On the product of projectors and generalised inverses

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We consider generalised inverses of linear operators on arbitrary vector spaces and study the question when their product in reverse order is again a generalised inverse. This problem is equivalent to the question when the product of projectors is again a projector, and we discuss necessary and sufficient conditions only in terms of the defining spaces. We give a new representation of the product of generalised inverses that does not require explicit knowledge of the factors. Our approach is based on implicit representations of subspaces via their orthogonals in the dual space. For Fredholm operators, the corresponding computations reduce to finite-dimensional problems. We illustrate our results with examples for matrices and linear ordinary boundary problems.

Keywords: generalised inverse; projector; reverse order law; Fredholm operator; linear boundary problem; duality

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1. Introduction

Analogues of the reverse order law \((AB)^{-1} = B^{-1}A^{-1}\) for bijective operators have been studied intensively for various kinds of generalised inverses. Most articles and books are concerned with the matrix case; see for example [2, 4, 7, 8, 14, 17, 21, 23, 25, 27]. For infinite-dimensional vector spaces, usually additional topological structures like Banach or Hilbert spaces are assumed; see for example [5, 6, 15]. In our approach, we systematically exploit duality results that hold in arbitrary vector spaces and a corresponding duality principle for statements about generalised inverses and projectors; see Appendix (A).

The validity of the reverse order law can be reduced to the question whether the product of two projectors is a projector (Section 2). This problem is studied in [10, 24, 26] for finite-dimensional vector spaces. We discuss necessary and sufficient conditions that carry over to arbitrary vector spaces and can be expressed only in terms of kernel and image of the respective operators (Section 4). Applying the duality principle leads to new conditions and a characterisation of the commutativity of two projectors that generalises a result from [1].

In Section 5, we translate the results for projectors to generalised inverses and obtain necessary and sufficient conditions for the reverse order law in arbitrary vector spaces. Based on these conditions, we give a short proof for the characterisation of two operators such that the reverse holds for all inner inverses (also called g-inverses or \{1\}-inverses). Moreover, we show that there always exist algebraic generalised inverses (also called \{1,2\}-inverses) of two operators \(A\) and \(B\) such that their product in reverse order is an algebraic generalised inverse of \(AB\).

Assuming the reverse order law to hold, Theorem 6.2 gives a representation of the product of two outer inverses (\{2\}-inverses) that can be computed using only

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kernel and image of the outer inverses of the factors. In this representation, we rely on a description of the kernel of a composition using inner inverses (Section 3) and implicit representations of subspaces via their orthogonals in the dual space. Moreover, we avoid the computation of generalised inverses by using the associated transpose map. Examples for matrices illustrating the results are given in Section 7.

An important application for our results is given by linear boundary problems (Section 9). Their solution operators (Green’s operators) are generalised inverses, and it is natural to express infinite dimensional solution spaces implicitly via the (homogeneous) boundary conditions they satisfy. Green’s operators for ordinary boundary problems are Fredholm operators, for which we can check the conditions for the reverse order law algorithmically and compute the implicit representation of the product (Section 8). Hence we can test if the product of two (generalised) Green’s operators is again a Green’s operator, and we can determine which boundary problem it solves.

2. Generalised inverses

In this section, we first recall basic properties of generalised inverses. For further details and proofs, we refer to [15, 16] and the references therein. Throughout this article, $U$, $V$, and $W$ always denote vector spaces over the same field $F$.

**Definition 2.1:** Let $T: V \to W$ be linear. We call a linear map $G: W \to V$ an inner inverse of $T$ if $TGT = T$ and an outer inverse of $T$ if $GTG = G$. If $G$ is an inner and an outer inverse of $T$, we call $G$ an algebraic generalised inverse of $T$.

This terminology of generalised inverses is adopted from [16]; other sources refer to inner inverses as generalised inverses or g-inverses, whereas algebraic generalised inverses are also called reflexive generalised inverses. Also the notations ${1}$-inverse (resp. ${2}$- and ${1,2}$-inverse) are used, which refer to the corresponding Moore-Penrose equations the generalised inverse satisfies.

**Proposition 2.2:** Let $T: V \to W$ and $G: W \to V$ be linear. The following statements are equivalent:

1. $G$ is an outer inverse of $T$,
2. $GT$ is a projector and $\text{Im} GT = \text{Im} G$,
3. $GT$ is a projector and $V = \text{Im} G + \text{Ker} GT$,
4. $GT$ is a projector and $W = \text{Im} T + \text{Ker} G$,
5. $TG$ is a projector and $\text{Ker} TG = \text{Ker} G$,
6. $TG$ is a projector and $W = \text{Ker} G + \text{Im} TG$,
7. $TG$ is a projector and $\text{Im} G \cap \text{Ker} T = \{0\}$.

Corresponding to (vii) and (vi), for subspaces $B \leq V$ and $E \leq W$ with

$$B \cap \text{Ker} T = \{0\} \quad \text{and} \quad W = E + T(B),$$

we can construct an outer inverse $G$ of $T$ with $\text{Im} G = B$ and $\text{Ker} G = E$ as follows; cf. [15, Cor. 8.2]. We consider the projector $Q$ with

$$\text{Im} Q = T(B), \quad \text{Ker} Q = E. \quad (1)$$

The restriction $T|_B: B \to T(B)$ is bijective since $B \cap \text{Ker} T = \{0\}$, and we can define $G = (T|_B)^{-1}Q$. One easily verifies that $G$ is an outer inverse of $T$ with $\text{Im} G = B$ and $\text{Ker} G = E$. Since by (iii) we have $V = B + T^{-1}(E)$, we define the
projector $P$ in analogy to $Q$ by

$$\text{Im } P = T^{-1}(E), \quad \text{Ker } P = B. \tag{2}$$

Then, by definition and by Proposition 2.2, we have

$$GTG = G, \quad TG = Q, \quad \text{and } GT = 1 - P,$$

and $G$ is determined uniquely by these equations. Hence an outer inverse depends only on the choice of $B$ and $E$, and we use the notations $G = O(T, B, E)$ and $G = O(T, P, Q)$ for $P$ and $Q$ as in (2) and (1).

Obviously, $G$ is an outer inverse of $T$ if and only if $T$ is an inner inverse of $G$. Therefore, we get a result analogous to Proposition 2.2 for inner inverses by interchanging the role of $T$ and $G$. The construction of inner inverses is not completely analogous to outer inverses, see [16, Prop. 1.3]. For subspaces $B \leq V$ and $E \leq W$ such that

$$V = \text{Ker } T + B \quad \text{and} \quad W = \text{Im } T + E, \tag{3}$$

an inner inverse $G$ of $T$ is given on $\text{Im } T$ by $(T|_B)^{-1}$ and can be chosen arbitrarily on $E$. For such an inner inverse with $B = \text{Im } GT$ and $E = \text{Ker } TG$, we write $G \in I(T, B, E)$.

For constructing algebraic generalised inverses, we start with direct sums as in (3), but require $\text{Ker } G = E$ and $\text{Im } G = B$. We use the notation $G = G(T, B, E)$.

The following result for inner inverses is well-known in the matrix case [20, 21, 26] and the proof remains valid for arbitrary vector spaces.

**Proposition 2.3:** Let $T_1: V \to W$ and $T_2: U \to V$ be linear with outer (resp. inner) inverses $G_1$ and $G_2$. Let $P = G_1T_1$ and $Q = T_2G_2$. Then $G_2G_1$ is an outer (resp. inner) inverse of $T_1T_2$ if and only if $QP$ (resp. $PQ$) is a projector.

**Proof:** Let $G_2G_1$ be an outer inverse of $T_1T_2$, that is,

$$G_2G_1 = G_2G_1T_1T_2G_2G_1.$$

Multiplying with $T_2$ from the left and with $T_1$ from the right yields

$$T_2G_2G_1T_1 = T_2G_2G_1T_1T_2G_2G_1T_1,$$

thus $QP = T_2G_2G_1T_1$ is a projector. For the other direction, we multiply the previous equation with $G_2$ from the left and $G_1$ from the right and use that $G_1T_1G_1 = G_1$ and $G_2T_2G_2 = G_2$. The proof for inner inverses follows by interchanging the roles of $T_1$ and $T_2$. \hfill \qed

3. Kernel of compositions

In this section, we describe the inverse image of the composition of two linear maps using inner inverses. For projectors, kernel and image of the composition can be expressed only in terms of kernel and image of the corresponding factors. Note that a projector is an inner inverse of itself.
Proposition 3.1: Let $T_1: V \to W$ and $T_2: U \to V$ be linear and $G_2$ an inner inverse of $T_2$. For a subspace $W_1 \leq W$, we have

$$(T_1T_2)^{-1}(W_1) = G_2(T_1^{-1}(W_1) \cap \text{Im } T_2) \oplus \text{Ker } T_2$$

for the inverse image of the composite. In particular, we have

$$\text{Ker } T_1T_2 = G_2(\text{Ker } T_1 \cap \text{Im } T_2) \oplus \text{Ker } T_2.$$ 

Proof: Since $T_2G_2$ is a projector onto $\text{Im } T_2$ by Proposition 2.2 (ii) (interchanging the role of $T$ and $G$), we have

$$T_1T_2(G_2(T_1^{-1}(W_1) \cap \text{Im } T_2) \oplus \text{Ker } T_2) = T_1Q_2(T_1^{-1}(W_1) \cap \text{Im } T_2) + 0$$

$$= T_1(T_1^{-1}(W_1) \cap \text{Im } T_2) \leq W_1 \cap \text{Im } T_1T_2 \leq W_1.$$ 

Conversely, let $u \in (T_1T_2)^{-1}(W_1)$. Then $T_2u = v$ with $v \in T_1^{-1}(W_1)$. Since also $v \in \text{Im } T_2$, we have

$$T_2(u - G_2v) = T_2u - Q_2v = T_2u - v = v - v = 0,$$

that is, $u - G_2v \in \text{Ker } T_2$. Writing $u = G_2v + u - G_2v$ yields $u \in G_2(T_1^{-1}(W_1) \cap \text{Im } T_2) \oplus \text{Ker } T_2$. The sum is direct since by Proposition 2.2 (vi) (interchanging the role of $T$ and $G$), we have $U = \text{Ker } T_2 \oplus \text{Im } G_2T_2$. □

Corollary 3.2: Let $T: V \to W$ be linear and let $P: V \to V$ and $Q: W \to W$ be projectors. Then we have

$$\text{Ker } TQ = (\text{Ker } T \cap \text{Im } Q) \oplus \text{Ker } Q \quad \text{and} \quad \text{Im } PT = (\text{Im } T + \text{Ker } P) \cap \text{Im } P.$$ 

Proof: Applying Proposition 3.1 yields

$$\text{Ker } TQ = Q(\text{Ker } T \cap \text{Im } Q) \oplus \text{Ker } Q = (\text{Ker } T \cap \text{Im } Q) \oplus \text{Ker } Q.$$ 

The statement for the image follows from the duality principle $A1$. □

This result generalises [26, Lemma 2.2], where kernel and image of a product $PQ$ of two projectors are computed like above for the case of $PQ$ again being a projector.

4. Products of projectors

In view of Proposition 2.3, we study necessary and sufficient conditions for the product of two projectors being a projector. Throughout this section let $P, Q: V \to V$ denote projectors.

The first of the following necessary and sufficient conditions for the product of $P$ and $Q$ to be a projector is mentioned as an exercise without proof in [3, p. 339]. In [10, Lemma 3] the same result is formulated for matrices but the proof is valid for arbitrary vector spaces. The second necessary and sufficient condition for the matrix case is given in [26, Lemma 2.2]. The simpler proof from [24] carries over to arbitrary vector spaces.
Lemma 4.1: The composition $PQ$ is a projector if and only if

$$\text{Im } PQ \leq \text{Im } Q + (\text{Ker } P \cap \text{Ker } Q)$$

if and only if

$$\text{Im } Q \leq \text{Im } P + (\text{Ker } P \cap \text{Im } Q) + (\text{Ker } P \cap \text{Ker } Q).$$

Using Corollary 3.2 and the duality principle A1, we obtain the following characterisation of the idempotency of $PQ$ only in terms of kernel and image of $P$ and $Q$.

Theorem 4.2: The following statements are equivalent:

(i) The composition $PQ$ is a projector,
(ii) $\text{Im } P \cap (\text{Im } Q + \text{Ker } P) \leq \text{Im } Q + (\text{Ker } P \cap \text{Ker } Q)$,
(iii) $\text{Im } Q \leq \text{Im } P + (\text{Ker } P \cap \text{Im } Q) + (\text{Ker } P \cap \text{Ker } Q)$,
(iv) $\text{Ker } Q + (\text{Ker } P \cap \text{Im } Q) \geq \text{Ker } P \cap (\text{Im } Q + \text{Im } P)$,
(v) $\text{Ker } P \geq \text{Ker } Q \cap (\text{Im } Q + \text{Ker } P) \cap (\text{Im } Q + \text{Im } P)$.

For algebraic generalised inverses, it is also interesting to have sufficient conditions for $PQ$ as well as $QP$ being projectors; for example, if $P$ and $Q$ commute. This can again be characterised only in terms of image and kernel of $P$ and $Q$. If $PQ = QP$, one sees with Corollary 3.2 that

$$\text{Im } PQ = \text{Im } P \cap \text{Im } Q \quad \text{and} \quad \text{Ker } PQ = \text{Ker } P + \text{Ker } Q. \quad (4)$$

In general, these conditions are necessary but not sufficient for commutativity of $P$ and $Q$, see [10, Ex. 1].

Using Corollary 3.2, modularity (A1), and (A2), one obtains the following characterisation of projectors with respectively image and kernel as in (4); for further details see [11]. For the commutativity of projectors see also [3, p. 339].

Proposition 4.3: The composition $PQ$ is a projector if and only if

(i) $\text{Im } PQ = \text{Im } P \cap \text{Im } Q$ if and only if

$$\text{Im } Q = (\text{Im } P \cap \text{Im } Q) \cap (\text{Ker } P \cap \text{Im } Q).$$

(ii) $\text{Ker } PQ = \text{Ker } P + \text{Ker } Q$ if and only if

$$\text{Ker } P = (\text{Ker } P \cap \text{Ker } Q) \cap (\text{Ker } P \cap \text{Im } Q).$$

Corollary 4.4: We have $PQ = QP$ if and only if

$$\text{Im } Q = (\text{Im } P \cap \text{Im } Q) \cap (\text{Ker } P \cap \text{Im } Q)$$

and

$$\text{Ker } Q = (\text{Im } P \cap \text{Ker } Q) \cap (\text{Ker } P \cap \text{Ker } Q).$$

In [10, Thm. 4] and [1, Thm. 3.2] different necessary and sufficient conditions for the commutativity of two projectors are given, but both require the computation of $PQ$ as well as of $QP$. 
5. Reverse order law for generalised inverses

Combining Proposition 2.3 and Theorem 4.2 gives necessary and sufficient conditions for the reverse order law for outer inverses to hold, only in terms of the defining spaces.

**Theorem 5.1:** Let \( T_1 : V \to W \) and \( T_2 : U \to V \) be linear with outer inverses \( G_1 = O(T_1, B_1, E_1) \) and \( G_2 = O(T_2, B_2, E_2) \). The following conditions are equivalent:

(i) \( G_2 G_1 \) is an outer inverse of \( T_1 T_2 \),
(ii) \( T_2(B_2) \cap (B_1 + E_2) \leq B_1 + (E_2 \cap T_1^{-1}(E_1)) \),
(iii) \( B_1 \leq T_2(B_2) + (E_2 \cap B_1) + (E_2 \cap T_1^{-1}(E_1)) \),
(iv) \( T_1^{-1}(E_1) \cap (E_2 \cap B_1) \geq E_2 \cap (B_1 + T_2(B_2)) \),
(v) \( E_2 \geq T_1^{-1}(E_1) \cap (B_1 + E_2) \cap (B_1 + T_2(B_2)) \).

**Proof:** Recall that \( \text{Im} G_i = B_i \) and \( \text{Ker} G_i = E_i \), and \( Q = T_2 G_2 \) and \( P = G_1 T_1 \) are projectors with

\[
\text{Im} P = B_1, \quad \text{Ker} P = T_1^{-1}(E_1), \quad \text{Im} Q = T_2(B_2), \quad \text{and} \quad \text{Ker} Q = E_2.
\]

By Proposition 2.3, \( G_2 G_1 \) is an outer inverse if and only if \( QP \) is a projector. Applying Theorem 4.2 proves the claim. \( \square \)

In the following theorem, we give the analogous conditions for inner inverses, where \( P = G_1 T_1 \) and \( Q = T_2 G_2 \) are the projectors corresponding to the direct sums in (3). Note that the conditions for inner inverses only depend on the choice of \( B_1 \) and \( E_2 \), but not on \( B_2 \) and \( E_1 \).

The characterisation (iii) and the orthogonal of (v) in the following theorem generalise [26, Thm. 2.3] to arbitrary vector spaces.

**Theorem 5.2:** Let \( T_1 : V \to W \) and \( T_2 : U \to V \) be linear with inner inverses \( G_1 \in I(T_1, B_1, E_1) \) and \( G_2 \in I(T_2, B_2, E_2) \). The following conditions are equivalent:

(i) \( G_2 G_1 \) is an inner inverse of \( T_1 T_2 \),
(ii) \( B_1 \cap (\text{Im} T_2 + \text{Ker} T_1) \leq \text{Im} T_2 + (\text{Ker} T_1 \cap E_2) \),
(iii) \( \text{Im} T_2 \leq B_1 + (\text{Ker} T_1 \cap \text{Im} T_2) + (\text{Ker} T_1 \cap E_2) \),
(iv) \( E_2 + (\text{Ker} T_1 \cap \text{Im} T_2) \geq \text{Ker} T_1 \cap (\text{Im} T_2 + B_1) \),
(v) \( \text{Ker} T_1 \geq E_2 \cap (\text{Im} T_2 + \text{Ker} T_1) \cap (\text{Im} T_2 + B_1) \).

The question when the reverse order law holds for all inner inverses of \( T_1 \) and \( T_2 \) was answered for matrices in [27, Thm. 2.3], and an alternative proof was given in [9]. Using the previous characterisations, we give a short proof that generalises the result to arbitrary vector spaces.

**Theorem 5.3:** Let \( T_1 : V \to W \) and \( T_2 : U \to V \) be linear. Then \( G_2 G_1 \) is an inner inverse of \( T_1 T_2 \) for all inner inverses \( G_1 \) of \( T_1 \) and \( G_2 \) of \( T_2 \) if and only if \( T_1 T_2 = 0 \) or \( \text{Ker} T_1 \leq \text{Im} T_2 \).

**Proof:** If \( \text{Ker} T_1 \leq \text{Im} T_2 \) then \( \text{Ker} T_1 \cap \text{Im} T_2 = \text{Ker} T_1 \) and (iii) in the previous theorem is satisfied since \( \text{Ker} T_1 + B_1 = V \). The case \( T_1 T_2 = 0 \) is trivial.

For the reverse implication, assume that \( \text{Im} T_2 \not\subseteq \text{Ker} T_1 \) and \( \text{Ker} T_1 \not\subseteq \text{Im} T_2 \). Choose \( V_1, V_2 \subseteq V \) such that we have two direct sums \( \text{Ker} T_1 = (\text{Im} T_2 \cap \text{Ker} T_1) + V_1 \) and \( \text{Im} T_2 = (\text{Im} T_2 \cap \text{Ker} T_1) + V_2 \). Then we have

\[
\text{Im} T_2 + \text{Ker} T_1 = (\text{Im} T_2 \cap \text{Ker} T_1) + V_1 + V_2.
\] (5)
By assumption, we can choose non-zero \( v_1 \in V_1 \) and \( v_2 \in V_2 \). Let \( v = v_1 + v_2 \). Then \( v \in \text{Im} T_2 + \text{Ker} T_1 \) and \( v \not\in \text{Ker} T_1 \), \( v \not\in \text{Im} T_2 \). Hence we can choose \( B_1 \) and \( E_2 \) such that \( v \in B_1 \) and \( v \in E_2 \) and \( V = \text{Ker} T_1 + B_1 = \text{Im} T_2 + E_2 \). Then

\[
v \in E_2 \cap (\text{Im} T_2 + \text{Ker} T_1) \cap (\text{Im} T_2 + B_1)
\]

but \( v \in \text{Ker} T_1 \). Hence (v) in the previous theorem is not satisfied for inner inverses with \( \text{Im} G_1 = B_1 \) and \( \text{Ker} G_2 = E_2 \).

Werner [26, Thm. 3.1] proves that for matrices it is always possible to construct inner inverses such that the reverse order law holds. Using the necessary and sufficient condition for outer inverses above, we extend this result to algebraic generalised inverses in arbitrary vector spaces. The special case of Moore-Penrose inverses is treated in [21, Thm. 3.2], and explicit solutions are constructed in [22, 28].

**Theorem 5.4:** Let \( T_1 : V \to W \) and \( T_2 : U \to V \) be linear. There always exist algebraic generalised inverses \( G_1 \) of \( T_1 \) and \( G_2 \) of \( T_2 \) such that \( G_2 G_1 \) is an algebraic generalised inverse of \( T_1 T_2 \).

**Proof:** Choose \( V_1, V_2 \leq V \) as in the previous proof such that (5) holds. Moreover, choose \( V_3 \leq V \) such that

\[
V = (\text{Im} T_2 + \text{Ker} T_1) + V_3 = (\text{Im} T_2 \cap \text{Ker} T_1) + V_1 + V_2 + V_3.
\]

Then \( B_1 = V_2 + V_3 \) is a direct complement of \( \text{Ker} T_1 \), and \( E_2 = V_1 + V_3 \) is a direct complement of \( \text{Im} T_2 \). Hence there exist respectively an algebraic generalised inverse \( G_1 \) of \( T_1 \) with \( \text{Im} G_1 = B_1 \) and \( G_2 \) of \( T_2 \) with \( \text{Ker} G_2 = E_2 \). We verify that such \( G_1 \) and \( G_2 \) satisfy Theorem 5.1 (iii), where \( T_1^{-1}(E_1) = \text{Ker} T_1 \) and \( T_2(B_2) = \text{Im} T_2 \) since \( G_1 \) and \( G_2 \) are algebraic generalised inverses:

\[
\text{Im} T_2 + (E_2 \cap B_1) \geq \text{Im} T_2 + V_3 = (\text{Im} T_2 \cap \text{Ker} T_1) + V_2 + V_3 \geq B_1.
\]

Similarly, we verify Theorem 5.2 (iii)

\[
B_1 + (\text{Ker} T_1 \cap \text{Im} T_2) = V_2 + V_3 + (\text{Ker} T_1 \cap \text{Im} T_2) \geq V_2 + (\text{Ker} T_1 \cap \text{Im} T_2) = \text{Im} T_2.
\]

Hence \( G_2 G_1 \) is an algebraic generalised inverse of \( T_1 T_2 \) for all \( G_1 = G(T_1, B_1, E_1) \) and \( G_2 = G(T_2, B_2, E_2) \), independent of the choice of \( E_1 \) and \( B_2 \). \( \square \)

6. Representing the product of outer inverses

In this section, we assume that for two linear maps \( T_1 : V \to W \) and \( T_2 : U \to V \) with outer inverses \( G_1 \) and \( G_2 \) the reverse order law holds. Our goal is to find a description of the product \( G_2 G_1 \) that does not require the explicit knowledge of \( G_1 \) and \( G_2 \). Using the representation via projectors, one immediately verifies that

\[
O(T_2, P_2, Q_2) O(T_1, P_1, Q_1) = O(T_1 T_2, P_2 - G_2 P_1 T_2, T_1 Q_2 G_1)
\]

but this expression involves both outer inverses \( G_1 \) and \( G_2 \). For the representation via defining spaces, we compute kernel and image of the product.
Lemma 6.1: Let $T_1: V \rightarrow W$ and $T_2: U \rightarrow V$ be linear with outer inverses $G_1 = O(T_1, B_1, E_1)$ and $G_2 = O(T_2, B_2, E_2)$. Then we have

$$\text{Ker} \ G_2G_1 = E_1 + T_1(B_1 \cap E_2) \quad \text{and} \quad \text{Im} \ G_2G_1 = G_2((B_1 + E_2) \cap \text{Im} \ T_2).$$

Proof: Recall that by definition Ker $G_i = E_i$ and Im $G_i = B_i$. Since $T_1$ is an inner inverse of $G_1$, the equation Ker $G_2G_1 = E_1 + T_1(B_1 \cap E_2)$ follows directly from Proposition 3.1.

For the statement about the image, we first recall that for a linear map $G$ and subspaces $V_1, V_2$, we have $G(V_1 \cap V_2) = G(V_1) \cap G(V_2)$ if Ker $G \leq V_1$. Hence

$$G_2((B_1 + E_2) \cap \text{Im} \ T_2) = G_2((\text{Im} \ G_1 + \text{Ker} \ G_2) \cap \text{Im} \ T_2) = G_2(\text{Im} \ G_1) \cap G_2(\text{Im} \ T_2) = \text{Im} \ G_2G_1 \cap \text{Im} \ G_2T_2 = \text{Im} \ G_2G_1 \cap \text{Im} \ G_2 = \text{Im} \ G_2G_1$$

since $\text{Im} \ G_2T_2 = \text{Im} \ G_2$ by Proposition 2.2 (ii).

Note that the expression for the image of the composition requires the explicit knowledge of $G_2$. In particular, the reverse order law takes the form

$$O(T_2, B_2, E_2) O(T_1, B_1, E_1) = O(T_1 T_2, G_2((B_1 + E_2) \cap \text{Im} \ T_2), E_1 + T_1(B_1 \cap E_2)).$$

Werner [26, Thm. 2.4] gives a result in a similar spirit for inner inverses of matrices.

Using an implicit description of Im $G_i$, it is possible to state the reverse order law in a form that only depends on the kernels and images of the respective outer inverses. This approach is motivated by our application to linear boundary problems (Section 9), where it is natural to define solution spaces via the boundary conditions they satisfy.

In more detail, the Galois connection from Appendix A allows to represent a subspace $B$ implicitly via the orthogonally closed subspace $\mathcal{B} = B^\perp$ of the dual space. We will therefore use the notation $G = O(T, \mathcal{B}, E)$ for the outer inverse with $\text{Im} \ G = \mathcal{B}^\perp$ and Ker $G = E$ as well as the analogue for inner inverses.

Theorem 6.2: Let $T_1: V \rightarrow W$ and $T_2: U \rightarrow V$ be linear with outer inverses $G_1 = O(T_1, \mathcal{B}_1, E_1)$ and $G_2 = O(T_2, \mathcal{B}_2, E_2)$. If $G_2G_1$ is an outer inverse of $T_1T_2$, we have

$$O(T_2, \mathcal{B}_2, E_2) O(T_1, \mathcal{B}_1, E_1) = O(T_1 T_2, \mathcal{B}_2 + T_2^* (\mathcal{B}_1 \cap E_2^\perp), E_1 + T_1(\mathcal{B}_1^\perp \cap E_2)), \quad (6)$$

where $T_2^*$ denotes the transpose of $T_2$.

Proof: From Lemma 6.1 we already know that Ker $G_2G_1 = E_1 + T_1(\mathcal{B}_1^\perp \cap E_2)$. From Proposition A.2 and 3.1 we get

$$(\text{Im} \ G_2G_1)^\perp = \text{Ker} \ G_1^*G_2^* = T_2^* (\text{Ker} \ G_1^* \cap \text{Im} \ G_2^*) = \text{Ker} \ G_2^*$$

$$= T_2^* ((\text{Im} \ G_1)^\perp \cap (\text{Ker} \ G_2)^\perp) + (\text{Im} \ G_2)^\perp = T_2^* (\mathcal{B}_1 \cap E_2^\perp) + \mathcal{B}_2,$$

and thus (6) holds.

A computational advantage of this representation is that one can determine $G_2G_1$ directly by computing only one outer inverse instead of computing both $G_1$ and $G_2$; see the next section for an example.
7. Examples for matrices

In this section, we illustrate our results for finite-dimensional vector spaces. In particular, we show how to compute directly the composition of two generalised inverses using the reverse order law in the form (6).

Consider the following linear maps $T_1 : \mathbb{Q}^4 \to \mathbb{Q}^3$ and $T_2 : \mathbb{Q}^3 \to \mathbb{Q}^4$ given by

$$T_1 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 3 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 2 \\ -1 & 5 & 4 \\ -1 & 5 & 4 \end{pmatrix}.$$

We first use Theorem 5.1 and 5.2 to check whether for algebraic generalised inverses $G_1 = G(T_1, B_1, E_1)$ and $G_2 = G(T_2, B_2, E_2)$ the composition $G_2 G_1$ is an algebraic generalised inverse of $T_1 T_2$.

For testing the conditions, we only need to fix $B_1 = \text{Im} G_1$ and $E_2 = \text{Ker} G_2$, such that $B_1 + \text{Ker} T_1 = \mathbb{Q}^4 = E_2 + \text{Im} T_2$. We have

$$\text{Ker} T_1 = \text{span}((0, 1, 0, 1)^T, (0, 0, 1, 1)^T), \quad \text{Im} T_2 = \text{span}((1, 0, -2, -2)^T, (0, 1, 1, 1)^T),$$

so we may choose for example

$$B_1 = \text{span}((1, 0, 0, 0)^T, (0, 1, 0, 0)^T), \quad E_2 = \text{span}((1, 0, 0, 0)^T, (0, 0, 1, 0)^T).$$

For algebraic generalised inverses, we obtain as a necessary and sufficient condition for being an outer inverse

$$B_1 \leq \text{Im} T_2 \oplus (E_2 \cap B_1) \oplus (E_2 \cap \text{Ker} T_1)$$

from Theorem 5.1 (iii).

Since $E_2 \cap \text{Ker} T_1 = \{0\}$ and $E_2 \cap B_1 = \text{span}((1, 0, 0, 0)^T)$, the right hand side yields $\text{span}((1, 0, 0, 0)^T, (0, 1, 0, 0)^T, (0, 0, 1, 1)^T) \geq B_1$. Thus for all algebraic generalised inverses $G_1$ and $G_2$ with $\text{Im} G_1 = B_1$ and $\text{Ker} G_2 = E_2$, the product $G_2 G_1$ is an outer inverse of $T_1 T_2$.

The corresponding condition for inner inverses by Theorem 5.2 (iii) is

$$\text{Im} T_2 \leq B_1 + (\text{Ker} T_1 \cap \text{Im} T_2) \oplus (\text{Ker} T_1 \cap E_2).$$

Since $\text{Ker} T_1 \cap \text{Im} T_2 = \{0\}$, the right hand side yields $B_1$, which does not contain $\text{Im} T_2$. Hence for the above choices of $G_1$ and $G_2$, the product $G_2 G_1$ is never an inner inverse of $T_1 T_2$.

Since $G_2 G_1$ is an outer inverse, Theorem 6.2 allows to determine $G_2 G_1$ directly without knowing the factors. Identifying the dual space with row vectors, the orthogonals of $B_1$ and $E_2$ are given by

$$B_1^\perp = \mathcal{B}_1 = \text{span}((0, 0, 1, 0), (0, 0, 0, 1)), \quad E_2^\perp = \text{span}((0, 1, 0, 0), (0, 0, 0, 1)),$$

so we have $\mathcal{B}_1 \cap E_2 = \text{span}((1, 0, 0, 0)^T)$ and $\mathcal{B}_1 \cap E_2^\perp = \text{span}((0, 0, 0, 1))$. For explicitly computing $G_2 G_1$, we also have to choose $B_2 = \text{Im} G_2$ and $E_1 = \text{Ker} G_1$. Since we have

$$\text{Im} T_1 = \text{span}((1, 0, 3)^T, (0, 1, 2)^T), \quad \text{Ker} T_2 = \text{span}((1, 1, -1)^T),$$

we choose $B_2 = \text{Im} G_2 = \text{span}((1, 0, 3)^T, (0, 1, 2)^T)$ and $E_1 = \text{Ker} G_1 = \text{span}((1, 1, -1)^T)$. Then $G_2 G_1$ is given by

$$G_2 G_1 = (G_2 G_1)_{\text{Im} T_1} \oplus (G_2 G_1)_{\text{Ker} T_2},$$

where

$$(G_2 G_1)_{\text{Im} T_1} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & -1 \end{pmatrix}$$

and

$$(G_2 G_1)_{\text{Ker} T_2} = \begin{pmatrix} -1 & 2 \\ -5 & 4 \\ -5 & 4 \end{pmatrix}.$$
we may choose the complements $E_1 = \text{Ker} G_1$ and $B_2 = \text{Im} G_2$ as

$$E_1 = \text{span}((0,0,1)^T) \quad \text{and} \quad B_2 = \text{span}((1,0,0)^T, (0,1,0)^T).$$

Using (6), we can determine the kernel

$$E = \text{Ker} G_2 G_1 = E_1 + T_1(\mathcal{B}_1 \cap E_2) = \text{span}((1,0,0)^T, (0,0,1)^T).$$

The image of $G_2 G_1$ is by (6) given via the orthogonal

$$(\text{Im} G_2 G_1)^\bot = B_2 + T_2^* (B_1 \cap E_2^\bot) = \text{span}((0,0,1), (-1,5,4)),$$

which means that $B = \text{Im} G_2 G_1 = \text{span}((5,1,0)^T)$. Therefore we can directly determine $G$ as the unique outer inverse

$$G = O(T_1 T_2, B, E) = \begin{pmatrix} 0 & \frac{5}{12} & 0 \\ \frac{5}{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

One easily checks that $G$ is an outer inverse of $T$.

8. Fredholm operators

We now turn to algorithmic aspects of the previous results. As already emphasised, for arbitrary vector spaces we can express conditions for the reverse order law only in terms of the input data. Nevertheless, in general it will not be possible to compute sums and intersections of infinite-dimensional subspaces. For algorithmically checking the conditions of Theorem 5.1 or 5.2 and for computing the reverse order law in the form (6), we consider finite (co)dimensional spaces and Fredholm operators.

Recall that a linear map $T$ between vector spaces is called Fredholm operator if $\dim \text{Ker} T < \infty$ and $\text{codim} \text{Im} T < \infty$. Moreover, for finite codimensional subspaces $V_1 \leq V$, we have $\text{codim} V_1 = \dim V_1^\bot$. In this case, $V_1$ can be implicitly represented by the finite-dimensional subspace $V_1^\bot \leq V^\ast$. For an application to linear ordinary boundary problems, see the next section.

We assume that for finite-dimensional subspaces, we can compute sums and intersections and check inclusions, both in vector spaces and in their duals. With the following lemma, also the intersection of a finite-dimensional with a finite codimensional subspace in $V$, respectively $V^\ast$, is reduced to computing kernels of matrices.

**Definition 8.1:** Let $u = (u_1, \ldots, u_m)^T \in V^m$ and $\beta = (\beta_1, \ldots, \beta_n)^T \in (V^\ast)^n$. We call

$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \ldots & \beta_1(u_m) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \ldots & \beta_n(u_m) \end{pmatrix} \in F^{n \times m}$$

the evaluation matrix of $\beta$ and $u$.

**Lemma 8.2:** Let $U \leq V$ and $\mathcal{B} \leq V^\ast$ be generated respectively by $u = (u_1, \ldots, u_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Let $k^1, \ldots, k^r \in F^m$ be a basis of $\text{Ker} \beta(u)$, and $\kappa^1, \ldots, \kappa^s \in F^n$ a basis of $\text{Ker} (\beta(u))^T$. Then
(i) $U \cap B^\perp$ is generated by $\sum_{i=1}^m k_i^1 u_i, \ldots, \sum_{i=1}^m k_i^r u_i$ and
(ii) $U^\perp \cap B$ is generated by $\sum_{i=1}^n k_i^1 \beta_i, \ldots, \sum_{i=1}^n k_i^r \beta_i$.

Proof: A linear combination $v = \sum_{i=1}^m c_i u_i$ is in $B^\perp$ if and only if $\beta_i(v) = 0$ for
1 \leq i \leq n$, that is, $\sum_{i=1}^m c_i \beta_i(u_i) = 0$ for 1 \leq i \leq n$. Hence $\beta(u) \cdot (c_1, \ldots, c_m)^T = 0$.

Analogously, one sees that the coefficients of linear combination in $U^\perp \cap B$ are in
the kernel of $(\beta(u))^T$. \hfill \square

We reformulate the conditions of Theorem 5.1 such that for Fredholm operators
they only involve operations on finite-dimensional subspaces and intersections like
in the previous lemma. Similarly, one can rewrite the conditions of Theorem 5.2.

Corollary 8.3: Let $T_1: V \rightarrow W$ and $T_2: U \rightarrow V$ be linear with outer
inverses $G_1 = O(T_1, B_1, E_1)$ and $G_2 = O(T_2, B_2, E_2)$. Let $G_2 = T_2(B_2^\perp)^\perp$ and
$K_1 = T_1^{-1}(E_1)$. The following conditions are equivalent:

(i) $G_2 G_1$ is an outer inverse of $T_1 T_2$,
(ii) $C_2 \oplus (B_1 \cap E_2^\perp) \supseteq B_1 \cap (E_2 \cap K_1)^\perp$,
(iii) $B_1 \supseteq C_2 \cap (E_2 \cap B_2^\perp)^\perp \cap (E_2 \cap K_1)^\perp$,
(iv) $K_1 + (E_2 \cap B_1^\perp) \supseteq E_2 \cap (B_1 \cap C_2)^\perp$,
(v) $E_2 \supseteq K_1 \cap (B_1 \cap E_2^\perp)^\perp \cap (B_1 \cap C_2)^\perp$.

Proof: Taking the orthogonal of both sides of respectively 5.1 (ii), (iii) and applying
Proposition A.1 gives (ii) and (iii). For (iv) and (v), we can apply Proposition
A.1 directly to the corresponding conditions of Theorem 5.1. \hfill \square

We note that using Lemma 8.2, it also possible to determine constructively the
implicit representation (6) of a product of generalised inverses; see the next section.

9. Examples for linear ordinary boundary problems

As an example involving infinite dimensional spaces and Fredholm operators, we
consider solution (Green’s) operators for linear ordinary boundary problems. Alge-
braically, linear boundary problems can be represented as a pair $(T, B)$, where
$T: V \rightarrow W$ is a surjective linear map, and $B \leq V^*$ is an orthogonally closed
subspace of (homogeneous) boundary conditions. We say that $v \in V$ is a solution of
$(T, B)$ for a given $w \in W$ if $Tv = w$ and $v \in B^\perp$.

For a regular boundary problem (having a unique solution for every right-hand
side), the Green’s operator is defined as the unique right inverse $G$ of $T$ with
$\text{Im } G = B^\perp$; see [18] for further details. The product $G_2 G_1$ of the Green’s operators
of two boundary problems $(T_1, B_1)$ and $(T_2, B_2)$ is then the Green’s operator of
the regular boundary problem $(T_1 T_2, B_2 + T_2^* (B_1))$, see also Theorem 6.2.

For boundary problems having at most one solution, that is $B^\perp \cap \text{Ker } T = \{0\}$,
the linear algebraic setting has been generalised in [11] by defining generalised
Green’s operators as outer inverses $G = O(T, B, E)$, where $E \leq W$ is a suitable
exceptional space. The question when the product of two outer inverses is again
an outer inverse, is the basis for factoring boundary problems into lower order
problems; see [18, 19] for the case of regular boundary problems. This, in turn,
provides a method to factor certain integral operators.

As an example, let us consider the boundary problem

$$
\begin{align*}
    u'' &= f \\
    u'(0) &= u'(1) = u(1) = 0.
\end{align*}
$$

(7)
In the above setting, this means we consider the pair \((T_1, \mathcal{B}_1)\) with \(T_1 = D^2\) and \(\mathcal{B}_1 = \text{span}(E_0, D, E_1, D, E_1)\), where \(D\) denotes the usual derivation on smooth functions and \(E_0\) the evaluation at \(c \in \mathbb{R}\). The boundary problem is only solvable for forcing functions \(f\) satisfying the compatibility condition \(\int_0^1 f(\xi) \, d\xi = 0\); more abstractly, we have \(T_1(\mathcal{B}_1^⊥) = \mathcal{C}_1^⊥\) with \(\mathcal{C}_1 = \text{span}(f_0^1)\), where \(f_0^1\) denotes the functional \(f \mapsto \int_0^1 f(\xi) \, d\xi\). For computing a generalised Green’s operator of \((T_1, \mathcal{B}_1, E_1)\), we have to project \(f\) onto \(\mathcal{C}_1^⊥\) along a fixed complement \(E_1\). In [12], we computed the generalised Green’s operator

\[
G_1(f) = x \int_0^x f(\xi) \, d\xi - \int_0^x \xi f(\xi) \, d\xi - \frac{1}{2}(x^2 + 1) \int_0^1 f(\xi) \, d\xi + \frac{1}{2} \xi f(\xi) \, d\xi
\]

of (7) for \(E_1 = \mathbb{R}\) being the constant functions. It is easy to see that in this case we have \(T_1^{-1}(E_1) = \text{span}(1, x, x^2)\).

As a second boundary problem, we consider

\[
\begin{align*}
  u'' - u &= f \\
  u'(0) &= u'(1) = u(1) = 0,
\end{align*}
\]

or \((T_2, \mathcal{B}_2)\) with \(T_2 = D^2 - 1\) and \(\mathcal{B}_2 = \text{span}(E_0, D, E_1, D, E_1)\). For the corresponding generalised Green’s operator \(G_2\) with exceptional space \(E_2 = \text{span}(x)\), we will now check if the products \(G_1 G_2\) and \(G_2 G_1\) are again generalised Green’s operators of \(T_1 T_2 = T_2 T_1 = D^4 - D^2\), using condition (ii) of Corollary 8.3.

We use the algorithm from [12], implemented in the package IntDiffOp for the computer algebra system Maple, to compute the compatibility conditions. We obtain \(\mathcal{C}_2 = \text{span}(f_0^1(\exp(-x) + \exp(x)))\), where \(f_0^1(\exp(-x) + \exp(x))\) denotes the functional \(f \mapsto \int_0^1 (\exp(-\xi) + \exp(\xi)) f(\xi) \, d\xi\).

The space \(T_2^{-1}(E_2) = \text{span}(x, \exp(x), \exp(-x))\) can be computed using Proposition 3.1 and a right inverse of the differential operator; this is also implemented in the IntDiffOp package. Hence we have \(E_1 \cap T_2^{-1}(E_2) = \{0\}\) and therefore \(\mathcal{B}_2 \cap (E_1 \cap T_2^{-1}(E_2))^⊥ = \mathcal{B}_2\). Computing \(\mathcal{B}_2 \cap E_2^⊥\) with Lemma 8.2 yields \(\mathcal{B}_2 \cap E_2^⊥ = \text{span}(E_0, D, E_1, D)\); thus \(G_1 G_2\) is not an outer inverse of the product \(T_2 T_1 = D^4 - D^2\) by Corollary 8.3 (ii).

On the other hand, we have \(E_2 \cap T_1^{-1}(E_1) = \text{span}(x) = E_2\), hence we know by Corollary 8.3 (ii) that \(G_2 G_1\) is an outer inverse of \(T_1 T_2 = D^4 - D^2\). Furthermore, by Theorem 6.2 we can determine which boundary problem it solves without computing \(G_1\) and \(G_2\). With Lemma 8.2 we obtain \(\mathcal{B}_1^⊥ \cap E_2 = \{0\}\) and \(\mathcal{B}_1 \cap E_2^⊥ = \text{span}(E_0, D - E_1, E_1, D - E_1)\). Since applying the transpose \(T_2^*\) to \(E_0, D - E_1, E_1, D - E_1\) corresponds to multiplying \(T_2 = D^2 - 1\) from the right, \(G_2 G_1\) is the generalised Green’s operator of

\[
(D^4 - D^2, \text{span}(E_0, D, E_1, E_0 D^3 - E_1 D^2, E_1 D^3 - E_1 D^2), \mathbb{R})
\]

by (6); or, in traditional notation, it solves the boundary problem

\[
\begin{align*}
  u''' - u'' &= f \\
  u'(0) &= u'(1) = u(1) = u'''(0) - u''(1) = u'''(1) - u''(1) = 0,
\end{align*}
\]

with exceptional space \(\mathbb{R}\).
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References

Appendix A. Duality

In the appendix, we summarise duality results for arbitrary vector spaces and their duals that generalise the standard duality for finite-dimensional vector spaces but do not require any topological assumptions; see [13, Section 9.2 and 9.3] and [18] for further details. The notation should also remind of the analogous and well-known results for Hilbert spaces.

Let $V$ and $W$ be vector spaces over a field $F$ and $\langle \cdot, \cdot \rangle : V \times W \to F$ be a bilinear map. For $V_1 \leq V$, we define the orthogonal

$$V_1^\perp = \{ w \in W \mid \langle v, w \rangle = 0 \text{ for all } v \in V_1 \} \leq W.$$ 

The orthogonal $W_1^\perp$ for $W_1 \leq W$ is defined analogously. A subspace $U$ is called orthogonally closed if $U = U_\perp \cup \perp$. It follows directly from the definition that for all subsets $X_1, X_2 \subseteq V$, we have $X_1 \subseteq X_2 \Rightarrow X_1^\perp \supseteq X_2^\perp$ and $X_1 \subseteq X_1^\perp \cup \perp$; and the same holds for subsets of $W$. Let $\mathbb{P}(V)$ denote the projective geometry of $V$, that is, the partially ordered set (poset) of all subspaces ordered by inclusion. Then we have an order-reversing Galois connection between $\mathbb{P}(V)$ and $\mathbb{P}(W)$ defined by $U \mapsto U^\perp$. We now consider the canonical bilinear form $V \times V^* \to F$ of a vector space $V$ and its dual $V^*$ defined by $\langle v, \beta \rangle \mapsto \beta(v)$. Then every subspace $V_1 \leq V$ is orthogonally closed with respect to the canonical bilinear form, and every finite-dimensional subspace $\mathcal{B} \leq V^*$ is orthogonally closed. The Galois connection gives an order-reversing bijection between $\mathbb{P}(V)$ and the poset of all orthogonally closed subspaces of $V^*$. So we can describe any subspace $V_1 \leq V$ implicitly by the corresponding orthogonally closed subspace $V_1^\perp$. We denote the poset of all orthogonally closed subspaces of $V^*$ with $\mathbb{P}(V^*)$.

The projective geometry $\mathbb{P}(V)$ is a modular lattice, where join and meet are defined as the sum and intersection of subspaces. Modularity means that for all $V_1, V_2, V_3 \in \mathbb{P}(V)$ with $V_1 \leq V_3$ we have

$$V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap V_3. \quad (A1)$$

Moreover, for spaces $V_1 \leq V_3$ and $V_2 \leq V_4$, we have

$$V = V_1 + V_2 = V_3 + V_4 \Rightarrow V_1 = V_3 \text{ and } V_2 = V_4, \quad (A2)$$

since $V_3 \cap V_4 = \{0\}$ implies $V_3 = (V_1 + V_2) \cap V_3 = V_1$ and $V_4 = (V_1 + V_2) \cap V_4 = V_2$.

One can also show that $\mathbb{P}(V^*)$ is a modular lattice, where the meet is the intersection and the join is the orthogonal closure of the sum of subspaces. Using this fact, one can prove in particular that the sum of two orthogonally closed subspaces is orthogonally closed. The following theorem summarises Section 9.3 of [13].

**Proposition A.1:** The map $V_1 \mapsto V_1^\perp$ gives an order-reversing lattice isomorphism with inverse $\mathcal{B}_1 \mapsto \mathcal{B}_1^\perp$ between the complemented modular lattices $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$. In particular, the intersection of orthogonally closed subspaces in $V^*$ is orthogonally closed and

$$(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp \quad \text{and} \quad (\mathcal{B}_1 \cap \mathcal{B}_2)^\perp = \mathcal{B}_1^\perp + \mathcal{B}_2^\perp.$$ 

The sum of two orthogonally closed subspaces in $V^*$ is orthogonally closed and

$$(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp \quad \text{and} \quad (\mathcal{B}_1 + \mathcal{B}_2)^\perp = \mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp.$$
Furthermore, orthogonality preserves direct sums, such that

\[ V = V_1 + V_2 \Rightarrow V^* = V_1^\perp + V_2^\perp \quad \text{and} \quad V^* = B_1 + B_2 \Rightarrow V = B_1^\perp + B_2^\perp. \]

For a linear map \( A: V \rightarrow W \) between vector spaces, the transpose \( A^*: W^* \rightarrow V^* \) is defined by \( \gamma \mapsto \gamma \circ A \). The transposition map \( A \mapsto A^* \) from \( L(V,W) \) to \( L(W^*,V^*) \) is linear, and it is injective since for all \( w \neq 0 \) there exists a linear map \( h \in W^* \) with \( h(w) \neq 0 \). Moreover, the transpose of a composition is given by \( (A_1 A_2)^* = A_2^* A_1^* \).

The image of an orthogonally closed space under the transpose map is orthogonally closed, and we have following identities, see, for example, [18, Prop. A.6].

**Proposition A.2:** Let \( V \) and \( W \) be vector spaces and \( A: V \rightarrow W \) be linear. Then we have

\[
A(V_1)^\perp = (A^*)^{-1}(V_1^\perp), \quad A(B_1^\perp) = (A^*)^{-1}(B_1)^\perp,
\]

\[
A^*(C_1)^\perp = A^{-1}(C_1^\perp), \quad A^*(W_1^\perp) = A^{-1}(W_1)^\perp,
\]

for subspaces \( V_1 \leq V, W_1 \leq W, C_1 \leq W^* \) and orthogonally closed subspaces \( B_1 \leq V^* \). In particular, we have

\[
(\text{Im } A)^\perp = \ker A^*, \quad \text{Im } A = (\ker A^*)^\perp, \quad (\text{Im } A^*)^\perp = \ker A, \quad \text{Im } A^* = (\ker A)^\perp,
\]

for the image and kernel of \( A \) and \( A^* \).

The property of being a projector, outer/inner/algebraic generalised inverse carries over to the transpose.

**Proposition A.3:** A linear map \( P: V \rightarrow V \) is a projector if and only if its transpose \( P^* \) is a projector. A linear map \( G: W \rightarrow V \) is an outer/inner/algebraic generalised inverse of \( T: V \rightarrow W \) if and only if \( G^* \) is an outer/inner/algebraic generalised inverse of \( T^* \).

**Proof:** This follows from the defining equations for these properties. For example, if \( G \) is an outer inverse of \( T \), we have \( G^* T^* G^* = (GTG)^* = G^* \), and the reverse implication follows from the injectivity of the transposition map. \( \square \)

With the results of this section, we obtain the following duality principle for generalised inverses.

**Remark A1:** Given a valid statement for linear maps on arbitrary vector spaces \( V \) over a common field involving inclusions, \( \{0\} \) and \( V \), sums and intersections, direct sums, kernels and images, projectors, and outer/inner/algebraic generalised inverses, we obtain a valid dual statement by

- reversing the order of the linear maps and the corresponding domains and codomains,
- reversing inclusions and interchanging \( V \) and \( \{0\} \),
- interchanging sums and intersections,
- interchanging kernels and images.

For example, one easily checks that in Proposition 2.2, the statements (v) – (vii) are the duals of (ii) – (iv) in this sense.