Symbolic Computation with Integro-Differential Operators

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ABSTRACT

The algebraic and algorithmic study of integro-differential algebras and operators has only started in the past decade. Integro-differential operators allow us in particular to study initial value and boundary problems for linear ODEs from an algebraic point of view. Differential operators already provide a rich algebraic structure with a wealth of results and algorithmic methods. Adding integral operators and evaluations, many new phenomena appear, including zero devisors and non-finitely generated ideals.

In this tutorial, we give an introduction to symbolic methods for integro-differential operators and boundary problems developed over the last years. In particular, we discuss normal forms, basic algebraic properties, and the computation of polynomial solutions for ordinary integro-differential equations with polynomial coefficients. We will also outline methods for manipulating and solving linear boundary problems and illustrate them with an implementation.

Keywords

integro-differential operators; integro-differential algebra; linear boundary problem; integro-differential equations; polynomial solutions

1. OVERVIEW

Boundary (value) problems and integro-differential equations are ubiquitous in science, engineering, and applied mathematics; see, e.g., [2, 11, 34]. While algebraic structures and computer algebra for differential equations per se are very well developed, the investigation of their integro-differential counterparts has started only recently. For studying linear differential equations, the basic algebraic structure is the noncommutative ring of differential operators over a differential algebra. More generally, skew polynomials are used for an algebraic and algorithmic treatment of many common operator algebras; see, e.g., [22, 9, 35, 8] and the survey [14] describing also implementations in computer algebra systems. However, if we add an integral operator, the resulting algebra cannot be modeled that way.

The basic algebraic identities between the derivation and the integral operator are the fundamental theorem of calculus and integration by parts. Based on these identities, a new operator approach for symbolic computation with linear ordinary boundary problems was introduced in [26, 27] and generalized to a differential algebra setting in [31, 30]. For further references and an implementation in Theorema, see [32]. We refer to [29, 19, 17, 20, 13, 33, 28, 12, 18] for recent developments including partial differential equations, generalized Green's operators and functions, free integro-differential algebras, singular coefficients, and a finite tensor reduction system for integro-differential operators.

The notion of integro-differential algebras combines a differential algebra with an integral operator and the corresponding multiplicative evaluation. The associated integro-differential operators over an ordinary integro-differential algebra allow to express and compute with boundary problems (differential equation plus boundary conditions) and their solution (Green's) operators in a single algebraic structure. For the related notion of differential Rota-Baxter algebras, we refer to [16] and to [17] for a detailed comparison. A Rota-Baxter algebra is an algebra with one linear operator that generalizes the algebra of continuous functions with the integral operator; see [15] and the references therein.

The simplest integro-differential algebra is given by the univariate polynomials over a field of characteristic zero with the usual derivation and integration. However, integro-differential operators with polynomial coefficients reflect many aspects and properties of the general constructions, and we focus in the tutorial on this case. We will discuss a parametrized noncommutative Gröbner basis of the defining relations, the corresponding normal forms, and basic algebraic properties in comparison with the Weyl algebra [10]. In particular, the algebra of integro-differential operators contains zero-divisors and is non-Noetherian. Polynomial coefficients also allow for some special constructions, for example, as a factor algebra of a skew polynomial ring [25]. Using generalized Weyl algebras [4], numerous important results on integro-differential operators with polynomial coefficients are shown in [5, 6].

Computing polynomial solutions of ordinary differential equations is well-studied in symbolic computation; see, e.g., [1, 3, 7]. Building on these methods, we describe an approach [23] for computing polynomial solutions for a class of algorithmic Fredholm operators that includes also integro-

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differential operators. Using the Maple package `IntDiffOp` [21], we also illustrate how to compute Green's operators and to factor boundary problems [24] into lower order problems along a factorization of the differential operator.

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