}$$

where $n^{\underline{k}} = n(n-1)\dots(n-k+1)$ *is the falling factorial.*

PROOF. Applying the Leibniz rule in both directions, one shows by induction that

$$x^n f = \sum_{k=0}^{n} \binom{n}{k}\,\delta^k(f)\,x^{n-k},$$

$$f x^n = \sum_{k=0}^{n} \binom{n}{k}\,(-1)^k\,x^{n-k}\,\delta^k(f)$$

for all $f \in K[\ell]$. Setting $f = \ell^m$ and applying $\binom{n}{k} = n^{\underline{k}}/k!$, the claim follows since $\delta^k(\ell^m) = (-1)^k\,(-m)^{\underline{k}}\,\ell^{m+k}$. $\square$

The formulae in Lemma 10 are written in such a way that the similarity to the corresponding formulae for $\mathrm{A}_1(\partial)$ becomes apparent. In fact, Equation (1.4) of [30] coincides with (13) if we allow $m \in \mathbb{Z}$ and identify ℓ with ∂^{-1}. These heuristic observations are made precise in Section 6 by the isomorphism of Proposition 16.

While the left and right bases of $\mathrm{A}_1(\ell)$ are special to the skew polynomial setting, the general ring of integral operators $\mathcal{F}[\int]$ from Section 3 has the K-basis $(\tilde{b}\int b)$. In the present setting, this leads to the *mid basis* $(x^m, x^m\ell x^n)$. As we shall see immediately, its role as a K-basis is justified by the following commutator relation.

LEMMA 11. *We have* $[x^n, \ell] = n\,\ell x^{n-1}\ell$.

PROOF. The case $n = 0$ being trivial, we prove the identity for arbitrary $n+1$. Substituting $m = 1$ in (13) and multiplying with $(n+1)\,\ell$ from the left yields

$$\begin{aligned}(n+1)\,\ell x^n\ell &= \sum_{k=0}^{n}(n+1)^{\underline{k+1}}\,\ell^{k+2}x^{n-k}\\ &= -\ell x^{n+1} + \sum_{k=0}^{n+1}(n+1)^{\underline{k}}\,\ell^{k+1}x^{n-k+1};\end{aligned}$$

we conclude by substituting $(n+1, 1)$ for (n, m) in (13). □

COROLLARY 12. *The monomials* (x^m) *and* $(x^m\ell x^n)$ *form a* K*-basis of* $\mathrm{A}_1(\ell)$.

PROOF. In analogy to the differential Weyl algebra, one sees immediately that $\mathrm{A}_1(\ell)$ is isomorphic to the free K-algebra in X and L modulo the ideal generated by $XL - LX - L^2$. Lemma 11 implies that the polynomials

$$LX^nL - (n+1)^{-1}\,[X^{n+1}, L]$$

belong to the ideal. They form a Gröbner basis with respect to the following admissible order [31, p. 268]: Words are compared in L-degree, then in total degree, and finally lexicographically (letters ordered either way). A routine calculation shows that the overlaps LX^nLX^mL are resolvable. The residue classes of the monomials X^n and X^mLX^n form a K-basis of the quotient ring [31, Thm. 7]. □

The transition between the left/right basis and the mid basis is governed by the following formulae.

LEMMA 13. *We have the identities*

$$x^m\ell x^n = \sum_{k=0}^{m}\frac{m!}{k!}\,\ell^{m-k+1}x^{k+n}, \tag{15}$$

$$x^m\ell x^n = \sum_{k=0}^{n}\frac{n!}{k!}\,(-1)^{n-k}x^{m+k}\ell^{n-k+1}, \tag{16}$$

$$\ell^{m+1} = \sum_{k=0}^{m}\frac{(-1)^k}{k!\,(m-k)!}\,x^{m-k}\ell x^k \tag{17}$$

for changing between the left/right and the mid basis.

PROOF. For proving the first formula, it suffices to set $n = 0$. Substituting $(m, 1)$ for (n, m) in (13), one obtains Equation (15) after an index transformation. Analogously, one proves the second formula with $m = 0$ by substituting 1 for m in (14).

We prove the third formula by induction. The base case $m = 0$ is trivial, so assume (17) for $m \geq 0$. Multiplying it with ℓ from the right and using Lemma 11 yields

$$\ell^{m+2} = \sum_{k=0}^{m}\frac{(-1)^k}{(k+1)!\,(m-k)!}\,(x^{m+1}\ell - x^{m-k}\ell x^{k+1}).$$

After expanding the parenthesis and extracting $x^{m+1}\ell$, one is left with the simple binomial sum

$$\sum_{k=0}^{m}\frac{(-1)^k}{(k+1)!\,(m-k)!} = \frac{1}{(m+1)!},$$

so we obtain

$$\begin{aligned}\frac{1}{(m+1)!}\,x^{m+1}\ell &+ \sum_{k=1}^{m+1}\frac{(-1)^k}{k!\,(m-k+1)!}\,x^{m-k+1}\ell x^k\\ &= \sum_{k=0}^{m+1}\frac{(-1)^k}{k!\,(m-k+1)!}\,x^{m-k+1}\ell x^k\end{aligned}$$

for ℓ^{m+2}, which is indeed (17) for $m+1$. □

We note that (17) can be regarded as an algebraic version of the well-known Cauchy formula for repeated integration [23, p. 38].

In view of the transition formulae (15) and (17), one can use the K-basis $(x^m\ell x^n)$ of $\mathrm{A}_1(\ell)\ell$ for setting up a concrete isomorphism (of algebras without unit) to $K[x][\int]$ with its K-basis $(x^m\int x^n)$. Confer Theorem 20 for an analogous statement for the full integro-differential Weyl algebra.

As for $\mathrm{A}_1(\partial, \ell)$, we see that $\mathrm{A}_1(\ell)$ is not a simple ring by the following characterization of the δ-ideals in $K[\ell]$.

LEMMA 14. *An ideal* $I \leq K[\ell]$ *is a nontrivial* δ*-ideal if and only if* $I = (\ell^n)$ *with* $n > 0$.

PROOF. Since $\delta(\ell^n) = n\ell^{n+1}$, ideals generated by ℓ^n are obviously δ-ideals. Conversely, let $I = (q)$ be a nontrivial δ-ideal with $q = \sum_{i=k}^{n} a_i\ell^i \in K[\ell]$ a polynomial of degree $n > 0$ and order k, meaning $a_k \neq 0$. Hence $\delta(q) = rq$ for some $r \in K[\ell]$ so that

$$\delta(q) = \sum_{i=k}^{n} a_i i\,\ell^{i+1} = r\sum_{i=k}^{n} a_i\,\ell^i$$

with $r = b_1\ell + b_0$. Equating the coefficients of ℓ^k and ℓ^{n+1} implies respectively $b_0 = 0$ and $b_1 = n$, the latter since K has characteristic 0. If $k < n$, equating the coefficients of ℓ^{k+1} implies $(n-k)a_k = 0$, in contradiction to our assumption on the characteristic of K. □

PROPOSITION 15. *The ring* $\mathrm{A}_1(\ell)$ *is not simple.*

PROOF. By the previous lemma there are nontrivial δ-ideals in $K[\ell]$. Since δ cannot be an inner derivation, the claim follows as in Proposition 8 from [18, Th. 3.15]. □

6. LOCALIZATION AND EVALUATION

By the construction of $\mathrm{A}_1(\partial, \ell)$, we have set up ℓ as an integral that is a right inverse for ∂. This still leaves some ambiguity for the choice of ℓ, which we will now remove. There are two extreme possibilities: When we require ℓ to be a two-sided inverse, we obtain a localization. On the other hand, we may insist ℓ to be a proper integral by fixing the integration constant; this leads us back to the ring of integro-differential operators $K[x][\partial, \int]$.

Let us start with the localization. Extending the derivation to the Laurent polynomial ring $K[\partial, \partial^{-1}]$ as in Proposition 6, we form the skew polynomial ring $K[\partial, \partial^{-1}][x; \delta]$. Of course, we may also localize $K[\ell]$ to obtain $K[\ell, \ell^{-1}][x; \delta]$ by using an analogous construction. These two rings are naturally isomorphic, as we will now prove. In the following proofs, we will make use of the universal property of skew polynomial rings [9, Prop. 3.6] [20, §1.2.5] that allows to lift differential homomorphisms from coefficients to skew polynomials.

Proposition 16. *The map*

$$K[\partial, \partial^{-1}][x; \delta] \xrightarrow{\sim} K[\ell, \ell^{-1}][x; \delta]$$

induced by $\partial \mapsto \ell^{-1}$ *is an isomorphism.*

Proof. The map φ induced by $\partial \mapsto \ell^{-1}$ is a differential homomorphism between $K[\partial, \partial^{-1}]$ and $K[\ell, \ell^{-1}]$ since $\delta(\varphi(\partial)) = \delta(\ell^{-1}) = -\ell^2/\ell^2 = -1 = \varphi(\delta(\partial))$. By the universal property, its extension to $K[\partial, \partial^{-1}][x; \delta]$ is also a homomorphism, and it is clearly bijective. □

The following lemma allows to transfer the skew polynomial structure across quotients as in the commutative case, compare also [9, Prop. 3.15].

Lemma 17. *Let* (R, δ) *be a differential ring and* $I \leq R$ *a differential ideal. Then*

$$(R/I)[x; \tilde{\delta}] \cong R[x; \delta]/(I)$$

as rings where (I) *denotes the ideal generated by* I *in* $R[x; \delta]$ *and* $\tilde{\delta}$ *is the derivation induced by* δ.

Proof. The proof is as in the commutative case. First we note that (I) consists exactly of the skew polynomials with coefficients in I. The canonical map $R \to R/I$ is a differential epimorphism and extends therefore to an epimorphism $R[x; \delta] \to (R/I)[x; \tilde{\delta}]$ by the universal property. Its kernel are all skew polynomials whose coefficients are in I. □

The next step is to explore the relation between $\mathrm{A}_1(\partial, \ell)$ and $K[\partial, \partial^{-1}][x; \delta]$. It is very natural—the latter arises from the former by making ℓ also a left inverse of ∂.

Theorem 18. *We have*

$$\mathrm{A}_1(\partial, \ell)/(\mathtt{E}) \cong K[\partial, \partial^{-1}][x; \delta]$$

as rings.

Proof. In Proposition 6 we have proved that there exists an isomorphism $\varphi\colon K\langle\partial, \ell\rangle/(\mathtt{E}) \to K[\partial, \partial^{-1}]$. Using again the universal property, there is a corresponding isomorphism $\tilde{\varphi}$ between the skew polynomial rings $(K\langle\partial, \ell\rangle/(\mathtt{E}))[x; \tilde{\delta}]$ and $K[\partial, \partial^{-1}][x; \delta]$, where $\tilde{\delta}$ denote the derivative induced by δ. The claim now follows from Lemma 17. □

We note that the localization can also be applied in the setting of Section 3 by factoring out $(\mathtt{E})$, leading to the isomorphism $\mathcal{F}[\partial, \textstyle\int]/(\mathtt{E}) \cong \mathcal{F}[\partial] \dotplus \mathcal{F}[\textstyle\int]$.

For reconstructing the ring $K[x][\partial, \textstyle\int]$ of Section 3 from $\mathrm{A}_1(\partial, \ell)$, we need a decomposition analogous to (12). Since the decomposition in Proposition 3 carries over coefficientwise to $\mathrm{A}_1(\partial, \ell)$, we obtain

$$\mathrm{A}_1(\partial, \ell) = \mathrm{A}_1(\partial) \dotplus \mathrm{A}_1(\ell)\ell \dotplus (\mathtt{E}), \tag{18}$$

where $(\mathtt{E})$ is the *evaluation ideal* in $\mathrm{A}_1(\partial, \ell)$. Note that this ideal consists of the skew polynomials with coefficients in $(\mathtt{E}) \subseteq K\langle\partial, \ell\rangle$ as observed before Proposition 8.

The key tool for fixing the integration constant $c \in K$ is the following refinement of the above decomposition. In analogy to the space $K[x][\mathtt{E}]$ introduced in Section 3, we consider the K-vector space $B \leq \mathrm{A}_1(\partial, \ell)$ with basis $(x^k \mathtt{E} \partial^j)$. Note that here and in the following we make use of the right basis $(x^k \partial^i, x^k \ell^i, x^k e_{ij})$ of $\mathrm{A}_1(\partial, \ell)$.

Lemma 19. *In* $\mathrm{A}_1(\partial, \ell)$, *we have for every* $c \in K$ *the decomposition*

$$(\mathtt{E}) = B \dotplus (\eta),$$

and $(x^k \ell^i \eta \partial^j)$ *is a* K*-basis for* (η), *where* $\eta = \mathtt{E}x - c\mathtt{E}$.

Proof. One can easily see $\mathtt{E}x = (x - \ell)\mathtt{E}$. This implies $\ell^{i-1}\eta = x\ell^{i-1}\mathtt{E} - i\ell^i\mathtt{E} - c\ell^{i-1}\mathtt{E} \in (\eta)$ and hence

$$x^k e_{ij} + \frac{c}{i}\, x^k e_{i-1,j} - \frac{1}{i}\, x^{k+1} e_{i-1,j} \in (\eta)$$

for $i \geq 1$. This allows to replace $x^k e_{ij}$ by terms with smaller powers of ℓ, eventually eliminating all occurrences of ℓ. This means that every element in $(\mathtt{E})$ may be represented as K-linear combination of elements of the form $x^k e_{0j} = x^k \mathtt{E} \partial^j$ and some element in (η).

We write η_{ij} for $\ell^i \eta \partial^j$ and H for the K-vector space generated by $x^k \eta_{ij}$. Obviously H is is a subspace of (η). The product of an element $x^k \eta_{ij}$ by ∂ or ℓ from the right is again in H. By Lemma 10 and by the Leibniz rule we may commute products of the form ℓx^k and ∂x^k, so left multiplication by ℓ and ∂ does not leave H either. Finally, H is also closed under right multiplication by x since $\eta x = (x - \ell)\eta$. Hence H is an ideal, which implies $H = (\eta)$.

For proving directness assume

$$\sum_{m,n} a_{mn} x^k e_{0n} = \sum_{i,j,k} b_{ijk} x^k \eta_{ij} \tag{19}$$

for suitable $a_{mn}, b_{ijk} \in K$. Converting the right-hand side to the basis $(x^k e_{ij})$ by $x^k \eta_{ij} = x^{k+1} e_{ij} - (i+1) x^k e_{i+1,j} - c x^k e_{ij}$ and choosing i maximal, we see that the terms b_{ijk} must all vanish because $(x^k e_{ij})$ is a K-basis and the left-hand side does not contain terms of the form $e_{i+1,j}$. Repeating this for smaller i, it follows that the sum is direct. Using the same argument with 0 as the left-hand side, we conclude that $(x^k \eta_{ij})$ is a K-basis of (η). □

Using the direct sum from Lemma 19, it is now immediate to draw the connection to the ring $K[x][\partial, \textstyle\int]$ of Section 3.

Theorem 20. *If* $\textstyle\int$ *is an integral operator for the standard derivation* ∂ *on* $K[x]$, *we have*

$$\mathrm{A}_1(\partial, \ell)/(\mathtt{E}x - c\mathtt{E}) \;\cong\; K[x][\partial, \textstyle\int]$$

with $c = \mathtt{E} \bullet x \in K$ *as the constant of integration.*

Proof. Using Lemma 19 and (18) we see that

$$\mathrm{A}_1(\partial, \ell)/(\eta) = \mathrm{A}_1(\partial) \dotplus \mathrm{A}_1(\ell)\ell \dotplus B.$$

As K-bases we can choose $(x^k \partial^i)$, the mid basis $(x^m \ell x^n)$ and $(x^k \mathtt{E} \partial^j)$, respectively. They map directly to the corresponding basis elements in $K[x][\partial, \textstyle\int]$ detailed in Section 3. This yields a K-linear isomorphism.

For proving that it is also an isomorphism of K-algebras, it suffices to verify that all identities in Table 1 are satisfied. The first six are immediate, for the $\textstyle\int f \int$ rule one uses Lemma 11, for the remaining two rules one can apply the identity $\ell x^k \mathtt{E} \equiv (x^{k+1} - c^{k+1})/(k+1)$ modulo $(\mathtt{E}x - c\mathtt{E})$. □

An alternative proof of Theorem 20 takes the detour via the free algebra $K\langle D, L, X\rangle$. Using its construction, one can show that $\mathrm{A}_1(\partial, \ell)/(\mathtt{E}x - c\mathtt{E})$ is isomorphic to the free algebra modulo the four relations $DL = 1$, $XD = DX - 1$, $XL = LX + L^2$, and $(1 - LD)X = c(1 - LD)$. It remains to prove that these four relations generate the identities of Table 1, which is laborious but straightforward.

7. CONCLUSION

The integro-differential Weyl algebra exhibits an interesting algebraic structure that deserves further study. Encoding integro-differential operators in a skew polynomial setting, it allows to recast our algebraic approach to linear boundary problems in a new language. We hope this will advance the algorithmic treatment of various operations [28], for example the computation of Green's operators and the factorization into lower-order problems.

The current formulation is still very limited in scope. Since we have taken only one character (necessarily the evaluation), boundary problems—both their formulation and their solution—are restricted initial value problems. Adjoining more characters in a skew polynomial setting will be an interesting task.

A more challenging extension concerns the transition from ODE to PDE, analogous to the classical Weyl algebra in several variables. As reported in [25], our algebraic setup (including the factorization) extends to boundary problems for PDE; the task is now to develop an algorithmic framework for relevant classes of such boundary problems. The skew polynomial approach initiated here could provide an appropriate vantage point.

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