Composing and Factoring Generalized Green's Operators and Ordinary Boundary Problems

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Abstract. We consider solution operators of linear ordinary boundary problems with "too many" boundary conditions, which are not always solvable. These generalized Green's operators are a certain kind of generalized inverses of differential operators. We answer the question when the product of two generalized Green's operators is again a generalized Green's operator for the product of the corresponding differential operators and which boundary problem it solves. Moreover, we show that—provided a factorization of the underlying differential operator—a generalized boundary problem can be factored into lower order problems corresponding to a factorization of the respective Green's operators. We illustrate our results by examples using the MAPLE package `IntDiffOp`, where the presented algorithms are implemented.

Keywords: Linear boundary problem, singular boundary problem, generalized Green's operator, reverse order law, integro-differential operator, factorization, ordinary differential equation.

1 Introduction

Although linear boundary problems play an important role in applied mathematics [1–4], there is little algebraic theory and algorithmic treatment of boundary problems. Current computer algebra systems provide many symbolic tools for differential equations, but boundary conditions are usually left to a backward solving procedure, which—depending on the forcing function and on the conditions—may or may not work.

In [5], a new operator based approach for symbolic computation with linear ordinary boundary problems was presented, which has constantly been extended over the last years [6–8]; see also [9] for a recent overview. The results needed are summarized in Section 2.

The most recent algorithms for regular boundary problems (that are uniquely solvable) are implemented in the TH∃OREM∀ system [8, 10], and in the MAPLE package `IntDiffOp` [11–13]. They do not only allow to compute solution operators (Green's operators), but also to factor regular boundary problems into lower order problems provided a factorization of the underlying differential operator. The factorization of boundary problems relies on the multiplicative structure

M. Barkatou et al. (Eds.): AADIOS 2012, LNCS 8372, pp. 116–134, 2014.

introduced in [6], which for regular problems corresponds to the multiplication of the respective solution operators in reverse order.

The `IntDiffOp` package also provides support for the class of singular problems treated in this paper: We consider boundary problems where the differential equation per se is solvable, but where inconsistent boundary conditions allow solutions only for forcing functions satisfying suitable *compatibility conditions*. As a simple example, consider the boundary problem

$$\boxed{\begin{array}{l} u''(x) = f(x) \\ u(1) = u'(1) = u'(0) = 0, \end{array}} \tag{1}$$

where the forcing function f clearly has to satisfy the compatibility condition $\int_0^1 f(\xi)\,\mathrm{d}\xi = 0$; see Example 1 for the corresponding code.

While Green's operators for regular boundary problems are right inverses of the differential operator, solution operators for singular problems can be described as generalized inverses. Algorithms for computing such generalized Green's operators and the compatibility conditions of a singular boundary problem are presented in [11]; we briefly recall the basic results in Section 3. For singular boundary problems and generalized or modified Green's functions in analysis, we refer for example to [4] and [14], and in the context of generalized inverses to [15, Sec. 9.4], [16], and [17, Sec. H].

The goal of this paper is to extend the factorization algorithm for regular boundary problems [6, 8] to generalized boundary problems. To this end, we have earlier investigated the multiplicative structure of generalized inverses in [18]. It turns out that, in contrast to the regular case, the product of generalized Green's operators is not, in general, itself a generalized Green's operator.

In Section 4, we define a new composition of generalized boundary problems, which includes the composition of regular problems, based on results from [13, 18]. Then we discuss algorithmic methods for testing when the so-called reverse order law holds, that is, when the product of two generalized Green's operators is again a generalized Green's operator of the product of the corresponding differential operators. Moreover, if the reverse order law holds, we can algorithmically determine which boundary problem is solved by the product of two generalized Green's operators. We present a first implementation of the new algorithms in the framework of integro-differential operators and illustrate our results by examples, carried out using the `IntDiffOp` package[1].

Building on results of [13], we discuss in Section 5 a new algorithm and implementation for factoring a generalized boundary problem, such that the factorization corresponds to the product of the respective generalized Green's operators. We illustrate the algorithm also with examples for differential equations with non-constant coefficients. The right-hand factor computed by this method will always be a regular boundary problem. However, we also present some first steps for obtaining other possible factorizations in Section 6.

[1] Available at `http://www.risc.jku.at/people/akorpora/index.html`.

2 Symbolic Computation for Boundary Problems

The algebraic framework for treating linear ordinary boundary problems with symbolic methods is given by the algebra of integro-differential operators over an ordinary integro-differential algebra. This algebra was introduced in [5, 7] as a uniform language to express boundary problems—meaning differential equations and boundary conditions—as well as their Green's operators, which are integral operators. We review the basic properties and refer the reader to [7] and [9] for additional details. See also [19, 20] for an extensive study on algebraic properties of integro-differential operators with polynomial coefficients and a single evaluation (corresponding to initial value problems).

Extending *differential algebras* [21], where derivations are linear operators satisfying the Leibniz rule, *integro-differential algebras* $(\mathcal{F}, \partial, \int)$ are defined as a differential algebra $(\mathcal{F}, \partial)$ along with an "integral" $\int$ that is a linear right inverse of the derivation ∂ and satisfies an algebraic version of the *integration by parts* formula. For the similar notion of differential Rota-Baxter algebras, see [22] and for a detailed comparison [23].

We call an integro-differential algebra over a field K *ordinary* if $\ker \partial = K$. The standard example of an ordinary integro-differential algebra is given by $\mathcal{F} = C^\infty(\mathbb{R})$ with the usual derivation and the integral operator $\int\colon f \mapsto \int_a^x f(\xi)\,\mathrm{d}\xi$ for a fixed $a \in \mathbb{R}$. We call $\mathtt{E} = 1 - \int \circ\, \partial$ the *evaluation* of $\mathcal{F}$. For representing not only initial value problems, but arbitrary boundary problems, we include additional characters (multiplicative linear functionals) $\mathtt{E}_c\colon f \mapsto f(c)$ at various evaluation points $c \in \mathbb{R}$.

We write $\mathcal{F}\langle\partial\rangle$ for the the ring of differential operators with coefficients in $\mathcal{F}$ and $\mathcal{F}\langle\int\rangle$ for integral operators of the form $\sum_{i=1}^n f_i \int g_i$ with $f_i, g_i \in \mathcal{F}$. For a set of characters Φ, the corresponding (two-sided ideal of) *boundary operators* (Φ) are finite sums

$$\sum_{\varphi\in\Phi}\left(\sum_{i\in\mathbb{N}} f_{i,\varphi}\,\varphi\partial^i + \sum_{j\in\mathbb{N}} g_{j,\varphi}\varphi{\textstyle\int} h_{j,\varphi}\right)$$

with $f_{i,\varphi}, g_{j,\varphi}, h_{j,\varphi} \in \mathcal{F}$. *Stieltjes boundary conditions* are boundary operators where all the $f_{i,\varphi} \in K$ are constants and $g_{j,\varphi} = 1$, so that they act on $\mathcal{F}$ as linear functionals.

The integro-differential operators $\mathcal{F}_\Phi\langle\partial, \int\rangle$ are given as a direct sum of K-vector spaces

$$\mathcal{F}_\Phi\langle\partial, {\textstyle\int}\rangle = \mathcal{F}\langle\partial\rangle \dotplus \mathcal{F}\langle{\textstyle\int}\rangle \dotplus (\Phi).$$

The representation of integral operators and boundary conditions is not unique due to linearity of $\int$, for normal forms of integro-differential operators; see [7].

For solving boundary problems, we restrict ourselves to monic (i.e., having leading coefficient 1) differential operators that have a *regular fundamental system* $u_1, \ldots, u_n$, which means that the associated Wronskian matrix

$$W(u_1,\dots,u_n)=\begin{pmatrix} u_1 & \cdots & u_n \\ u_1' & \cdots & u_n' \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{pmatrix}$$

is regular. Then the *fundamental right inverse* (solving the initial value problem) can be computed as an integro-differential operator in $\mathcal{F}_\Phi\langle\partial,\textstyle\int\rangle$ by the *variation of constants* formula

$$T^\blacklozenge=\sum_{i=1}^{n} u_i {\textstyle\int} d^{-1}\, d_i, \tag{2}$$

where d is the determinant of the Wronskian matrix, $d_i=\det W_i$, and W_i is the matrix obtained from W by replacing the ith column by the nth unit vector.

Algorithms for solving and factoring regular ordinary boundary problems are described in [6, 7]. We recall the basic definitions and results. A *boundary problem* is defined as a pair $(T,\mathscr{B})$ consisting of a monic differential operator T of order n with a regular fundamental system and a space of boundary conditions $\mathscr{B}=\operatorname{span}(\beta_1,\dots,\beta_n)$ generated by n linearly independent Stieltjes boundary conditions. We think of T as a surjective linear operator between suitable K-vector spaces of "functions" $T\colon V\to W$ and the boundary conditions β_i acting as linear functionals from the dual space V^*. We describe the subspace of functions satisfying the boundary conditions via the orthogonal

$$\mathscr{B}^\perp=\{f\in V\mid \beta(f)=0 \text{ for all } \beta\in\mathscr{B}\},$$

i.e., $u\in V$ is a solution of $(T,\mathscr{B})$ for a given forcing function $f\in W$, if

$$Tu=f \quad\text{and}\quad u\in\mathscr{B}^\perp.$$

One easily checks that a boundary problem has a unique solution for each forcing function f if and only if $\operatorname{Ker}T\dotplus\mathscr{B}^\perp=V$: The sum $\operatorname{Ker}T+\mathscr{B}^\perp=V$ ensures the existence of a solution, the trivial intersection $\operatorname{Ker}T\cap\mathscr{B}^\perp=\{0\}$ implies its uniqueness. Such boundary problems are called *regular*, and the solution operator G that maps each $f\in W$ to its unique solution $u\in V$ is called *Green's operator*. The Green's operator is a right inverse of T and can be computed as $G=(1-P)T^\blacklozenge$, where P is the projector onto $\operatorname{Ker}T$ along $\mathscr{B}^\perp$, see also [7]. We will use the notation $G=(T,\mathscr{B})^{-1}$ for the Green's operator G of a regular problem $(T,\mathscr{B})$.

Regularity of a boundary problem can be tested via its *evaluation matrix*: Let $(u_1,\dots,u_n)$ be a basis of $\operatorname{Ker}T$ and $(\beta_1,\dots,\beta_n)$ a basis of $\mathscr{B}$. Then $(T,\mathscr{B})$ is regular iff

$$\beta(u)=\begin{pmatrix}\beta_1(u_1) & \dots & \beta_1(u_n)\\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \dots & \beta_n(u_n)\end{pmatrix}\in K^{n\times n}$$

is regular.

For factoring regular boundary problems and their Green's operators, we recall the composition of boundary problems in [6]. This composition corresponds to the product of their Green's operators in reverse order. Since $G_1 = (T_1, \mathscr{B}_1)^{-1}$ and $G_2 = (T_2, \mathscr{B}_2)^{-1}$ are right inverses of $T_1 \colon V \to W$ and $T_2 \colon U \to V$, the product $G_2 G_1$ obviously is a right inverse of $T_1 T_2$. Defining the composition of boundary problems as

$$(T_1, \mathscr{B}_1) \circ (T_2, \mathscr{B}_2) = (T_1 T_2, \mathscr{B}_2 + T_2^*(\mathscr{B}_1)), \tag{3}$$

where $T_2^* \colon W^* \to V^*$ denotes the transpose map $\beta \mapsto \beta \circ T_2$, the reverse order law

$$\big((T_1, \mathscr{B}_1) \circ (T_2, \mathscr{B}_2)\big)^{-1} = (T_2, \mathscr{B}_2)^{-1} (T_1, \mathscr{B}_1)^{-1} \tag{4}$$

always holds for regular problems. Using this multiplicative structure of boundary problems, it is always possible to split a regular boundary problem $(T, \mathscr{B})$ into regular lower order problems, provided there exists a factorization $T = T_1 T_2$ of the differential operator; see [8] for a constructive proof that requires only a fundamental system of T_2.

We conclude this section with a remark on the "function spaces" V and W on which we let the differential operator T act. The assumption $V = W = \mathcal{F}$ used for example in [7, 11]—ensuring well-definedness of arbitrary operations—for some applications is too restrictive. For a given boundary problem $(T, \mathscr{B})$ of order n, it is for example sufficient to consider n times continuously differentiable functions, i.e., $V = C^n[a, b]$ and $W = C[a, b]$, where a and b are the minimal and maximal evaluation point appearing in the Stieltjes conditions of $\mathscr{B}$. Similarly, for composing boundary problems $(T_1, \mathscr{B}_1)$ and $(T_2, \mathscr{B}_2)$ of order m and n, it suffices to restrict the domains as to consider $T_2 \colon C^{m+n}[a, b] \to C^m[a, b]$ and $T_1 \colon C^m[a, b] \to C[a, b]$ (with suitable choices of a and b).

3 Generalized Green's Operators

In [11], several methods of the previous section are generalized to boundary problems with "too many" boundary conditions, meaning that $\operatorname{ord} T < \dim \mathscr{B}$. These problems are not solvable for all forcing functions f; but we keep the condition $\operatorname{Ker} T \cap \mathscr{B}^\perp = \{0\}$ to ensure unique solutions. We briefly recall the basic results from [11].

Definition 1. *We call a boundary problem* $(T, \mathscr{B})$ semi-regular *if*

$$\operatorname{Ker} T \cap \mathscr{B}^\perp = \{0\}. \tag{5}$$

Let $(T, \mathscr{B})$ *be a boundary problem with* $T \colon V \to W$ *and* $E \leq W$*. We call the triple* $(T, \mathscr{B}, E)$ *regular, if* $(T, \mathscr{B})$ *is semi-regular and*

$$T(\mathscr{B}^\perp) \dotplus E = W. \tag{6}$$

In this case, we call E *an* exceptional space *for* $(T, \mathscr{B})$ *and* $(T, \mathscr{B}, E)$ *a* generalized boundary problem.

For simplicity, we also refer to the triple $(T, \mathscr{B}, E)$ as a boundary problem and identify $(T, \mathscr{B})$ with $(T, \mathscr{B}, \{0\})$. One easily checks that regularity of $(T, \mathscr{B})$ implies regularity of $(T, \mathscr{B}, \{0\})$ and vice versa, see also [13, Sec. 4.1].

For solving generalized boundary problems in analysis the forcing function f is projected onto $T(\mathscr{B}^\perp)$, the space of *admissible forcing functions*; see for example [4, 14]. This leads to the following algebraic definition of generalized Green's operators.

Definition 2. *Let $(T, \mathscr{B}, E)$ be regular with $T\colon V \to W$, and let Q be the projector onto $T(\mathscr{B}^\perp)$ along E. Then $u \in V$ is called a* solution *for $f \in W$ if*

$$Tu = Qf \quad \text{and} \quad u \in \mathscr{B}^\perp. \tag{7}$$

The generalized Green's operator *maps each $f \in W$ to the unique solution according to* (7).

As in the regular case, we use the notation $G = (T, \mathscr{B}, E)^{-1}$ for the generalized Green's operator for a regular boundary problem $(T, \mathscr{B}, E)$.

For testing semi-regularity of a boundary problem $(T, \mathscr{B})$ with $\operatorname{ord} T = m$ and $\dim \mathscr{B} = n$, we have to check whether the associated evaluation matrix $\beta(u) \in K^{n \times m}$ has full column rank, see also [11, Lem. 1]. Condition (6) can be checked analogously, provided an implicit description of $T(\mathscr{B}^\perp)$ via *compatibility conditions* $\mathscr{C} = T(\mathscr{B}^\perp)^\perp$: If $(\gamma_1, \ldots, \gamma_r)$ is a basis of $\mathscr{C}$ and $(e_1, \ldots, e_r)$ is a basis of E, then the corresponding evaluation matrix $\gamma(e) \in K^{r \times r}$ has to be regular. Moreover, in [11, Prop. 1] a method is presented to compute a basis of $\mathscr{C}$; we have

$$\mathscr{C} = G^*(\mathscr{B} \cap (\operatorname{Ker} T)^\perp) \tag{8}$$

for any right inverse G of T. Note that equation (8) requires computing the intersection of the finite dimensional space $\mathscr{B}$ with the finite codimensional space $(\operatorname{Ker} T)^\perp$, which can be done using the following observation; see for example [18].

Lemma 1. *Let the subspaces $U \leq V$ and $\mathscr{B} \leq V^*$ be generated respectively by $u = (u_1, \ldots, u_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Let $k^1, \ldots, k^r \in F^m$ be a basis of* $\operatorname{Ker} \beta(u)$, *and $\kappa^1, \ldots, \kappa^s \in F^n$ a basis of* $\operatorname{Ker}(\beta(u))^T$. *Then*

(i) $U \cap \mathscr{B}^\perp$ is generated by $\sum_{i=1}^m k_i^1 u_i, \ldots, \sum_{i=1}^m k_i^r u_i$ and
(ii) $U^\perp \cap \mathscr{B}$ is generated by $\sum_{i=1}^n \kappa_i^1 \beta_i, \ldots, \sum_{i=1}^n \kappa_i^s \beta_i$.

Example 1. We consider the boundary problem (1)

$$\boxed{\begin{array}{l} u''(x) = f(x) \\ u(1) = u'(1) = u'(0) = 0, \end{array}}$$

which reads in operator notation as $(\partial^2, \operatorname{span}(\mathtt{E}_1, \mathtt{E}_1\partial, \mathtt{E}_0\partial))$. We employ the standard integro-differential algebra $C^\infty(\mathbb{R})$ with the usual derivation and integral operator $\int\colon f \mapsto \int_0^x f(\xi)\, d\xi$ that is implemented in the Maple package `IntDiffOp` as presented in [11, 12].

In [11], we have already computed the compatibility conditions and the generalized Green's operator with respect to the exceptional space $E = \mathbb{R}$. We again carry out the computations in the `IntDiffOp` package, but using the interface `IntDiffOperations` for input of integro-differential operators. There, we use the symbols d and a for input of the differential and integral operator, and $e(c)$ for the evaluation at $c \in \mathbb{R}$. For the Maple output, the respective capital letters D, A, and $\mathtt{E}[c]$ are used, and the non-commutative multiplication of integro-differential operators is denoted by ".". The constructors BP, GBP, BC and ES are used for input of respectively boundary problems, generalized boundary problems, boundary conditions, and exceptional spaces.

```
> with(IntDiffOp):
> with(IntDiffOperations):
> t1 := d^2:    #input differential operator
> b1 := e(1): b2 := e(1).d: b3 := e(0).d:    #boundary conditions
> B1 := BC(b1, b2, b3):
> C1 := CompatibilityConditions(BP(t1, B1));
```

$$\mathrm{BC}(\mathtt{E}[1].\,\mathrm{A})$$

```
> E1 := ES(1):    #exceptional space
> bp1 := GBP(t1, B1, E1):    #generalized boundary problem
> g1 := GreensOperator(bp1);
```

$$x.\,\mathrm{A} - \mathrm{A}\,.x + \left(-\frac{1}{2}x^2 - \frac{1}{2}\right).\mathtt{E}_1.\,\mathrm{A} + \mathtt{E}_1.\,\mathrm{A}\,.x$$

4 Composition of Generalized Green's Operators

In Section 2, we have already recalled the multiplicative structure for boundary problems introduced in [6, 7]. In contrast to the regular case, where Green's operators always satisfy the reverse order law, the situation is more involved for generalized Green's operators since they are not right inverses of the differential operator.

Proposition 1. *Let $(T, \mathscr{B}, E)$ be regular with $T\colon V \to W$ and $G = (T, \mathscr{B}, E)^{-1}$ its generalized Green's operator. Then $GTG = G$, that is, G is an outer inverse of T.*

Proof. By definition of generalized Green's operators, we have $Tu = Qf$ for all $f \in W$, as well as $Gf = GQf = u$, where Q denotes the projector onto $T(\mathscr{B}^\perp)$ along E. Hence $TGf = Tu = Qf$, and $GTGf = GQf = Gf$ for all $f \in W$.

In terms of generalized inverses, the Green's operator of a regular problem $(T, \mathscr{B}, E)$ can therefore also be defined as the unique outer inverse G of T with $\operatorname{Im} G = \mathscr{B}^\perp$ and $\operatorname{Ker} G = E$. In particular, for an outer inverse G of T, the boundary problem $(T, (\operatorname{Im} G)^\perp, \operatorname{Ker} G)$ is regular, and G is its generalized Green's operator; see also [13, Rmk. 4.7].

The composition G_2G_1 of two outer inverses of T_1 and T_2 is in general not an outer inverse of the product T_1T_2. However, from the above considerations it is clear that if G_2G_1 is an outer inverse of T_1T_2, then computing its kernel and image yields the boundary problem it solves. For a proof of the following result, see [18, Thm. 6.2] or [13, Thm. 3.27].

Theorem 1. *Let $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$ be regular with $T_1 \colon V \to W$, $T_2 \colon U \to V$ and $G_1 = (T_1, \mathscr{B}_1, E_1)^{-1}$, $G_2 = (T_2, \mathscr{B}_2, E_2)^{-1}$ their generalized Green's operators. If G_2G_1 is an outer inverse of T_1T_2, the boundary problem*

$$\big(T_1T_2, \mathscr{B}_2 + T_2^*(\mathscr{B}_1 \cap E_2^\perp), E_1 + T_1(\mathscr{B}_1^\perp \cap E_2)\big) \tag{9}$$

is regular with generalized Green's operator G_2G_1. Furthermore, the two sums $\mathscr{B}_2 \dotplus T_2^*(\mathscr{B}_1 \cap E_2^\perp)$ and $E_1 \dotplus T_1(\mathscr{B}_1^\perp \cap E_2)$ are direct.

Based on Equation (9), we define the composition of two arbitrary boundary problems as follows.

Definition 3. *The* composition *of two boundary problems is defined as*

$$(T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2) = \big(T_1T_2, \mathscr{B}_2 + T_2^*(\mathscr{B}_1 \cap E_2^\perp), E_1 + T_1(\mathscr{B}_1^\perp \cap E_2)\big), \tag{10}$$

assuming that the composition T_1T_2 is defined.

This definition clearly reduces to the composition of regular boundary problems (3) when $E_1 = E_2 = \{0\}$. The composition of generalized boundary problems is implemented in the following algorithm.

Algorithm 1 (Composition).

Input *Two boundary problems $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$,*
$\beta_1, \ldots, \beta_n$ and $\tilde{\beta}_1, \ldots, \tilde{\beta}_\nu$ bases of $\mathscr{B}_1$ and $\mathscr{B}_2$,
$e_1, \ldots, e_t$ and $\tilde{e}_1, \ldots, \tilde{e}_\tau$ bases of E_1 and E_2.
Output *The composite boundary problem $(T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2)$.*

1. *Multiply $T = T_1T_2 \in \mathcal{F}_\Phi\langle \partial, \textstyle\int \rangle$.*
2. *Compute a basis $\gamma_1, \ldots, \gamma_k$ of $\mathscr{B}_1 \cap E_2^\perp$ using Lemma 1.*
3. *Compute a basis $v_1, \ldots, v_\ell$ of $\mathscr{B}_1^\perp \cap E_2$ using Lemma 1.*
4. *For $1 \le i \le k$ multiply $\delta_i = \gamma_i T_2 \in \mathcal{F}_\Phi\langle \partial, \textstyle\int \rangle$.*
5. *For $1 \le j \le \ell$ compute $t_j = T_1(v_j)$.*
6. *Compute a basis $\alpha_1, \ldots, \alpha_q$ of $\operatorname{span}(\tilde{\beta}_1, \ldots, \tilde{\beta}_\nu, \delta_1, \ldots, \delta_k)$.*
7. *Compute a basis $f_1, \ldots, f_r$ of $\operatorname{span}(e_1, \ldots, e_t, t_1, \ldots, t_\ell)$.*
8. *Return $(T, (\alpha_1, \ldots, \alpha_q), (f_1, \ldots, f_r))$.*

We are especially interested in the situation of Theorem 1, that is, the case where the composition of boundary problems corresponds to the composition of their generalized Green's operators. For testing when G_2G_1 is an outer inverse of T_1T_2, we use the following characterization from [13, 18], which is based on results from [24] and [25]. It gives necessary and sufficient conditions on the subspaces $\mathscr{B}_1^\perp$, $T_2(\mathscr{B}_2^\perp)$, E_2, and $T_1^{-1}(E_1)$ such that the revers order law

$$((T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2))^{-1} = (T_2, \mathscr{B}_2, E_2)^{-1}(T_1, \mathscr{B}_1, E_1)^{-1} \tag{11}$$

for the respective generalized Green's operators holds. The conditions can be checked using only the input data, in particular, without computing G_2 or G_1.

Theorem 2. *Let $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$ be regular with $T_1 \colon V \to W$, $T_2 \colon U \to V$ and $G_1 = (T_1, \mathscr{B}_1, E_1)^{-1}$, $G_2 = (T_2, \mathscr{B}_2, E_2)^{-1}$ their generalized Green's operators. Let $\mathscr{C}_2 = T_2(\mathscr{B}_2^\perp)^\perp$ and $K_1 = T_1^{-1}(E_1)$. The following conditions are equivalent:*

(i) G_2G_1 *is an outer inverse of* T_1T_2,
(ii) $(T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2)$ *is regular with Green's operator* G_2G_1,
(iii) $\mathscr{C}_2 + (\mathscr{B}_1 \cap E_2^\perp) \geq \mathscr{B}_1 \cap (E_2 \cap K_1)^\perp$,
(iv) $\mathscr{B}_1 \geq \mathscr{C}_2 \cap (E_2 \cap \mathscr{B}_1^\perp)^\perp \cap (E_2 \cap K_1)^\perp$,
(v) $K_1 \dotplus (E_2 \cap \mathscr{B}_1^\perp) \geq E_2 \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$,
(vi) $E_2 \geq K_1 \cap (\mathscr{B}_1 \cap E_2^\perp)^\perp \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$.

All conditions are formulated so that they can be checked algorithmically with the methods from the end of Section 2. For non-algorithmic purposes, also the equivalent dual conditions are useful, as for example

$$\mathscr{B}_1^\perp \leq \mathscr{C}_2^\perp \dotplus (E_2 \cap \mathscr{B}_1^\perp) \dotplus (E_2 \cap K_1), \tag{12}$$

which is the dual statement of Condition (iv).

For testing the conditions of Theorem 2, it is necessary to determine the space $K_1 = T_1^{-1}(E_1)$. This can be done using the identity

$$T^{-1}(E) = G(E) \dotplus \operatorname{Ker} T \tag{13}$$

for the inverse image of a subspace, where G is an arbitrary right inverse of T. It can be verified directly, see also [6, Prop. A.12]. As always, we assume a regular fundamental system to be given, which allows to construct the fundamental right inverse. Since any right inverse of T is injective and because of the direct sum in (13), the output of the following algorithm is indeed a basis of $T^{-1}(E)$.

Algorithm 2 (Inverse Image).

Input *A monic differential operator $T \colon V \to W$ and a basis $e_1, \ldots, e_r$ of $E \leq W$. A regular fundamental system $s_1, \ldots, s_m$ of T.*
Output *A basis of $T^{-1}(E)$.*

1. *Compute the fundamental right inverse $T^\blacklozenge$ according to* (2).
2. *For $1 \leq i \leq r$ compute $k_i = T^\blacklozenge(e_i)$.*
3. *Return $s_1, \ldots, s_m, k_1, \ldots, k_r$.*

For testing the necessary and sufficient conditions of Theorem 2, we assume that for finite-dimensional subspaces of V or V^*, we can compute sums and intersections and check inclusions. For the heuristic approach used in the `IntDiffOp` package, see [13, Sec. 7.4].

The following algorithm implements a test of Condition (iii), the others can be implemented similarly. (None of them seems to be particularly preferable from computational aspects.)

Algorithm 3 (Check Reverse Order Law).

Input *Two boundary problems* $(T_1, \mathscr{B}_1, E_1)$ *and* $(T_2, \mathscr{B}_2, E_2)$,
$\beta_1, \ldots, \beta_n$ *and* $\tilde{\beta}_1, \ldots, \tilde{\beta}_\nu$ *bases of* $\mathscr{B}_1$ *and* $\mathscr{B}_2$,
$e_1, \ldots, e_t$ *and* $\tilde{e}_1, \ldots, \tilde{e}_\tau$ *bases of* E_1 *and* E_2.
Regular fundamental systems $s_1, \ldots, s_m$ *of* T_1 *and* $\tilde{s}_1, \ldots, \tilde{s}_\ell$ *of* T_2.
Output ***true*** *if the reverse order law holds and* ***false*** *otherwise.*

1. *Compute a basis of* $T_1^{-1}(E_1)$ *with Algorithm 2.*
2. *Compute a basis of* $J = E_2 \cap T_1^{-1}(E_1)$.
3. *Compute a basis of* $B = \mathscr{B}_1 \cap J^\perp$ *using Lemma 1.*
4. *Compute a basis of* $K = \mathscr{B}_1 \cap E_2^\perp$ *using Lemma 1.*
5. *Compute the compatibility conditions* $\gamma_1, \ldots, \gamma_r$ *of* $(T_2, \mathscr{B}_2)$ *due to* (8).
6. *Compute* $C = \operatorname{span}(\gamma_1, \ldots, \gamma_r) + K$.
7. *If* $B \leq C$ *return* ***true****, else return* ***false****.*

Example 2. As a first example, we consider the generalized boundary problem $(\mathrm{D}^2, \operatorname{span}(\mathtt{E}_1, \mathtt{E}_1\,\mathrm{D}, \mathtt{E}_0\,\mathrm{D}), \mathbb{R})$ from Example 1. As a second boundary problem, we consider the differential operator $T_2 = \mathrm{D}^2 - 1$, also with the boundary conditions $\operatorname{span}(\mathtt{E}_1, \mathtt{E}_1\,\mathrm{D}, \mathtt{E}_0\,\mathrm{D})$, or, in usual notation

$$\boxed{\begin{array}{l} u''(x) - u(x) = f(x) \\ u(1) = u'(1) = u'(0) = 0. \end{array}}$$

As we will see below, the forcing function f has to satisfy the compatibility condition

$$\textstyle\int_0^1 \exp(-\xi) f(\xi)\,\mathrm{d}\xi + \int_0^1 \exp(\xi) f(\xi)\,\mathrm{d}\xi = 0,$$

so that $E_2 = \operatorname{span}(x)$ is a suitable exceptional space for the second boundary problem. Moreover, we compute the inverse image $K_1 = T_1^{-1}(E_1)$ for testing the conditions of Theorem 2.

```
> t2 := d^2-1: B2 := B1:
> C2 := CompatibilityConditions(BP(t2, B2)));
```
$$\mathrm{BC}(\mathtt{E}[1].\,\mathrm{A}\,.\exp(-x) + \mathtt{E}[1].\,\mathrm{A}\,.\exp(x))$$
```
> E2 := ES(x): bp2 := GBP(t2, B2, E2):
> IsRegular(GBP(t2, B2, E2));
```
$$true$$
```
> K1 := InverseImage(t1, E1);
```
$$\mathrm{ES}(1, x, x^2)$$

In this case $E_2 = \mathrm{span}(x) \leq \mathrm{span}(1, x, x^2) = K_1$, so Condition (iv) of Theorem 2 is trivially satisfied. We check that the reverse order law holds for the respective Green's operators.

```
> p1 := MultiplyBoundaryProblem(bp1, bp2);
```
$$\mathrm{GBP}(\mathrm{D}^4 - \mathrm{D}^2, \mathrm{BC}(\mathtt{E}[0].\,\mathrm{D}, \mathtt{E}[0].\,\mathrm{D}^3 - \mathtt{E}[1].\,\mathrm{D}^3, \mathtt{E}[1], \mathtt{E}[1].\,\mathrm{D}, \mathtt{E}[1].\,\mathrm{D}^2 - \mathtt{E}[1].\,\mathrm{D}^3), \mathrm{ES}(1))$$
```
> g := GreensOperator(p1):
> g2 := GreensOperator(bp2):
> IsZero(g-g2.g1);    #check reverse order law
```
$$true$$

Since for the boundary problems in the previous example, we have $T_1T_2 = T_2T_1 = \mathrm{D}^4 - \mathrm{D}^2$, we can also consider the product of Green's operators in reverse order, that is, test if $(T_1, \mathscr{B}_1, E_1)^{-1}(T_2, \mathscr{B}_2, E_2)^{-1}$ is an outer inverse of T_1T_2.

Example 3. We follow the steps of Algorithm 3, interchanging the indices accordingly. Recall from Example 1 that $\mathscr{C}_1$ is generated by $\mathtt{E}_1\,\mathrm{A}$ and that we have chosen the exceptional space $E_1 = \mathbb{R}$.

```
> K2 := InverseImage(t2, E2);
```
$$\mathrm{ES}(x, \exp(x), \exp(-x))$$
```
> J := Intersection(E1, K2);
```
$$\mathrm{ES}()$$
```
> B := Intersection(B2, J, space = dual);
```
$$\mathrm{BC}(\mathtt{E}[1],\ \mathtt{E}[1].\,\mathrm{D},\ \mathtt{E}[0].\,\mathrm{D})$$
```
> K := Intersection(B2, E1, space = dual);
```
$$\mathrm{BC}(\mathtt{E}[1].\,\mathrm{D},\ \mathtt{E}[0].\,\mathrm{D})$$
```
> TestComposition(bp2, bp1);    #check reverse order law
```
$$false$$

We see that in the above example Algorithm 3 returns *false* since the inclusion

$$\mathscr{C}_1 + (\mathscr{B}_2 \cap E_1^\perp) \geq \mathscr{B}_2 \cap (E_1 \cap K_2)^\perp$$

from Theorem 2 (ii) is not satisfied: the left-hand side does not contain the evaluation $\mathtt{E}_1$. We also verify directly that the product G_1G_2 indeed is not an outer inverse of T_1T_2.

```
> t := t2.t1: g:= g1.g2:
> IsZero((g.t).g-g);
```

$$false$$

```
> p2 := MultiplyBoundaryProblem(bp2, bp1);
```

$$\mathrm{GBP}(\mathrm{D}^4 - \mathrm{D}^2, \mathrm{BC}(\mathtt{E}[0].\,\mathrm{D}, \mathtt{E}[0].\,\mathrm{D}^3, \mathtt{E}[1], \mathtt{E}[1].\,\mathrm{D}, \mathtt{E}[1].\,\mathrm{D}^3), \mathrm{ES}(x))$$

```
> IsRegular(p2);
```

$$true$$

Although G_1G_2 is not an outer inverse of T_2T_1, in this case the composite boundary problem $(T_2, \mathscr{B}_2, E_2) \circ (T_1, \mathscr{B}_1, E_1)$ is regular. In general, we do not even obtain semi-regularity of $(T_1T_2, \mathscr{B}_2 + T_2^*(\mathscr{B}_1 \cap E_2^\perp))$, see [13, Thm. 4.21] for more details.

5 Factorization of Generalized Boundary Problems

In this section, we discuss certain classes of factorizations of generalized boundary problems. We start with a regular boundary problem $(T, \mathscr{B}, E)$ and a factorization $T = T_1T_2$ of the underlying differential operator. The factorization algorithm presented in this section works for arbitrary factorizations of T, regardless of the coefficient domain, as long as a regular fundamental system of T_2 is known. In our package, we use the function `DFactor` from the Maple `DETools` package to factor differential operators with rational coefficients (Example 6), unless a particular factorization is specified in the input.

As in [6] for regular boundary problems, the overall goal would be to characterize all regular boundary problems $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$ such that

$$(T, \mathscr{B}, E) = (T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2).$$

For generalized boundary problems, we additionally have to require that the reverse order law

$$((T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2))^{-1} = (T_2, \mathscr{B}_2, E_2)^{-1}(T_1, \mathscr{B}_1, E_1)^{-1}$$

holds, which is always valid for regular boundary problems (4). Due to the structure of the composite (10), where information about $\mathscr{B}_1$ gets lost when intersecting with E_2, and the rather involved interactions of the subspaces $\mathscr{B}_1^\perp, K_1, \mathscr{C}_2^\perp$ and E_2 in Theorem 2, it is a difficult task to describe *all* possible factorizations.

In the following, we discuss a special case of factoring a regular boundary problem $(T, \mathscr{B}, E)$: As a right factor, we strive for a regular problem $(T_2, \mathscr{B}_2)$, meaning that $E_2 = \{0\}$. Then the reverse order law is trivially satisfied, and the composite takes the easier form

$$(T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2) = (T_1 T_2, \mathscr{B}_2 \dotplus T_2^*(\mathscr{B}_1), E_1).$$

The existence of such a factorization was proven in [13]. It relies on the fact that for a semi-regular boundary problem $(T, \mathscr{B})$ and a factorization $T = T_1 T_2$, there always exists $\mathscr{B}_2 \leq \mathscr{B}$ such that $(T_2, \mathscr{B})$ is regular. (For ordinary differential equations, this can be seen immediately by inspecting the evaluation matrix.)

Theorem 3. *Let $(T, \mathscr{B})$ be a semi-regular boundary problem and let $T = T_1 T_2$ be a factorization of T into monic differential operators. For each exceptional space E of $(T, \mathscr{B})$ there exists a unique regular boundary problem $(T_1, \mathscr{B}_1, E)$ such that for each $\mathscr{B}_2 \leq \mathscr{B}$ for which $(T_2, \mathscr{B}_2)$ is regular, we have*

$$(T, \mathscr{B}, E) = (T_1, \mathscr{B}_1, E) \circ (T_2, \mathscr{B}_2) = (T_1 T_2, \mathscr{B}_2 \dotplus T_2^*(\mathscr{B}_1), E) \tag{14}$$

and $(T, \mathscr{B}, E)^{-1} = (T_2, \mathscr{B}_2)^{-1}(T_1, \mathscr{B}_1, E)^{-1}$. The boundary conditions of the left factor are given by $\mathscr{B}_1 = H_2^*(\mathscr{B} \cap (\operatorname{Ker} T_2)^{\perp})$, where H_2 is an arbitrary right inverse of T_2.

Example 4. We consider the factorization of the boundary problem `p2` from Example 3 with $\mathrm{D}^4 - \mathrm{D}^2 = (\mathrm{D}^2 - 1) \cdot \mathrm{D}^2$:

$$\begin{aligned}(\mathrm{D}^4 - \mathrm{D}^2, \operatorname{span}(\mathtt{E}_0\,\mathrm{D}, \mathtt{E}_0\,\mathrm{D}^3, \mathtt{E}_1, \mathtt{E}_1\,\mathrm{D}, \mathtt{E}_1\,\mathrm{D}^3), \operatorname{span}(x))\\ = (\mathrm{D}^2 - 1, \operatorname{span}(\mathtt{E}_0\,\mathrm{D}, \mathtt{E}_1\,\mathrm{D}, \mathtt{E}_1\,\mathrm{A}), \operatorname{span}(x)) \circ (\mathrm{D}^2, \operatorname{span}(\mathtt{E}_0\,\mathrm{D}, \mathtt{E}_1)).\end{aligned}$$

The following Maple session shows that the reverse order law holds for the respective Green's operators.

```
> p2;
```

$$\mathrm{GBP}(\mathrm{D}^4 - \mathrm{D}^2, \mathrm{BC}(\mathtt{E}[0].\,\mathrm{D}, \mathtt{E}[0].\,\mathrm{D}^3, \mathtt{E}[1], \mathtt{E}[1].\,\mathrm{D}, \mathtt{E}[1].\,\mathrm{D}^3), \mathrm{ES}(x))$$

```
> g := GreensOperator(p2):
> f1, f2 := FactorBoundaryProblem(p2, t2, t1);
```

$$\mathrm{GBP}(D^2 - 1, \mathrm{BC}(\mathtt{E}[0].\,\mathrm{D}, \mathtt{E}[1].\,\mathrm{D}, \mathtt{E}[1].\,\mathrm{A}), \mathrm{ES}(x)), \mathrm{BP}(\mathrm{D}^2, \mathrm{BC}(\mathtt{E}[0].\,\mathrm{D}, \mathtt{E}[1]))$$

```
> g1 := GreensOperator(f1): g2:= GreensOperator(f2):
> IsZero(g-g2.g1);
```

$$true$$

We now show how to construct the above factorization algorithmically, generalizing the method presented in [8]. We assume a given factorization $T = T_1 T_2$ of the differential operator and a regular fundamental system of T_2, which is needed in Step 4 to construct a right inverse of T_2.

Algorithm 4 (Right Regular Factorization).

Input *A regular boundary problem* $(T, \mathscr{B}, E)$,
$\beta_1, \ldots, \beta_n$ *a basis of* $\mathscr{B}$, $e_1, \ldots, e_r$ *a basis of* E.
A factorization $T = T_1 T_2$, *a reg. fundamental system* $s_1, \ldots, s_\mu$ *of* T_2.
Output *Two regular boundary problems* $(T_1, \mathscr{B}_1, E)$ *and* $(T_2, \mathscr{B}_2)$ *with* $(T, \mathscr{B}, E) = (T_1, \mathscr{B}_1, E) \circ (T_2, \mathscr{B}_2)$.

1. *Compute the evaluation matrix* $M = \beta(s) \in F^{n \times \mu}$.
2. *Compute* $S = (s_{i,j}) \in F^{n \times n}$ *s.t.* SM *is in reduced row echelon form.*
3. *For* $1 \leq i \leq n$ *set* $\tilde{\beta}_i = \sum_{k=1}^{n} s_{i,k} \beta_k$.
4. *Compute a right inverse* H_2 *of* T_2 *according to* (2).
5. *For* $\mu + 1 \leq j \leq n$ *multiply* $\alpha_{j-\mu} = \tilde{\beta}_j H_2 \in \mathcal{F}_\Phi\langle \partial, \textstyle\int \rangle$.
6. *Return* $(T_1, (\alpha_1, \ldots, \alpha_{n-\mu}), (e_1, \ldots, e_r))$ *and* $(T_2, (\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu))$.

Theorem 4. *Let* $(T, \mathscr{B}, E)$ *be regular and* $T = T_1 T_2$. *The boundary problems* $(T_1, \mathscr{B}_1, E)$ *and* $(T_2, \mathscr{B}_2)$ *computed by Algorithm 4 are regular and satisfy*

$$(T, \mathscr{B}, E) = (T_1, \mathscr{B}_1, E) \circ (T_2, \mathscr{B}_2).$$

Proof. In view of Theorem 3, we only have to show that $(T_2, \operatorname{span}(\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu))$ is regular with $\tilde{\beta}_i \in \mathscr{B}$ for $1 \leq i \leq \mu$, and that the Stieltjes conditions $\tilde{\beta}_{\mu+1}, \ldots, \tilde{\beta}_n$ computed in Step 3, are a basis of $\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp$.

First we observe that in Step 3 we obviously have $\tilde{\beta}_i \in \mathscr{B}$ for all i. Since $\operatorname{Ker} T_2 \leq \operatorname{Ker} T$ and since $(T, \mathscr{B})$ is semi-regular, $(T_2, \mathscr{B})$ is also semi-regular. Hence the evaluation matrix M computed in Step 1 has rank μ, and the first μ rows of SM in Step 2 give the $\mu \times \mu$ identity matrix. Since SM is the evaluation matrix $\tilde{\beta}(s)$—meaning that the $\tilde{\beta}$ are applied to the fundamental system of $\operatorname{Ker} T_2$—the boundary problem $(T_2, \operatorname{span}(\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu))$ is regular.

The lower part of SM is the $(n - \mu) \times \mu$ zero matrix, hence we know that

$$\operatorname{span}(\tilde{\beta}_{\mu+1}, \ldots, \tilde{\beta}_n) \leq \mathscr{B} \cap (\operatorname{Ker} T_2)^\perp.$$

Moreover, $\tilde{\beta}_{\mu+1}, \ldots, \tilde{\beta}_n$ are linearly independent, since S is regular. For proving that the spaces are equal, it suffices to show that $\dim(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp) = n - \mu$. Since $\mathscr{B}$ is finite dimensional, also $\dim(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp) < \infty$, and we have

$$\dim(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp) = \operatorname{codim}(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp)^\perp = \operatorname{codim}(\mathscr{B}^\perp + \operatorname{Ker} T_2),$$

where we use the duality principle for switching between a vector space and its dual as explained in [6, Sec. A.1]. Furthermore, [6, Cor. A.15] yields

$$\operatorname{codim}(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp)^\perp = \dim(\operatorname{Ker} T_2 \cap \mathscr{B}^\perp) + \operatorname{codim} \mathscr{B}^\perp - \dim \operatorname{Ker} T_2.$$

Since $(T_2, \mathscr{B})$ is semi-regular, the first summand vanishes, and moreover using that $\operatorname{codim} \mathscr{B}^\perp = \dim \mathscr{B}$, we obtain $\dim(\mathscr{B} \cap (\operatorname{Ker} T_2)^\perp) = n - \mu$.

Example 5. We demonstrate the steps of the algorithm in detail, continuing Example 4. Obviously, a regular fundamental system of D^2 is given by $(1, x)$ and a right inverse of D^2 is A^2. The evaluation matrix M and the transformation matrix S are given as

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence we compute the $\tilde{\beta}$ as follows

$$\tilde{\beta}_1 = \mathtt{E}_1 - \mathtt{E}_0 \mathrm{D},\ \tilde{\beta}_2 = \mathtt{E}_0 \mathrm{D},\ \tilde{\beta}_3 = \mathtt{E}_0 \mathrm{D}^3,\ \tilde{\beta}_4 = \mathtt{E}_1 \mathrm{D} - \mathtt{E}_0 \mathrm{D},\ \tilde{\beta}_5 = \mathtt{E}_1 \mathrm{D}^3 .$$

The space of boundary conditions for the right-hand factor is spanned by $\tilde{\beta}_1$ and $\tilde{\beta}_2$, which means that $\mathscr{B}_2 = \mathrm{span}(\mathtt{E}_1,\ \mathtt{E}_0 \mathrm{D})$. We multiply $\tilde{\beta}_3, \tilde{\beta}_4$ and $\tilde{\beta}_5$ by A^2 to obtain a basis of $\mathscr{B}_1$. Since $\mathrm{D}\,\mathrm{A} = 1$ and $\mathtt{E}_0 \mathrm{A} = 0$, this yields $\mathscr{B}_1 = \mathrm{span}(\mathtt{E}_0 \mathrm{D}, \mathtt{E}_1 \mathrm{A}, \mathtt{E}_1 \mathrm{D})$, as already seen in Example 4.

The previous example shows that the computation of the boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$ reduces to a linear algebra problem, which in our implementation is solved by the MAPLE procedures from the `LinearAlgebra` package. In the `IntDiffOp` package, the most time-consuming operations—factoring the differential operator and computing a fundamental system for the right factor—are done in preprocessing steps via the MAPLE commands `dsolve` and `DFactor`.

Example 6. On the interval $[0, 1]$, we consider the differential operator

$$\begin{aligned} \mathrm{D}^4 + \frac{5x^2 + 4x + 1}{(x+1)(x^2+1)} \mathrm{D}^3 + \frac{x^7 + x^6 + 2x^5 + 2x^4 - x^3 - 5x^2 + 14x + 10}{(x+1)(x^2+1)^2} \mathrm{D}^2 \\ + \frac{2(2x^8 + 2x^7 + 4x^6 + 4x^5 + x^4 + 2x^3 - 14x^2 - 16x + 3)}{(x^2+1)^3(x+1)} \mathrm{D} \\ + \frac{2(x^7 + x^6 + 2x^5 + 2x^4 + 5x^3 + 7x^2 - 4x - 2)}{(x^2+1)^3(x+1)} \end{aligned}$$

with coefficients in $\mathbb{Q}(x)$.

```
> a3 := (5*x^2+4*x+1)/((x+1)*(x^2+1)):
> a2 := (x^7 + x^6 + 2*x^5 + 2*x^4-x^3-5*x^2+14*x+10)/((x+1)*(x^2+1)^2):
> a1 := 2*(2*x^8+2*x^7+4*x^6+4*x^5+x^4+2*x^3-14*x^2-16*x+3)/((x^2+1)^3*(x+1)):
> a0 := 2*(x^7+x^6+2*x^5+2*x^4+5*x^3+7*x^2-4*x-2)/((x^2+1)^3*(x+1)):
> t := d^4 + a3.(d^3) + a2.(d^2) + a1.d + a0:
> b1 := e(0): b2 := e(0).d: b3 :=e(0).(d^2): b4 := e(1): b5 := e(1).d:
> bp := GBP(t, BC(b1, b2, b3, b4, b5), ES(1)):
> FactorBoundaryProblem(bp);
```

$$\mathrm{GBP}(x^2 + \frac{1}{1+x}.\,\mathrm{D} + \mathrm{D}^2, \mathrm{BC}(\mathtt{E}[0], \mathtt{E}[1].\,\mathrm{A}\,.x^2 + \mathtt{E}[1].\,\mathrm{A}, \mathtt{E}[1].\,\mathrm{A}\,.x^3 + \mathtt{E}[1].\,\mathrm{A}\,.x), \mathrm{ES}(1)),$$

$$\mathrm{BP}(\frac{2x}{x^2+1} + \mathrm{D}, \mathrm{BC}(\mathtt{E}[0])),\quad \mathrm{BP}(\frac{2x}{x^2+1} + \mathrm{D}, \mathrm{BC}(\mathtt{E}[0]))$$

We conclude this section with an example for a more general differential operator, which cannot be factored with the `DFactor` command. In this case, a factorization of the differential operator has to be provided in the input data.

Example 7. We consider a differential operator on $\Omega = (0, \infty)$ with coefficients in $C^\infty(\Omega)$.

$$\mathrm{D}^2 - \frac{\exp(x) + \exp(2x) - 1}{\exp(x) - 1}\,\mathrm{D} + \frac{\exp(2x)}{\exp(x) - 1}$$

with boundary conditions $\mathtt{E}_1, \mathtt{E}_2, \mathtt{E}_3$, and the factorization

$$T_1 = \mathrm{D} - \frac{\exp(2x)}{\exp(x) - 1} \quad \text{and} \quad T_2 = \mathrm{D} - 1.$$

```
> t := d^2-(exp(x)+exp(2*x)-1)/(exp(x)-1).d+exp(2*x)/(exp(x)-1):
> t1 := d-exp(2*x)/(exp(x)-1):
> t2 := d-1:
> b1 := e(1): b2 := e(2): b3 := e(3):
> bp := GBP(t, BC(b1, b2, b3), ES(1)):
> FactorBoundaryProblem(bp, t1, t2);
```

$$\mathrm{GBP}(\frac{-\exp(2x)}{\exp(x)-1} + \mathrm{D}, \mathrm{BC}(\mathtt{E}[1].\,\mathrm{A}\,.\exp(-x), \mathtt{E}[2].\,\mathrm{A}\,.\exp(-x) - \mathtt{E}[3].\,\mathrm{A}\,.\exp(-x)), \mathrm{ES}(1)),$$

$$\mathrm{BP}(-1 + \mathrm{D}, \mathrm{BC}(\mathtt{E}[1]))$$

6 Conclusion and Outlook

As outlined in the previous section, describing all possible factorizations of a generalized boundary problem along a fixed factorization of the differential operator is quite involved. Nevertheless, it would sometimes be preferable to have the generalized factor on the right, for which we assume symbolic solutions for the differential operator. In this section, we describe some first steps in this direction that rely on the following result.

Theorem 5. *Let $(T_1, \mathscr{B}_1)$ and $(T_2, \mathscr{B}_2)$ be semi-regular with $T_1 \colon V \to W$ and $T_2 \colon U \to V$. Then there exists an exceptional space $E_2 \leq V$ of $(T_2, \mathscr{B}_2)$ such that the respective generalized Green's operators satisfy the reverse order law for all possible exceptional spaces $E_1 \leq W$ of $(T_1, \mathscr{B}_1)$.*

Proof. Let V_1 be a complement of $\mathscr{B}_1^\perp \cap T_2(\mathscr{B}_2^\perp)$ in $\mathscr{B}_1^\perp$, i.e.,

$$\mathscr{B}_1^\perp = \big(\mathscr{B}_1^\perp \cap T_2(\mathscr{B}_2^\perp)\big) \dotplus V_1.$$

Since $V_1 \leq \mathscr{B}_1^\perp$ and because of the direct sum, we then have

$$V_1 \cap T_2(\mathscr{B}_2^\perp) = \big(V_1 \cap T_2(\mathscr{B}_2^\perp)\big) \cap \mathscr{B}_1^\perp = V_1 \cap \big(T_2(\mathscr{B}_2^\perp) \cap \mathscr{B}_1^\perp\big) = \{0\}.$$

Enlarging V_1 to a complement E_2 of $T_2(\mathscr{B}_2^\perp)$, so that $V_1 \leq E_2$ and $T_2(\mathscr{B}_2^\perp) \dotplus E_2 = V$ yields

$$\mathscr{B}_1^\perp = \big(\mathscr{B}_1^\perp \cap T_2(\mathscr{B}_2^\perp)\big) \dotplus V_1 \leq T_2(\mathscr{B}_2^\perp) + (E_2 \cap \mathscr{B}_1^\perp).$$

Hence Condition (12) is satisfied.

The previous proof is not constructive; choosing exceptional spaces for arbitrary Stieltjes boundary conditions leads to an interpolation problem with integral conditions. However, one can apply the following strategy to obtain a factorization of a semi-regular boundary problem $(T, \mathscr{B})$ into a regular problem $(T_1, \mathscr{B}_1)$ as a left factor and a generalized boundary problem as a right factor. In this case we will ignore the exceptional space in the beginning and (try to) choose it accordingly in the end.

1. Apply (a version of) Algorithm 4 to compute a semi-regular factor $(T_1, \mathscr{B}_1)$ and a regular factor $(T_2, \mathscr{B}_2)$.
2. Let $\alpha_1, \ldots, \alpha_n$ be a basis of $\mathscr{B}_1$. Choose a subset (wlog we write $\alpha_1, \ldots, \alpha_m$), such that $(T_1, \operatorname{span}(\alpha_1, \ldots, \alpha_m))$ is regular. Such a subset exists by Lemma 1 and can also be computed.
3. Compute $T_2^*(\alpha_{m+1}), \ldots, T_2^*(\alpha_n)$.
4. Since $(T_2, \mathscr{B}_2)$ is regular, $(T_2, \mathscr{B}_2 + \operatorname{span}(T_2^*(\alpha_{m+1}), \ldots, T_2^*(\alpha_n))$ is semi-regular, and from Theorem 5 we obtain an appropriate exceptional space.

Example 8. In Example 5, we have computed the spaces of boundary conditions

$$\mathscr{B}_1 = \operatorname{span}(\mathtt{E}_0\, \mathrm{D}, \mathtt{E}_1\, \mathrm{D}, \mathtt{E}_1\, \mathrm{A}) \quad \text{and} \quad \mathscr{B}_2 = \operatorname{span}(\mathtt{E}_0\, \mathrm{D}, \mathtt{E}_1)$$

for the differential operators $T_1 = \mathrm{D}^2 - 1$ and $T_2 = \mathrm{D}^2$. The boundary problem $(T_1, \operatorname{span} \mathtt{E}_0\, \mathrm{D}, \mathtt{E}_1\, \mathrm{D})$ is regular, and we compute

$$T_2^*(\mathtt{E}_1\, \mathrm{A}) = \mathtt{E}_1\, \mathrm{A}\, \mathrm{D}^2 = \mathtt{E}_1\, \mathrm{D} - \mathtt{E}_0\, \mathrm{D}\,.$$

Computation of the compatibility conditions for the modified right-hand factor $(T_2, \operatorname{span}(\mathtt{E}_0\, \mathrm{D}, \mathtt{E}_1, \mathtt{E}_1\, \mathrm{D}))$ yields $\mathscr{C}_2 = \operatorname{span}(\mathtt{E}_1\, \mathrm{A})$, hence we may choose for example $E_2 = \operatorname{span}(\exp(x))$. Then, since $E_2 \leq \operatorname{Ker} T_1$, Condition (iii) of Theorem 2 is trivially fulfilled.

Closer investigation of Algorithm 4 and the above procedure indicates some redundancy in first applying H_2^* and afterwards T_2^*. However, we cannot decide in advance for which choices of α_i the boundary problem $(T_1, \operatorname{span}(\alpha_1, \ldots, \alpha_m))$ will be regular. We will study this and related question for obtaining different factorizations of generalized boundary problems in future work.

We are also investigating algebraic and algorithmic aspects of generalized Green's operators for certain classes of linear partial differential equations and boundary conditions. While the linear algebra setting used in this paper is in principle applicable to partial differential equations, examples and constructive methods for singular partial boundary problems are to be studied. In this context, we refer to [6, 8, 10, 26] for regular partial boundary problems. The setting in [26] includes also inhomogeneous boundary conditions and a rewrite system for partial integro-differential operators (PIDOS) including linear substitution.

Acknowledgements. We would like to thank the anonymous referees for their numerous helpful comments. G.R. was partially supported by the Austrian Science Fund (FWF): J 3030-N18.

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