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December 2018

Tensor reduction systems for rings of linear operators

Doctoral Thesis
to obtain the academic degree of
Doktor der Naturwissenschaften
in the Doctoral Program
Naturwissenschaften

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Linz, Dezember 2018
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Zusammenfassung

Für symbolisches Rechnen mit Systemen linearer Funktionalgleichungen (z.B. Integro-Differentialgleichungen, retardierten Differentialgleichungen, Rekursionsgleichungen, ...), benötigen wir ein algebraisches Framework für solche Systeme, das effektive Berechnungen in entsprechenden Operatorringen erlaubt. Um solche Frameworks zu finden, verwenden wir anstelle der bisher in der Literatur verwendeten parametrisierten Gröbnerbasen in freien Algebren das basenfreie Analogon von Bergman in Tensorringen, welches oft ein endliches Reduktionssystem mit eindeutigen Normalformen ermöglicht. Kurz gesagt schlagen wir einen allgemeinen algorithmischen Ansatz für nichtkommutative Operatoralgebren vor, die von additiven Operatoren erzeugt werden.

In dieser Dissertation wird eine vollständige Darstellung von Reduktionssystemen in Tensorringen mit den entsprechenden Grundlagen wie Tensorprodukt von Bimodulen, Tensorringen und Termersetzungssystemen vorgestellt. Wir verwenden unsere Verallgemeinerung von Bergmans Tensorsetting, um den Ring von Integrodifferentialoperatoren (IDO) mit (nichtkommutativen) Matrixkoeffizienten zu konstruieren. Außerdem erweitern wir diesen Ring zum Ring der Integrodifferentialoperatoren mit linearen Substitutionen (IDOLS). Als neues Beispiel konstruieren wir den Ring der Summations- und Differenzenoperatoren (SDO). Um Normalformen in diesen Ringen zu finden, vervollständigen wir die definierenden Reduktionssysteme zu konfluenten. Normalformen erlauben es, Operatorgleichungen mittels Ansatz zu lösen.

Wir zeigen, dass mit Tensorreduktionssystemen zum Beispiel die Methoden der Variation der Konstanten und des Schrittweisen Integrierens automatisch gefunden und bewiesen werden können. Wir zeigen wie Greensche Operatoren für lineare gewöhnliche Randwertprobleme für Systeme erster Ordnung mittels Normalformen von IDO berechnet werden können. Darüber hinaus automatisieren wir im Ring von IDOLS bestimmte Berechnungen mit retardierten Differentialgleichungen, beispielsweise die Artsteintransformation und eine Verallgemeinerung davon. Basierend auf dem Mathematica-Paket `TenReS` implementieren wir die Ringe von IDO, IDOLS, SDO, deren Normalformen und verwenden diese auch für Berechnungen in Anwendungen.

Abstract

In order to facilitate symbolic computations with systems of linear functional equations (e.g. integro-differential, differential time-delay, recurrence, ...), we require an algebraic framework for such systems which enables effective computations in corresponding rings of operators. For finding such frameworks, instead of using parametrized Gröbner bases in free algebras as has been done so far in the literature, we exploit and generalize Bergman's basis-free analog in tensor rings, which often allows for a finite reduction system with unique normal forms. In short, we propose a general algorithmic approach to noncommutative operator algebras generated by additive operators.

In this thesis, a self-contained treatment of reduction systems in tensor rings including preliminaries such as tensor product of bimodules, tensor rings, and basics of term rewriting is presented. We apply our generalization of Bergman's setting to construct the ring of integro-differential operators (IDO) having (noncommutative) matrix coefficients. Moreover, we extend this ring to the ring of integro-differential operators with linear substitutions (IDOLS). As a new instance of the tensor setting, we construct the ring of inversive sum-difference operators (SDO). For finding normal forms in these rings, we complete their defining reduction systems to obtain confluent ones. These normal forms allow to solve operator equations by ansatz.

We show that, by applying tensor reduction systems, results like the method of variation of constants in the ring of IDO and the method of steps in the ring of IDOLS can be found and proven in an automated way. Using normal forms in the ring of IDO, we illustrate how to compute Green's operators for first-order systems of linear ordinary boundary problems. Moreover, in the ring of IDOLS, we partly automatize certain computations related to differential time-delay systems, e.g. Artstein's transformation and its generalization. Using the Mathematica package `TenReS`, we implement the rings of IDO, IDOLS, SDO, and corresponding normal forms. We also use these implementations to perform computations for the applications treated in this thesis.

Acknowledgements

The writing of this thesis has been a challenging and creative experience. I would like to express my thanks to my formal supervisor Prof. Manuel Kauers, as well as to the examining board.

I also express my sincere gratitude to my co-supervisors Georg Regensburger and Clemens Gunter Raab for the continuous support of my PhD studies and related research, for their patience, motivation, and immense knowledge. The door to their office was always open whenever I ran into a trouble spot or had a question about my research or writing. They consistently allowed this thesis to be my own work, but steered me in the right the direction whenever they thought I needed it.

I acknowledge Thomas Cluzeau for his willingness to be my external reviewer. The collaboration with him and Alban Quadrat has given rise to fruitful discussions and provided important inputs for the direction of the thesis.

I extend my gratitude to all my colleagues in symbolic compuation group at RICAM and the algebra department at JKU, as well as my officemate Audie Warren, for the continuous support in the highs and lows. I always enjoyed our joint meetings and seminars.

This volume is dedicated to my wife, Sahar Faghfouri. Without her encouragements, assistance and sense of value, my PhD would have proved too difficult. Her faith in me never wavers and her joy in my efforts seems greater than my own.

Last but not least, I would like to thank my parents for their support in so many ways, which allowed me to focus more on my work. Without them my studies would have been less smooth.

Thanks to you all!

The research was supported by the Austrian Science Fund (FWF): project P27229, "Algebra and algorithms for integro-differential equations", lead by Georg Regensburger.

Contents

List of frequently used notations

Notation	Description	Page List
$\mathcal{K}X$	Free $\mathcal{K}$-module on a set X	10
$\mathcal{M}$	An arbitrary monoid	11
W	Word in a free monoid $\langle X \rangle$	11
$\langle X \rangle$	Free word monoid on a set X	11
ϵ	Empty tensor/ empty word	11, 44
$\mathcal{Z}(\mathcal{A})$	Center of a ring $\mathcal{A}$	12
$\mathcal{K}\langle X \rangle$	Free algebra over a ring $\mathcal{K}$ on a set X	13
$\bigoplus_{i \in I} M_i$	Direct sum of modules $(M_i)_{i \in I}$	15
$\mathbb{Z}$	Set of integer numbers	22
$\mathcal{K}^{\text{op}}$	the opposite of a ring $\mathcal{K}$	22
$\mathcal{K}X\mathcal{K}$	Free $\mathcal{K}$-bimodule on a set X	24
$\mathcal{R}$	Arbitrary unitary ring	25
$M^{\otimes n}$	Tensor product of a bimodule M with itself	28
$\mathcal{K}\langle M \rangle$	$\mathcal{K}$-tensor ring over a module M	29
$M_n(\mathcal{S})$	Ring of $n \times n$ matrices over a ring $\mathcal{S}$	31
$\mathcal{D}$	Noncommutative ring of functional operators	38
$\mathcal{F}$	A left $\mathcal{D}$-module	38
$(M_x)_{x \in X}$	Family of $\mathcal{K}$-bimodules indexed by a set X	44
M_W	Bimodule corresponding to a word W	44
Σ	Reduction system or summation operation/operator	45, 106
$\to_\Sigma$	Reduction relation induced by Σ	45
$\overset{*}{\to}_\Sigma$	Reflexive transitive closure of $\to_\Sigma$	45
$\langle X \rangle_{\text{irr}}$	Set of irreducible words in a word monoid $\langle X \rangle$	46

Notation	Description	Page List
$\mathcal{K}\langle M\rangle_{\text{irr}}$	$\mathcal{K}$-subbimodule of irreducible tensors in $\mathcal{K}\langle M\rangle$	46
$\leq$	Semigroup partial order on $\langle X\rangle$	46
$M_{<W}$	Direct sum of bimodules M_V where $W < W$ and $W \in \langle X\rangle$	47
SP	S-polynomials of ambiguities	48
I_Σ	Two-sided reduction ideal induced by Σ	48
$I_{\Sigma,W}$	$\mathcal{K}$-bimodule generated by all the terms $t-s$ where $t \in M_V$ for some $V < W$ and $t \to_\Sigma s$	49
$\mathcal{K}\langle M\rangle_{\text{un}}$	$\mathcal{K}$-bimodule of tensors with unique normal forms in $\mathcal{K}\langle M\rangle$	50
$t\downarrow_\Sigma$	Irreducible form of t with respect to Σ	51
$(\mathcal{R},\partial)$	Differential ring	55
M_{R}	$\mathcal{K}$-bimodule generated by $\mathcal{R}$	55
M_{D}	$\mathcal{K}$-bimodule generated by the operator ∂	55
∂	Derivation/differentiation operator	55
$\mathcal{R}\langle\partial\rangle$	Ring of differential operators	55
Z	An alphabet containing the alphabet X	58
$S(z)$	Set of specializations of $z \in Z$	58
$S(W)$	Set of specializations of a word W	59
Σ_X	Refined reduction system on the alphabet X	59
$\int$	Integration/Integral operator	68
$(\mathcal{R},\partial,\int)$	Integro-differential ring	68
E	Evaluation operator/operation	68
M_{K}	$\mathcal{K}$-bimodule generated by the constants $\mathcal{K}$	70
$M_{\tilde{\mathsf{R}}}$	K-bimodule generated by $\tilde{\mathcal{R}} = \int\mathcal{R}$	70
Φ	The set of multiplicative linear functionals on $\mathcal{R}$	70
M_{I}	$\mathcal{K}$-bimodule generated by the operator $\int$	71
M_{E}	$\mathcal{K}$-bimodule generated by the operator E	71
$M_{\tilde{\Phi}}$	$\mathcal{K}$-bimodule generated by $\tilde{\Phi} = \Phi \setminus \{\mathrm{E}\}$	71
M_{Φ}	$\mathcal{K}$-bimodule generated by functional in Φ	71
$\mathcal{R}\langle\partial,\int,\Phi\rangle$	Ring of integro-differential operators	71
$\bullet$	Multiplication on the module of functions $\mathcal{F}$	74
Σ_{IDO}	Reduction system for the ring of IDO	76
$\sigma_{a,b}$	linear substitution operator mapping a smooth function $f(x)$ to $f(ax-b)$	84

Notation	Description	Page List
$\mathcal{C}$	Intersection of the ring $\mathcal{K}$ with $\mathcal{Z}(\mathcal{R})$	84
$\mathcal{C}^*$	Group of units in $\mathcal{C}$	84
$\mathcal{S}$	Group of linear substitution operators $\sigma_{a,b}$ with $a \in \mathcal{C}^*$ and $b \in \mathcal{C}$	85
$(\mathcal{R}, \partial, \int, \mathcal{S})$	Integro-differential ring with linear substitutions	85
M_{G}	$\mathcal{K}$-bimodule generated by the set $\mathcal{S}$	85
M_{N}	$\mathcal{K}$-bimodule generated by $\sigma_{1,0} = \mathrm{id}$	85
$M_{\tilde{\mathsf{G}}}$	$\mathcal{K}$-bimodule generated by $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{\sigma_{1,0}\}$	85
$\mathcal{R}\langle \partial, \int, \Phi, \mathcal{S} \rangle$	Ring of integro-differential operators with linear substitutions	86
Σ_{LS}	Set of reduction rules containing linear substitutions	86
δ	Time-delay operator $\sigma_{1,h}$	86
Σ_{IDOLS}	Reduction system for the ring of IDOLS	89
Δ	Forward difference operator	106
∇	Backward difference operator	106
Σ_{SDO}	Reduction system for the ring of SDO	115

Chapter 1

Introduction

Many processes in science and engineering can be modelled by so-called linear functional systems, for example, systems of differential and difference equations. To represent and manipulate such systems algebraically, one computes with the corresponding matrices and linear operators. For concrete linear functional systems, so far, matrices of operators with scalar coefficients are used. However, for statements about families of linear systems, an algebraic framework for operators with undetermined matrix coefficients is required. In this thesis, we propose a general approach to model additive operators having noncommutative coefficients based on tensor reduction systems.

For effective symbolic computations with operators, normal forms are needed. They can be used to prove operator identities as well as to solve operator equations by ansatz. To find normal forms, we construct confluent reduction systems for integro-differential operators starting only from basic identities of differentiation, integration, and evaluation. We also work out normal forms for integro-differential operators with linear substitutions and sum-difference operators.

By normal form computations, results like the method of variation of constants, Green's operators for first-order linear systems, and the method of steps can be found and proven in an automated way. Using the Mathematica package `TenReS`, we implement normal forms for the rings of operators and perform computations for the applications considered in this thesis.

1.1 Integro-differential operators

We motivate and illustrate how to use our approach for treating differential and differential time-delay systems. First, we introduce informally integro-differential rings and the corresponding operators without referring explicitly to tensor reduction systems. Recall from analysis that for differentiable real-valued functions f and g, the Leibniz rule is expressed as

$$\frac{d}{dt}(f(t)g(t)) = (\frac{d}{dt}f(t))g(t) + f(t)(\frac{d}{dt}g(t)).$$

Moreover, by the fundamental theorem of calculus, for a continuous real-valued function f on a closed interval $[t_0, t]$ we have

$$\frac{d}{dt}\int_{t_0}^{t} f(s)\, ds = f(t),$$

and the evaluation at t_0 can be expressed in terms of differentiation and integration as

$$f(t_0) = f(t) - \int_{t_0}^{t} \frac{d}{ds} f(s)\, ds.$$

In addition, the evaluation at t_0 of a product is the product of the individual evaluations. Based on these identities, we define an integro-differential ring as a natural generalization of a differential ring by adding integration $\int$ and evaluation E to the derivation ∂, see Definition 4.1 which generalizes the definition given in [47, 51]. Note that the operations on these rings satisfy not only the Leibniz rule, but also other well-known identities of smooth functions like integration by parts and the Rota-Baxter axiom, see Table 4.2. For a given integro-differential ring, the ring of integro-differential operators is generated by ∂, $\int$, and E, see Section 4.1 for the formal construction via tensor reduction systems.

To illustrate computations with operators in this ring, we consider the simple example of a linear first-order differential system

$$\frac{d}{dt}x(t) - A(t)x(t) = f(t),$$

where $A(t)$ is a matrix of smooth functions of size $n \times n$ and $f(t)$ is a vector of smooth functions. If $\Phi(t)$ is a fundamental matrix for the homogeneous system, then a particular solution for this system is

$$x_0(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s)\, ds,$$

obtained via variation of constants. In terms of operators, we can write these equations as $Lx(t) = f(t)$ and $x(t) = Hf(t)$ where $L = \partial - A$ is the corresponding differential operator and $H = \Phi \cdot \int \cdot \Phi^{-1}$ is the corresponding integral operator, where by $\cdot$ we denote composition of operators. The fact that $x_0(t)$ is a particular solution of the differential system is equivalent to saying that the composition $L \cdot H$ equals to the identity. Proving the identity $L \cdot H = 1$ of operators not for a concrete differential system, but in full generality, requires an algebraic setting for integro-differential operators with undetermined matrix coefficients. Assuming such a framework, we have

$$\begin{aligned} L \cdot H &= (\partial - A) \cdot \Phi \cdot \textstyle\int \cdot \Phi^{-1} = \partial \cdot \Phi \cdot \textstyle\int \cdot \Phi^{-1} - A \cdot \Phi \cdot \textstyle\int \cdot \Phi^{-1} \\ &= (\Phi \cdot \partial + (\partial\Phi - A\Phi)) \cdot \textstyle\int \cdot \Phi^{-1} = \Phi \cdot \partial \cdot \textstyle\int \cdot \Phi^{-1} = \Phi \cdot \Phi^{-1} = 1. \end{aligned}$$

Note that the transition from matrices of operators to operators with matrix coefficients is not mere notation but an essential part of the algebraic framework in order to perform such a symbolic computation with matrix coefficients of generic size on a computer. Indeed, constructing such a framework including normal forms is one of the main points of the thesis, see Section 4.3.

Moreover, using the tensor setting, the ring of integro-differential operators can be extended by adding linear substitution operators, see Section 4.5. Formal computations with operators having rectangular matrices as their coefficients can be done in the ring of integro-differential operators with linear substitution, see [13, 40]. This ring can be used for studying systems of differential time-delay equations. As a concrete application, we apply the obtained normal forms to partly automatize certain computations related to differential time-delay systems, e.g. Artstein's transformation [2], see our joint paper [13], and its generalization in Section 5.2: by means of this transformation, a first-order linear differential system with delayed inputs

$$\frac{d}{dt}x(t) = A(t)x(t) + B_0(t)u(t) + B_1(t)u(t-h), \quad h > 0$$

is equivalent to a first-order linear differential system without delay under an invertible transformation which includes integral and time-delay operators, see Section 5.2 for further details.

1.2 Sum-difference operators

Difference equations occupy a central position in applied fields. For the theory of difference equations, we refer the reader to [1, 35], for example. To treat these equations algebraically with our tensor setting, we develop a formal construction of rings of shift and summation operators. We are not aware that reduction systems in tensor rings have been used so far in the literature for an algorithmic treatment of inversive sum-difference operators.

In the following, we briefly outline the defining properties of these operators. For a bi-infinite sequence $(A_n)_{n\in\mathbb{Z}}$ in a ring, the operations shift forward σ

and shift backward $\bar{\sigma}$ are inverses of each other, i.e.

$$\sigma\bar{\sigma}(A_n) = \bar{\sigma}\sigma(A_n) = (A_n).$$

We define the summation operation $\Sigma(A_n) = (\sum_{k=0}^{n-1} A_k)_{n\in\mathbb{Z}}$ by the sequence of partial sums. Then, the identity

$$\sigma\Sigma(A_n) = (A_n) + \Sigma(A_n)$$

holds. In addition, the evaluation E defined by $\mathrm{E}(A_n) = (A_0)$ can be expressed in terms of the shift forward and the summation by

$$\mathrm{E}(A_n) = (A_n) + \Sigma\big((A_n) - \sigma(A_n)\big).$$

Based on these properties, we consider inversive sum-difference rings, see Definition 6.3, which are generalization of difference rings [37]. The operations also satisfy additional identities such as summation by parts and the Rota-Baxter axiom with weight one, see Table 6.2. From a given inversive sum-difference ring, we can construct the ring of inversive sum-difference operators as the corresponding ring of operators generated by $\bar{\sigma}$, σ, Σ, E, and the elements of the coefficient ring, see Section 6.2 where we work out normal forms for these operators.

The ring of inversive sum-difference operators can be used for solving difference boundary problems. Moreover, it addresses the open Problem 6.2 in [10], where an operator framework for studying creative telescoping in the difference case is asked for.

1.3 Algebraic frameworks for linear operators

For an algorithmic treatment of many linear operators, e.g. differential and difference operators, skew polynomials are often used in the literature as a well-established algebraic tool; see e.g. the works [12, 38, 9, 11, 36] or the recent overview [22]. Normal forms for skew polynomials are given by the standard polynomial basis. However, not all common operator algebras are covered by this setting. For instance, normal forms for univariate integral operators are sums of terms of the form $f\int g$. To overcome this problem, we can exploit quotients of tensor algebras and tensor rings which provide useful algebraic modelling of, and algorithmic computations with, linear operators.

Tensor algebras can be seen as a generalization of free noncommutative polynomial algebras and inherit all their algorithmic obstructions. For tensor algebras, Bergman's paper [5] also provides a framework in which reduction systems and corresponding normal forms can be analysed, analogous to Gröbner bases. At the same time, parts of the tensor setting can be automated, in particular, verification of the confluence criterion and subsequent computations with normal forms. In the tensor setting, reduction rules are given by module homomorphisms. This often allows for finite reduction systems with unique normal forms. In practice, normal forms are needed for effective computation. Finding and proving the structure of normal forms is a difficult task, in fact the general problem is even undecidable.

Tensor rings naturally capture the multiadditivity of compositions of additive operators. In addition, they allow basis-free treatment of multiplication operators resp. coefficients. In particular, the coefficient ring is not required to be finitely presented. For further details on tensor rings and proofs see [16, 53]. A Gröbner basis theory for free bimodules has been presented in [33] and for bimodules over Poincaré-Birkhoff-Witt (PBW) algebras in [46, 36].

Tensor reduction systems in tensor rings have been exploited for the first time by us in [30] to treat operator algebras algorithmically. We also develop a generalization of Bergman's setting by defining the notion of ambiguities with specialization. The new setting allows overlapping domains of reduction homomorphisms, which also make the algorithmic verification of the confluence criterion more efficient.

In order to see applications of noncommutative Gröbner bases in the free polynomial algebra to operator algebras, we recommend the reader to read [26, 27, 48]. Integro-differential operators over a field of constants were introduced in [47, 51]. They were defined in terms of a parametrized Gröbner basis in infinitely many variables coming from a basis of the coefficient algebra; see also the survey [52] for an automated confluence proof and [42] for related references. For integro-differential operators with polynomial coefficients, generalized Weyl algebras [4], skew polynomials [43], and noncommutative Gröbner bases [41] have been used. An overview on Gröbner-Shirshov bases for various algebraic structures is given in [6]; see, in particular, [24, 21, 20, 19] in connection with differential type, integro-differential, and Rota-Baxter type operators.

1.4 Contributions and results

We summarize the main contributions of this thesis and the related publications. A detailed description of contributions is given at the beginning of each chapter. Chapter 2 includes all prerequisites needed for reading the thesis. Chapter 3 is a self-contained description of tensor reduction systems including a fully worked-out proof of Bergman's diamond lemma [5] for tensor rings and its generalization, see [30]. Chapter 4 on the ring of integro-differential operators (IDO) and the ring of integro-differential operators with linear substitutions (IDOLS) includes the main results of our paper [30]. In addition, we show how to obtain confluent reduction systems for IDO and IDOLS starting from basic identities and we construct the action of these rings of operators on modules of vector-valued functions. In Chapter 5, we use our algebraic framework for IDO to generalize the formula of Green's operators of first-order linear boundary problems to the system case. The rest of this chapter on Artstein's reduction presents our results in [13] and generalizes some of them. The content of Chapter 6 is a discrete analog of IDO presented for the first time in this thesis.

Our first results on tensor reduction systems were published in [28], where we used a two-level reduction system in a tensor algebra in order to model IDO with commutative coefficients. In particular, we compare the computational effort for the confluence check to Bergman's setting. In the case of IDO, we explain a computer-assisted completion process for obtaining a confluent reduction system in [29], where we also give a brief description of the package `TenReS`. Generalizing our results in [28], we develop tensor reduction systems with specialization in tensor rings in [30] where we also give confluent reduction systems for IDO and IDOLS with noncommutative coefficients. In our joint paper [13], we exploit this framework to recover Artstein's transformation in a computer-assisted way.

In order to support computations with tensor reduction systems, the Mathematica package `TenReS` has been implemented, see [29]. In this thesis, we use the package for verification of the confluence criterion, computing with normal forms, and solving operator identities in the examples and applications. We also make implementation of the package, along with all computations related to the thesis available at the website for the package:

`http://gregensburger.com/softw/tenres`

1.5 Outline

In Chapter 2, we recall definitions and properties of basic algebraic objects and constructions used in the thesis. We first recall the structure of free modules, monoids, algebras and their universal properties in Section 2.1. The direct sum and the tensor product of modules are described in Section 2.2. For the rest of that section, we focus on free bimodules viewed as free left modules. In Section 2.3, we summarize the structure of tensor rings over arbitrary rings and differential rings. We explain in Section 2.4 basics of term rewriting. Then, in Section 2.5, we briefly outline the algebraic analysis approach, which transforms systems of linear functional equations, needed later in Section 5.2.

In Chapter 3, we provide a self-contained presentation of reduction systems on tensor rings. To illustrate this approach, in Section 3.1, we explain informally the structure of the ring of differential operators over noncommutative differential rings. In Section 3.2, we provide a detailed proof for Bergman's diamond lemma in tensor rings, which is omitted in his paper. In Section 3.3, we describe a generalization of the tensor setting by introducing the concept of specialization.

In Chapter 4, we give two instances for modelling rings of operators using our tensor setting. First, in Section 4.1, we give the formal construction of IDO with noncommutative coefficients. Then, in Section 4.2, we show that the coefficient ring is a left module over the ring of IDO. We proceed by describing a completion process for IDO in Section 4.3, yielding confluent reduction systems and normal forms. Based on that, we illustrate how to solve operator equations by ansatz. Later, in Section 4.4, we introduce the module of vector-valued "functions" over the ring of IDO with matrix coefficients. The second instance of rings of operators, explained in Section 4.5, extends the ring of IDO by including linear substitutions. Another instance of the rings of operators is given in Chapter 6.

In Chapter 5, we give applications of the rings of IDO and IDOLS. Section 5.1 formulates first-order systems of linear boundary problems and their Green's operators in our framework. In Section 5.2, we discuss on computing with operators in the ring of IDOLS having different domains and codomains. Then, we recover the well-known Artstein's transformation and work out a generalization of it.

In Chapter 6, we develop an algebraic framework for the ring of inversive sum-difference operators (SDO). We construct the ring of SDO in Section 6.1, obtain a confluent reduction system in Section 6.2, and add some remarks on computational aspects.

Notational conventions

Throughout this thesis, we have the following conventions:

- By a $\mathcal{K}$-module, we always mean a left $\mathcal{K}$-module.
- We use operator notation, e.g. we write $\varphi 1$ instead of $\varphi(1)$ or

$$\partial AB = (\partial A)B + A\partial B$$

  for the Leibniz rule instead of $\partial(AB) = \partial(A)B + A\partial(B)$.

- All our operators act from the left, in particular, a product $L \cdot L'$ acts on A as $(L \cdot L')(A)$.
- By using the notation $\cdot$, we distinguish multiplication of operators from the action of operations. For instance, we denote by $\partial \cdot A$ product of the differential operator ∂ with the multiplication operator A, whereas ∂A is used for representing derivation of A.

Chapter 2

Preliminaries

For the convenience of the reader, in this chapter, we summarize and discuss algebraic notions, constructions, and results used in this thesis. We assume basic notions from algebra, in particular, groups, rings, ideals, modules over commutative rings, and isomorphism theorems to be known.

In Section 2.1, we review the construction of free modules, monoids, and algebras. Section 2.2 is dedicated to study basic properties of direct sums of modules, tensor products of modules over arbitrary rings, and the free bimodules. Using these notions, in Section 2.3, we discuss tensor rings over free bimodules, as basic objects in our tensor setting described in Chapter 3, see also [15, 16, 53]. We also explain with entries in a given commutative differential ring, how to construct a differential ring over its ring of matrices.

We recall in Section 2.4 basics of term rewriting that are needed for explaining tensor reduction systems. The definitions, remarks, lemmas and theorems which are recalled in this section have been taken from [3, Secs. 2.1 and 2.7], see also there for further details and proofs. In Section 2.5, we explain briefly the algebraic analysis approach which addresses systems of linear functional equations from an algebraic point of view. For more details, we refer the reader to [11] and the references therein.

Throughout this chapter $\mathcal{K}$ denotes a unitary (not necessarily commutative) ring, unless it is explicitly stated otherwise.

2.1 Free modules, monoids and algebras

In this section, we recall the notions of free left module, free monoid, and free algebra on a set, which are essential for studying more complicated algebraic structures explained in later sections.

2.1.1 Free modules

Recall that the endomorphism ring of an abelian group $\mathcal{G}$, denoted by $\mathrm{End}(\mathcal{G})$, is the set of all endomorphisms of $\mathcal{G}$ (i.e. the set of all homomorphisms of $\mathcal{G}$ into itself) endowed with an addition operation defined by pointwise addition of functions and a multiplication operation defined by function composition. A left $\mathcal{K}$-module is an abelian group M together with a homomorphism $\varphi\colon \mathcal{K} \to \mathrm{End}(M)$. The corresponding scalar multiplication is defined by $km = \varphi(k)(m)$, for all $k \in \mathcal{K}$ and $m \in M$. It is easy to check this definition is equivalent to the standard definition of a left module.

Definition 2.1. *Let $X = \{x_i \mid i \in I\}$ be a set indexed by a set I. The set of formal sums*

$$\mathcal{K}X = \{\sum_{i\in I} k_i x_i \mid k_i \in \mathcal{K} \text{ and almost all } k_i \text{ are zero}\},$$

together with the following addition and scalar multiplication

$$\sum_{i\in I} k_i x_i + \sum_{i\in I} l_i x_i = \sum_{i\in I} (k_i + l_i) x_i, \quad c\sum_{i\in I} k_i x_i = \sum_{i\in I} (ck_i) x_i \quad (c \in \mathcal{K}),$$

is a $\mathcal{K}$-module which is called the free $\mathcal{K}$-module *on the set X.*

Theorem 2.2. *(Factor theorem for modules) Let M and N be two $\mathcal{K}$-modules. Given a module homomorphism $\varphi\colon M \to N$ and a submodule M' of M with natural homomorphism $\pi\colon M \to M/M'$, such that $M' \subseteq \ker\varphi$, there is a unique $\mathcal{K}$-module homomorphism $\overline{\varphi}\colon M/M' \to N$ such that the following diagram is commutative.*

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/M' \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\overline{\varphi}} \\ & & N \end{array}$$

Moreover, $\overline{\varphi}$ is a $\mathcal{K}$-module monomorphism if and only if $M' = \ker\varphi$.

Theorem 2.3. *(First isomorphism theorem of modules) Let $\varphi\colon M \to M'$ be a $\mathcal{K}$-module homomorphism. Then the image of φ is isomorphic to the quotient module $M/\ker\varphi$.*

Remark 2.4. *Recall that analogues of Theorems 2.2 and 2.3 hold in any algebraic structure.*

Theorem 2.5. *(Universal property of free modules) Let $\mathcal{K}X$ be the free $\mathcal{K}$-module on a set $X = \{x_i \mid i \in I\}$ indexed by a set I, $\iota\colon X \hookrightarrow \mathcal{K}X$ be the* *inclusion map, and N be a $\mathcal{K}$-module. Then for each map $\varphi\colon X \to N$, there exists a unique $\mathcal{K}$-module homomorphism $\overline{\varphi}\colon \mathcal{K}X \to N$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{K}X \\ & {}_{\varphi}\searrow & \downarrow \overline{\varphi} \\ & & N \end{array}$$

In particular, $\overline{\varphi}$ satisfies $\overline{\varphi}(\sum_{i\in I} k_i x_i) = \sum_{i\in I} k_i \varphi(x_i)$.

Corollary 2.6. *Let $X = \{x_i \mid i \in I\}$ be a set indexed by a set I. Then, every $\mathcal{K}$-module that satisfies the universal property of free modules on X is isomorphic to $\mathcal{K}X$.*

Remark 2.7. *In general, universal objects are unique up to level of isomorphisms. In order to see details, see [31, Sec. 1.7] and [53, Sec. 1].*

2.1.2 Free monoids

In the following, we study the structure of the free monoid on a set whose elements are all the finite sequences of zero or more elements from the set. We start by recalling the definition of a monoid.

Definition 2.8. *A* semigroup *is a non-empty set $\mathcal{M}$ together with a binary operation $\cdot\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$, called multiplication, such that $(m\cdot n)\cdot l = m\cdot(n\cdot l)$ for all $m, n, l \in \mathcal{M}$. Moreover, if there exists an element $e \in \mathcal{M}$ satisfying $e \cdot m = m \cdot e = m$, for all $m \in \mathcal{M}$, then $(\mathcal{M}, \cdot)$ is called a* monoid.

Throughout this thesis by an *alphabet* we mean a set X. Let us explain notion of the *word monoid* on an alphabet.

Definition 2.9. *A* word W *on an* alphabet X *is a finite sequence of elements in X of the form $W = x_1 \cdots x_r$ such that $x_i \in X$, for $i = 1, \ldots, r$.*

We denote by $\langle X\rangle$ the set of all words over the alphabet X and equip it with a binary operation $*$ which is defined as concatenation of words, i.e.

$$x_1 \cdots x_r * y_1 \cdots y_s = x_1 \cdots x_r y_1 \cdots y_s.$$

Note that the operation $*$ is associative. The empty sequence is the *neutral* element for the operation $*$. We call it the *empty word* and denote it by ϵ. Therefore, $(\langle X\rangle, *)$ is a monoid which is called the *word monoid* on the alphabet X.

Example 2.10. *Suppose that* $X = \{a, b, c\}$*. Then the elements of the word monoid* $(\langle X \rangle, *)$ *are words of the form*

$$\epsilon, a, b, c, aa, bb, cc, ab, ba, ac, ca, bc, cb, aab, \dots.$$

From here on we omit the monoid operation. Therefore, we denote a monoid $(M, \cdot)$ only by M and write mn instead of $m \cdot n$.

Theorem 2.11. *(Universal property of word monoids) Let* X *be a set,* $\mathcal{M}$ *be a monoid, and* ι *be the inclusion map from* X *into the word monoid* $\langle X \rangle$*. Then for any map* $\varphi\colon X \to \mathcal{M}$*, there exists a unique monoid homomorphism* $\overline{\varphi}\colon \langle X \rangle \to \mathcal{M}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \langle X \rangle \\ & {}_{\varphi}\searrow & \downarrow \overline{\varphi} \\ & & \mathcal{M} \end{array}$$

In particular, $\overline{\varphi}$ *satisfies* $\overline{\varphi}(x_1 \cdots x_s) = \varphi(x_1) \cdots \varphi(x_s)$.

Because of the universal property of the word monoids, every word monoid is also called a *free monoid*.

2.1.3 Free algebras

Our goal in this subsection is to explain notion of *the free algebra* on a set. For the whole subsection we let $\mathcal{K}$ be a unitary commutative ring.

Definition 2.12. *A* $\mathcal{K}$-algebra $\mathcal{A}$ *is a unitary ring which is also a* $\mathcal{K}$*-module such that the multiplication operation in* $\mathcal{A}$ *is* $\mathcal{K}$*-bilinear, i.e., for any* $a, a' \in \mathcal{A}$ *and* $k \in \mathcal{K}$ *we have*

$$k \cdot (aa') = (k \cdot a)a' = a(k \cdot a')$$

where $\cdot$ *denotes the action of* $\mathcal{K}$ *on* $\mathcal{A}$.

Equivalently, we can define an algebra as follows:

Definition 2.13. *A* $\mathcal{K}$-algebra $\mathcal{A}$ *is a unitary ring together with a ring homomorphism* $\varphi\colon \mathcal{K} \to \mathcal{A}$ *such that* $\varphi(\mathcal{K}) \subseteq \mathcal{Z}(\mathcal{A})$ *where* $\mathcal{Z}(\mathcal{A})$ *denotes center of the ring* $\mathcal{A}$.

Proposition 2.14. *It is routine to check that both Definition 2.12 and Definition 2.13 are equivalent.*

Definition 2.15. *Let* $X = \{x_i \mid i \in I\}$ *be a set indexed by a set* I *and* $\langle X \rangle$ *be the word monoid on the set* X. *As a* $\mathcal{K}$*-module, the* free algebra $\mathcal{K}\langle X \rangle$ *on the set* X *over the ring* $\mathcal{K}$, *is the free* $\mathcal{K}$*-module on the set* $\langle X \rangle$, *i.e.,*

$$\mathcal{K}\langle X \rangle = \{ \sum_{w \in \langle X \rangle} k_w w \mid k_w \in \mathcal{K} \text{ and almost all } k_w \text{ are zero}\}.$$

This $\mathcal{K}$*-module becomes a* $\mathcal{K}$*-algebra by defining the following multiplication*

$$\Big(\sum_{w \in \langle X \rangle} k_w w\Big)\Big(\sum_{v \in \langle X \rangle} k'_v v\Big) = \sum_{u \in \langle X \rangle} \Big(\sum_{wv=u} k_w k'_v\Big) u$$

The elements of $\mathcal{K}\langle X \rangle$ *are called* noncommutative polynomials *and they can be written uniquely as*

$$\sum k_{j_1,\ldots,j_s} x_{j_1} \cdots x_{j_s} = \sum_J k_J x_J$$

where J *runs over all distinct finite sequences in* I, *and for* $J = (j_1, \ldots, j_s)$ *we define* $x_J = x_{j_1} \cdots x_{j_s}$.

Definition 2.16. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\mathcal{K}$*-algebras. A ring homomorphism* $\psi \colon \mathcal{A} \to \mathcal{B}$ *is called a* $\mathcal{K}$-algebra homomorphism *if* $\psi(ka) = k\psi(a)$, *for all* $k \in \mathcal{K}$ *and* $a \in \mathcal{A}$.

Lemma 2.17. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\mathcal{K}$*-algebras together with the ring homomorphisms* φ *and* φ' *respectively as in Definition 2.13. Then a ring homomorphism* $\psi \colon \mathcal{A} \to \mathcal{B}$ *is a* $\mathcal{K}$*-algebra homomorphism if and only if the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\varphi} & \mathcal{A} \\ & \searrow^{\varphi'} & \downarrow \psi \\ & & \mathcal{B} \end{array}$$

Definition 2.18. *A* $\mathcal{K}$-subalgebra *of a* $\mathcal{K}$*-algebra* $\mathcal{A}$ *is a subring* $\mathcal{B}$ *of* $\mathcal{A}$ *that is also a* $\mathcal{K}$*-module.*

Remark 2.19. *Every* $\mathcal{A}$*-submodule* $\mathcal{B}$ *of a* $\mathcal{K}$*-algebra* $\mathcal{A}$ *is a* $\mathcal{K}$*-submodule of* $\mathcal{A}$, *since we have* $k \cdot b = (k \cdot 1)b$ *for all* $k \in \mathcal{K}$ *and* $b \in \mathcal{B}$.

Definition 2.20. *A* left ideal *of a* $\mathcal{K}$*-algebra* $\mathcal{A}$ *is a left* $\mathcal{A}$*-submodule.*

Similarly, one can define a *right ideal* as a right $\mathcal{A}$-submodule. A *two-sided ideal* of $\mathcal{A}$ is a left ideal that is a right ideal as well.

Definition 2.21. *Let $\mathcal{A}$ be a $\mathcal{K}$-algebra and I be a* two-sided ideal *of $\mathcal{A}$. The quotient module*

$$\mathcal{A}/I = \{a + I \mid a \in \mathcal{A}\},$$

together with the following multiplication

$$(a_1 + I)(a_2 + I) = (a_1 a_2 + I) \quad (\textit{for all } \ a_1, a_2 \in \mathcal{A}),$$

is a $\mathcal{K}$-algebra which is called the quotient *algebra $\mathcal{A}/I$.*

Theorem 2.22. *(Universal property of free algebras) Let $\mathcal{K}\langle X\rangle$ be the free $\mathcal{K}$-algebra on a set $X = \{x_i \mid i \in I\}$ indexed by a set I and let $\mathcal{A}$ be a $\mathcal{K}$-algebra. Then for each map $\varphi\colon X \to \mathcal{A}$, there exists a unique $\mathcal{K}$-algebra homomorphism $\overline{\varphi}\colon \mathcal{K}\langle X\rangle \to \mathcal{A}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{K}\langle X\rangle \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\overline{\varphi}} \\ & & \mathcal{A} \end{array}$$

In particular, $\overline{\varphi}$ satisfies

$$\overline{\varphi}(\sum_J k_J x_{j_1} \cdots x_{j_s}) = \sum_J k_J \varphi(x_{j_1}) \cdots \varphi(x_{j_s}).$$

2.2 The direct sum and the tensor product of modules

2.2.1 The direct sum of modules

In this subsection we explain the *direct sum* of left modules over a ring. We assume I to be an index set.

Definition 2.23. *The* direct product *of a family $(M_i)_{i\in I}$ of $\mathcal{K}$-modules, is the Cartesian product of M_i, i.e, the set*

$$\Pi_{i\in I} M_i = \{(m_i)_{i\in I} \mid m_i \in M_i \textit{ for all } i \in I\},$$

where the operations are defined componentwise:

$$(m_i)_{i\in I} + (m'_i)_{i\in I} = (m_i + m'_i)_{i\in I}, \quad k(m_i) = (km_i)_{i\in I} \quad (k \in \mathcal{K})$$

With these operations, $\Pi_{i\in I} M_i$ is a $\mathcal{K}$-module. In particular, if $M_i = M$ for all $i \in I$, we obtain the direct power *of M, denoted by M^I.*

Definition 2.24. *The* external direct sum *of a family* $(M_i)_{i\in I}$ *of* $\mathcal{K}$*-modules, is the* $\mathcal{K}$*-submodule*

$$\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} \in \Pi_{i\in I} M_i \mid m_i = 0 \textit{ for almost all } i \in I\}$$

of $\Pi_{i\in I} M_i$. *In particular, when* $M_i = M$ *for all* $i \in I$, *we have a direct sum of copies of* M, *denoted by* $M^{(I)}$. *Any element of* $\bigoplus_{i\in I} M_i$ *is denoted by the sum* $\sum\limits_{i\in I} m_i$ *where* $m_i \in M_i$ *for all* $i \in I$ *and* $m_i = 0$ *for almost all* $i \in I$.

For each $i \in I$ we have the canonical injection $\iota_i \colon M_i \hookrightarrow \bigoplus_{i\in I} M_i$ which maps $x \in M_i$ to the family $(x_j)_{j\in I}$ where $x_j = \delta_{ij} x$. Moreover, for a finite index set I the direct sum and the direct product coincide as modules.

Definition 2.25. *A* $\mathcal{K}$*-module, denoted by* $\sum_I M_i$, *is called the* sum *of a family of* $\mathcal{K}$*-submodules, if*

$$\sum_I M_i = \{\sum_{i\in I} m_i \mid m_i \in M_i \textit{ and } m_i = 0 \textit{ for almost all } i \in I\}.$$

If each element of this sum can be expressed in only one way, we write $\sum_I M_i = \bigoplus_{i\in I} M_i$ *and call it the* direct sum.

The module $\sum_I M_i$ is sometimes called the *internal direct sum*, to distinguish it from the *external direct sum* or *coproduct* $\bigoplus_I M_i$. It is clear that $\bigoplus_I M_i$ is in fact the internal direct sum of the submodules $\operatorname{im} \iota_i$, where the maps ι_i are canonical injections.

Theorem 2.26. *(Universal property of the direct sum of modules) Let* $(M_i)_{i\in I}$ *be a family of* $\mathcal{K}$*-modules. Let* $(\varphi_i)_{i\in I} \colon M_i \to N$ *be a family of* $\mathcal{K}$*-module homomorphisms into a* $\mathcal{K}$*-module* N *and let* $\iota_i \colon M_i \hookrightarrow \bigoplus_{i\in I} M_i$ *be the* i*-th canonical injection. Then there exists a unique* $\mathcal{K}$*-module homomorphism* $\varphi \colon \bigoplus_{i\in I} M_i \to N$ *such that for all* $i \in I$ *the following diagram is commutative:*

$$\begin{array}{ccc} M_i & \overset{\iota_i}{\hookrightarrow} & \bigoplus_{i\in I} M_i \\ & {\scriptstyle \varphi_i}\searrow & \downarrow{\scriptstyle \varphi} \\ & & N \end{array}$$

In particular, $\varphi\big(\sum\limits_{i\in I} m_i\big) = \sum\limits_{i\in I} \varphi_i(m_i)$.

2.2.2 The tensor product over a commutative ring

The notion of tensor product may be defined for any pair of bimodules, but first we discuss it for the simpler case of modules over a commutative ring. In this subsection, we assume that $\mathcal{K}$ is a unitary commutative ring.

Definition 2.27. *Let M, N, and L be $\mathcal{K}$-modules. A map $\varphi\colon M \times N \to L$ is called* $\mathcal{K}$-bilinear *if it satisfies the following relations*

$$\varphi(m+m', n) = \varphi(m, n) + \varphi(m', n),\ \varphi(m, n+n') = \varphi(m, n) + \varphi(m, n'),$$
$$\varphi(km, n) = k\varphi(m, n),\ \varphi(m, kn) = k\varphi(m, n),$$

for all $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$.

Remark 2.28. *Note that in Definition 2.27 commutativity of the ring $\mathcal{K}$ is required, since for all $k, k' \in \mathcal{K}$, $m \in M$, and $n \in N$ we have*

$$\varphi(km, k'n) = k\varphi(m, k'n) = kk'\varphi(m, n)$$

and

$$\varphi(km, k'n) = k'\varphi(km, n) = k'k\varphi(m, n).$$

Definition 2.29. *Let M and N be $\mathcal{K}$-modules. Let F be the free $\mathcal{K}$-module on the set $M \times N$ and let T be the $\mathcal{K}$-submodule of F generated by all elements of the form*

$$(m+m', n) - (m, n) - (m', n),\ (m, n+n') - (m, n) - (m, n'),$$
$$(km, n) - k(m, n),\ (m, kn) - k(m, n),$$

for $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$. The quotient module

$$M \otimes_{\mathcal{K}} N = F/T$$

containing all equivalence classes $m \otimes n = (m, n) + T$, together with the map $\otimes\colon M \times N \to M \otimes_{\mathcal{K}} N$ defined by $\otimes(m, n) = m \otimes n$, is called the tensor product *of M and N over $\mathcal{K}$.*

By construction, the tensor product $M \otimes_{\mathcal{K}} N$ is the $\mathcal{K}$-module generated by the set of all *pure tensors* $\{m \otimes n \mid m \in M, n \in N\}$ with relations

$$(m+m') \otimes n = m \otimes n + m' \otimes n,\ m \otimes (n+n') = m \otimes n + m \otimes n',$$
$$km \otimes n = k(m \otimes n),\ m \otimes kn = k(m \otimes n),$$

for all $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$. Every element of $M \otimes_{\mathcal{K}} N$ can be written as a finite sum of pure tensors, i.e.,

$$M \otimes_{\mathcal{K}} N = \{\sum_{i=1}^{t} m_i \otimes n_i \mid t \in \mathbb{N}, m_i \in M, n_i \in N\}.$$

Lemma 2.30. *The map $\otimes$ defined in Definition 2.29 is a $\mathcal{K}$-bilinear map.*

Theorem 2.31. *(Universal property of the tensor product of modules over a commutative ring) Let M and N be $\mathcal{K}$-modules and let $(M \otimes_{\mathcal{K}} N, \otimes)$ be the tensor product of M and N together with the $\mathcal{K}$-bilinear map $\otimes$. Then for any $\mathcal{K}$-bilinear map $\varphi \colon M \times N \to L$, there exists a unique $\mathcal{K}$-linear map $\overline{\varphi} \colon M \otimes_{\mathcal{K}} N \to L$ such that the following diagram is commutative:*

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_{\mathcal{K}} N \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle \overline{\varphi}} \\ & & L \end{array}$$

In particular, the map $\overline{\varphi}$ satisfies $\overline{\varphi}(\sum_{i=1}^{t} m_i \otimes n_i) = \sum_{i=1}^{t} \varphi(m_i, n_i)$.

Proposition 2.32. *Let $\varphi \colon M \to M'$ and $\psi \colon N \to N'$ be two $\mathcal{K}$-linear maps. Then there exists a $\mathcal{K}$-linear map denoted*

$$\varphi \otimes \psi \colon M \otimes_{\mathcal{K}} N \to M' \otimes_{\mathcal{K}} N'$$

given by $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$.

2.2.3 The tensor product over a ring

In the following, we explain the tensor product of bimodules over an arbitrary unitary ring.

Definition 2.33. *A* $\mathcal{K}$-bimodule *is a left $\mathcal{K}$-module M which is also a right $\mathcal{K}$-module satisfying the associativity condition*

$$(km)k' = k(mk'),$$

for all $m \in M$ and $k, k' \in \mathcal{K}$.

Definition 2.34. *Let M be a right $\mathcal{K}$-module and N be a left $\mathcal{K}$-module. Given an abelian group L, we call $\psi \colon M \times N \to L$ a* balanced map *if it satisfies the following relations*

$$\begin{aligned} \psi(m + m', n) &= \psi(m, n) + \psi(m', n), \\ \psi(m, n + n') &= \psi(m, n) + \psi(m, n'), \\ \psi(mk, n) &= \psi(m, kn), \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$.

From the above relations it follows that

$$\psi(0,n) = \psi(0+0,n) = \psi(0,n) + \psi(0,n),$$

and now by cancellation law $\psi(0,n) = 0$ for all $n \in N$. Similarly $\psi(m,0) = 0$ for all $m \in M$. In addition, we have

$$0 = \psi(0,n) = \psi(m-m,n) = \psi(m,n) + \psi(-m,n).$$

Therefore, $\psi(-m,n) = -\psi(m,n)$ and similarly $\psi(m,-n) = -\psi(m,n)$ for all $m \in M$ and $n \in N$.

Definition 2.35. *Let M be a right $\mathcal{K}$-module and N be a left $\mathcal{K}$-module. Let $\mathcal{G}$ be freely generated (as a $\mathbb{Z}$-module) by the set $M \times N$ and let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ generated by all elements of the form*

$$(m+m',n) - (m,n) - (m',n),\ (m,n+n') - (m,n) - (m,n'),$$
$$(mk,n) - (m,kn),$$

for $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$. The abelian group

$$M \otimes_{\mathcal{K}} N = \mathcal{G}/\mathcal{H}$$

containing all equivalence classes $m \otimes n = (m,n) + \mathcal{H}$, together with the map $\otimes\colon M \times N \to M \otimes_{\mathcal{K}} N$ defined by $\otimes(m,n) = m \otimes n$, is called the tensor product *of M and N over $\mathcal{K}$.*

The tensor product $M \otimes_{\mathcal{K}} N$ is generated by the set of all *pure tensors* $\{m \otimes n \mid m \in M, n \in N\}$ with relations

$$\begin{aligned}(m+m') \otimes n &= m \otimes n + m' \otimes n,\\ m \otimes (n+n') &= m \otimes n + m \otimes n',\\ mk \otimes n &= m \otimes kn,\end{aligned}$$

for $m, m' \in M$, $n, n' \in N$, and $k \in \mathcal{K}$. Every element of $M \otimes_{\mathcal{K}} N$ can be written as a finite sum of pure tensors, i.e.

$$M \otimes_{\mathcal{K}} N = \{\sum_{i=1}^{t} m_i \otimes n_i \mid t \in \mathbb{N}, m_i \in M, n_i \in N\}.$$

Lemma 2.36. *The map $\otimes$ defined in Definition 2.35 is a balanced map.*

Proof. We have that

$$\begin{aligned}\otimes(m+m',n)-\otimes(m,n)-\otimes(m',n) &= (m+m')\otimes n - m\otimes n - m'\otimes n\\ &= ((m+m',n)+H)-((m,n)+H)-((m',n)+H)\\ &= ((m+m',n)-(m,n)-(m',n))+H = H = 0.\end{aligned}$$

Similarly, we have $\otimes(m,n+n') = \otimes(m,n)+\otimes(m,n')$, and $\otimes(mk,n) = \otimes(m,kn)$. Hence $\otimes$ is a balanced map. $\square$

Theorem 2.37. *(Universal property of the tensor product of bimodules) Let M be a right $\mathcal{K}$-module and N be a left $\mathcal{K}$-module and let $(M\otimes_{\mathcal{K}} N, \otimes)$ be the tensor product of M and N together with the balanced map $\otimes$. Then for any balanced map $\varphi\colon M\times N\to L$, there exists a unique group homomorphism $\overline{\varphi}\colon M\otimes_{\mathcal{K}} N\to L$ such that the following diagram is commutative:*

$$\begin{array}{ccc} M\times N & \xrightarrow{\ \otimes\ } & M\otimes_{\mathcal{K}} N \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\overline{\varphi}} \\ & & L \end{array}$$

In particular, the map $\overline{\varphi}$ satisfies $\overline{\varphi}(\sum_{i=1}^{t} m_i\otimes n_i) = \sum_{i=1}^{t}\varphi(m_i,n_i)$.

Proof. Since the $\mathcal{K}$-bimodule $M\otimes_{\mathcal{K}} N$ is generated as an abelian group by the set of all pure tensors and $\overline{\varphi}$ is free on this generating set, the group homomorphism $\overline{\varphi}$ is uniquely determined if it exists.

Let $\mathcal{G}$ and $\mathcal{H}$ be the abelian groups which were defined in Definition 2.35. By the universal property of free modules, the map $\varphi\colon M\times N\to L$ can be uniquely extended to a $\mathbb{Z}$-module homomorphism $\tilde{\varphi}\colon\mathcal{G}\to L$. Since the map φ is balanced we have

$$\begin{aligned}&\tilde{\varphi}((m+m',n)-(m,n)-(m',n))\\ &\qquad = \varphi(m+m',n)-\varphi(m,n)-\varphi(m',n) = 0,\end{aligned}$$

$$\tilde{\varphi}((mk,n)-(m,kn)) = \varphi(mk,n)-\varphi(m,kn) = 0,$$

and similarly $\tilde{\varphi}((m,n+n')-(m,n)-(m,n')) = 0$ for all $m,m'\in M$, $n,n'\in N$ and $k\in\mathcal{K}$ and hence $\mathcal{H}\subseteq\ker\tilde{\varphi}$. Now, by the factor theorem, the map $\tilde{\varphi}$ induces a unique group homomorphism $\overline{\varphi}\colon\mathcal{G}/\mathcal{H}\to L$ such that $\tilde{\varphi} = \overline{\varphi}\circ\pi$, where $\pi\colon\mathcal{G}\to\mathcal{G}/\mathcal{H}$ is the natural homomorphism. $\square$

Lemma 2.38. *If M and N are $\mathcal{K}$-bimodules, then $M\otimes_{\mathcal{K}} N$ is again a $\mathcal{K}$-bimodule with scalar multiplications*

$$(k, m\otimes n)\mapsto km\otimes n \quad \textit{and} \quad (m\otimes n, k)\mapsto m\otimes nk.$$

Proof. We first prove the left scalar multiplication defined above is well-defined: taking an arbitrary but fixed element $k \in \mathcal{K}$, we define a map $\varphi_k\colon M \times N \to M \otimes_{\mathcal{K}} N$ by $\varphi_k(m,n) = km \otimes n$. Note that the map φ_k is biadditive since

$$\begin{aligned}\varphi_k(m+m',n) &= k(m+m') \otimes n = (km+km') \otimes n \\ &= km \otimes n + km' \otimes n = \varphi_k(m,n) + \varphi_k(m',n),\end{aligned}$$

and similarly $\varphi_k(m, n+n') = \varphi_k(m,n) + \varphi_k(m,n')$ for all $m, m' \in M$ and $n, n' \in N$. It is also balanced since

$$\varphi_k(mk',n) = k(mk') \otimes n = (km)k' \otimes n = km \otimes k'n = \varphi_k(m,k'n),$$

for all $m \in M$, $n \in N$, and $k' \in \mathcal{K}$. Hence, by the universal property of the tensor product of bimodules, there exists a group homomorphism $\overline{\varphi}_k\colon M \otimes_{\mathcal{K}} N \to M \otimes_{\mathcal{K}} N$ such that the following diagram is commutative:

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_{\mathcal{K}} N \\ & \searrow^{\varphi_k} & \downarrow \overline{\varphi}_k \\ & & M \otimes_{\mathcal{K}} N \end{array}$$

In particular, $\overline{\varphi}_k(m \otimes n) = km \otimes n$. Hence the scalar multiplication $k(m \otimes n) = \overline{\varphi}_k(m \otimes n)$ is well-defined. Analogously, the right scalar multiplication defined above is well-defined as well. We have

$$(k+k')(m \otimes n) = (k+k')m \otimes n = km \otimes n + k'm \otimes n = k(m \otimes n) + k'(m \otimes n),$$

$$k(m \otimes n + m' \otimes n') = km \otimes n + km' \otimes n' = k(m \otimes n) + k(m' \otimes n'),$$
$$(kk')(m \otimes n) = (kk')m \otimes n = k(k'm) \otimes n = k(k'(m \otimes n)),$$
$$1(m \otimes n) = 1m \otimes n = m \otimes n,$$

for all $k, k' \in \mathcal{K}$, $m, m' \in M$, and $n, n' \in N$. Hence $M \otimes_{\mathcal{K}} N$ is a left $\mathcal{K}$-module. Analogously, $M \otimes_{\mathcal{K}} N$ is a right $\mathcal{K}$-module. In addition,

$$(k(m \otimes n))k' = (km \otimes n)k' = km \otimes nk' = k(m \otimes nk') = k((m \otimes n)k').$$

Hence, the associativity condition holds and $M \otimes_{\mathcal{K}} N$ is a $\mathcal{K}$-bimodule. $\square$

2.2.4 Bimodules as left modules

In this subsection, we describe bimodules as left modules over a bigger ring. This enables us to define the *free bimodule* on a set over a ring as a free left module.

Proposition 2.39. *Let $\mathcal{C}$ be a commutative ring and let $\mathcal{R}$ and $\mathcal{S}$ be two $\mathcal{C}$-algebras. Then $\mathcal{R} \otimes \mathcal{S}$ is a $\mathcal{C}$-algebra with the ring multiplication*

$$(r \otimes s)(r' \otimes s') = rr' \otimes ss',$$

for all $r, r' \in \mathcal{R}$ and $s, s' \in \mathcal{S}$.

Proof. Let $\tau\colon \mathcal{S} \times \mathcal{R} \to \mathcal{R} \otimes \mathcal{S}$ be map defined by $\tau(s, r) = r \otimes s$. The map τ is bilinear since

$$\tau(s + s', r) = r \otimes (s + s') = r \otimes s + r \otimes s' = \tau(s, r) + \tau(s', r),$$

and

$$\tau(cs, r) = r \otimes cs = rc \otimes s = cr \otimes s = c(r \otimes s) = c\tau(s, r),$$

for all $r \in \mathcal{R}$, $s \in \mathcal{S}$, and $c \in \mathcal{C}$. Similarly, $\tau(s, r + r') = \tau(s, r) + \tau(s, r')$ and $\tau(s, cr) = c\tau(s, r)$. Hence, by the universal property of tensor product of modules, it induces a $\mathcal{C}$-module homomorphism $\tilde{\tau}\colon \mathcal{S} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{S}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S} \times \mathcal{R} & \xrightarrow{\otimes} & \mathcal{S} \otimes \mathcal{R} \\ & \searrow^{\tau} & \downarrow{\tilde{\tau}} \\ & & \mathcal{R} \otimes \mathcal{S} \end{array}$$

In particular, $\tilde{\tau}(s \otimes r) = r \otimes s$, for all $r \in \mathcal{R}$ and $s \in \mathcal{S}$. This gives rise to the permutation map $\tau_1 = \mathrm{id}_{\mathcal{R}} \otimes \tilde{\tau} \otimes \mathrm{id}_{\mathcal{S}}\colon \mathcal{R} \otimes \mathcal{S} \otimes \mathcal{R} \otimes \mathcal{S} \to \mathcal{R} \otimes \mathcal{R} \otimes \mathcal{S} \otimes \mathcal{S}$, where

$$\tau_1(r \otimes s \otimes r' \otimes s') = r \otimes r' \otimes s \otimes s'.$$

Let $\mu\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $\nu\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ be ring multiplications. Since they are bilinear, by the universal property of tensor product of modules, they induce two $\mathcal{C}$-module homomorphisms $\overline{\mu}$ and $\overline{\nu}$ respectively, such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{R} \times \mathcal{R} & \xrightarrow{\otimes} & \mathcal{R} \otimes \mathcal{R} \\ & \searrow^{\mu} & \downarrow{\overline{\mu}} \\ & & \mathcal{R} \end{array} \qquad \begin{array}{ccc} \mathcal{S} \times \mathcal{S} & \xrightarrow{\otimes} & \mathcal{S} \otimes \mathcal{S} \\ & \searrow^{\nu} & \downarrow{\overline{\nu}} \\ & & \mathcal{S} \end{array}$$

Combining $\overline{\mu} \otimes \overline{\nu}$ with the map τ_1, we obtain a linear map

$$\pi = (\overline{\mu} \otimes \overline{\nu}) \circ \tau_1 \colon \mathcal{R} \otimes \mathcal{S} \otimes \mathcal{R} \otimes \mathcal{S} \to \mathcal{R} \otimes \mathcal{S}.$$

We claim that this multiplication is associative whenever $\overline{\mu}$ and $\overline{\nu}$ are. Put $\mathcal{U} = \mathcal{R} \otimes \mathcal{S}$, then the previous map can be written as $\pi \colon \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ where for any $r, r', r'' \in \mathcal{R}$, $s, s', s'' \in \mathcal{S}$ we have

$$\begin{aligned} \pi \otimes (\pi \otimes \mathrm{id}_{\mathcal{R}\otimes\mathcal{S}})&(r \otimes s \otimes r' \otimes s' \otimes r'' \otimes s'') \\ &= \pi(rr' \otimes ss' \otimes r'' \otimes s'') = (rr')r'' \otimes (ss')s''. \end{aligned}$$

Applying $(\pi \otimes \mathrm{id}_{\mathcal{R}\otimes\mathcal{S}}) \otimes \pi$, we obtain $r(r'r'') \otimes s(s's'')$, which is the same, by associativity in $\mathcal{R}$ and $\mathcal{S}$. Since the elements on the left span $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$, the associativity of π follows. In addition, $1 \otimes 1$ is the identity element of $\mathcal{R} \otimes \mathcal{S}$. In fact, for all $r, s \in \mathcal{R}, \mathcal{S}$ we have

$$(1 \otimes 1)(r \otimes s) = r \otimes s = (r \otimes s)(1 \otimes 1). \qquad \square$$

For the rest of this section, we focus on a special case where $\mathcal{C} = \mathbb{Z}$, $\mathcal{R} = \mathcal{K}$, and $\mathcal{S} = \mathcal{K}^{\mathrm{op}}$, where $\mathcal{K}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{K}$ with the multiplication $k \cdot k' := k'k$ for all $k, k' \in \mathcal{K}$. More precisely, we consider the tensor product of $\mathbb{Z}$-modules

$$\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}} = \{\sum_{i=1}^{p} k_i \otimes k_i' \mid p \in \mathbb{N} \text{ and } k, k' \in \mathcal{K}\},$$

as a ring together with the multiplication

$$(k \otimes k')(l \otimes l') = kl \otimes l'k'.$$

Definition 2.40. *Let M and N be $\mathcal{K}$-bimodules. An additive map $\varphi \colon M \to N$ is called a* $\mathcal{K}$-bimodule homomorphism *if $\varphi(kmk') = k\varphi(m)k'$ for all $m \in M$ and $k, k' \in \mathcal{K}$.*

Theorem 2.41. *Every $\mathcal{K}$-bimodule M can also be viewed as a left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module. Conversely, every left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module M has the structure of a $\mathcal{K}$-bimodule. Moreover, any $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module homomorphism induces a $\mathcal{K}$-bimodule homomorphism and any $\mathcal{K}$-bimodule homomorphism induces a $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module homomorphism.*

Proof. Let M be a $\mathcal{K}$-bimodule. We show that M is a left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module under the scalar multiplication $(k \otimes k')m = kmk'$ where $k, k' \in \mathcal{K}$ and $m \in M$. We first prove the scalar multiplication is well-defined: taking an arbitrary but fixed element $m \in M$, we define a map $\varphi_m \colon \mathcal{K} \times \mathcal{K}^{\mathrm{op}} \to M$ by $\varphi_m(k, k') = kmk'$. The map φ_m is biadditive since

$$\begin{aligned}\varphi_m(k + l, k') = (k + l)mk' &= (km + lm)k' \\ &= kmk' + lmk' = \varphi_m(k, k') + \varphi_m(l, k'),\end{aligned}$$

and similarly $\varphi_m(k, k' + l') = \varphi_m(k, k') + \varphi_m(k, l')$ for all $k, k', l, l' \in \mathcal{K}$. It is also balanced since

$$\varphi_m(kn, k') = (kn)mk' = k(nm)k' = k(mn)k' = km(nk') = \varphi_m(k, nk'),$$

for all $n \in \mathbb{Z}$. Hence, by the universal property of the tensor product of bimodules, there exists a group homomorphism $\overline{\varphi}_m \colon \mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}} \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{K}^{\mathrm{op}} & \xrightarrow{\otimes} & \mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}} \\ & \underset{\varphi_m}{\searrow} & \downarrow \overline{\varphi}_m \\ & & M \end{array}$$

In particular, $\overline{\varphi}_m(k \otimes k') = kmk'$. Hence the scalar multiplication $(k \otimes k')m = kmk'$ is well-defined. We have

$$\begin{aligned}(k \otimes k')(m + n) &= k(m + n)k' = (km + kn)k' \\ &= kmk' + knk' = (k \otimes k')m + (k \otimes k')n, \\ ((k \otimes k')(l \otimes l'))m &= (kl \otimes l'k')m = klml'k' = k(lml')k' \\ &= (k \otimes k')(lml') = (k \otimes k')((l \otimes l')m),\end{aligned}$$

for all $k, k', l, l' \in \mathcal{K}$ and $m, n \in M$. In addition,

$$(k \otimes k' + l \otimes l')m = kmk' + lml' = (k \otimes k')m + (l \otimes l')n$$

and $(1 \otimes 1)m = 1m1 = m$. Hence, M is a left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module. For the reverse implication, let M be a left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module. Considering the left and the right scalar multiplications $km = (k \otimes 1)m$ and $mk' = (1 \otimes k')m$, we will show that M is a left $\mathcal{K}$-module. We have

$$\begin{aligned}k(m + n) = (k \otimes 1)(m + n) &= (k \otimes 1)m + (k \otimes 1)n = km + kn, \\ (k + k')m = ((k + k') \otimes 1)m &= (k \otimes 1 + k' \otimes 1)m \\ &= (k \otimes 1)m + (k' \otimes 1)m = km + k'm,\end{aligned}$$

for all $k, k' \in \mathcal{K}$ and $m, n \in M$. Moreover,

$$(kk')m = (kk' \otimes 1)m = ((k \otimes 1)(k' \otimes 1))m = (k \otimes 1)(k'm) = k(k'm),$$

and $1m = (1 \otimes 1)m = m$. Analogously, M is a right $\mathcal{K}$-module and since

$$(km)k' = ((k \otimes 1)m)k' = (1 \otimes k')(k \otimes 1)m = (k \otimes 1)(1 \otimes k')m = k(mk'),$$

hence the associativity condition holds. Now let $\varphi\colon M \to N$ be a $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\text{op}}$-module homomorphism. Considering M and N as $\mathcal{K}$-bimodules we have

$$\varphi(kmk') = \varphi((k \otimes k')m) = (k \otimes k')\varphi(m) = k\varphi(m)k',$$

for all $k, k' \in \mathcal{K}$ and $m \in M$. Conversely, if $\varphi\colon M \to N$ is a $\mathcal{K}$-bimodule homomorphism, then viewing M, N as left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\text{op}}$-modules implies

$$\varphi((k \otimes k')m) = \varphi(kmk') = k\varphi(m)k' = (k \otimes k')\varphi(m),$$

for all $k, k' \in \mathcal{K}$ and $m \in M$. □

Definition 2.42. *Let $X = \{x_i \mid i \in I\}$ be a set indexed by a set I. The* free $\mathcal{K}$-bimodule *on the set X is defined as the left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{op}$-module (viewed as a $\mathcal{K}$-bimodule) and is denoted by $\mathcal{K}X\mathcal{K}$.*

Consequently, every element of the free $\mathcal{K}$-bimodule $\mathcal{K}X\mathcal{K}$ can be represented by a sum

$$\sum_{i \in I} \sum_{j=0}^{p_i} k_{i,j} x_i k'_{i,j}.$$

However, this representation is not unique. For example, for any $x \in X$ and $k, l, k' \in \mathcal{K}$,

$$(k + l)xk' = kxk' + lxk'$$

are the same element in $\mathcal{K}X\mathcal{K}$.

Theorem 2.43. *(Universal property of free bimodules) Let $X = \{x_i \mid i \in I\}$ be a set indexed by a set I and let $\iota\colon X \to \mathcal{K}X\mathcal{K}$ be the inclusion map. Then for any map $\varphi\colon X \to M$, where M is a $\mathcal{K}$-bimodule, there exists a unique $\mathcal{K}$-bimodule homomorphism $\overline{\varphi}\colon \mathcal{K}X\mathcal{K} \to M$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{K}X\mathcal{K} \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle\overline{\varphi}} \\ & & M \end{array}$$

In particular, $\overline{\varphi}\big(\sum_{i \in I} \sum_{j=0}^{p_i} k_{i,j} x_i k'_{i,j}\big) = \sum_{i \in I} \sum_{j=0}^{p_i} k_{i,j} \varphi(x_i) k'_{i,j}$.

Proof. As a $\mathcal{K}$-bimodule, the free $\mathcal{K}$-bimodule $\mathcal{K}X\mathcal{K}$ is generated by the set of all kx_ik' where $k, k' \in \mathcal{K}$ and $x_i \in X$ for all $i \in I$. Therefore, for a fixed map $\varphi\colon X \to M$, any $\mathcal{K}$-bimodule homomorphism $\psi\colon \mathcal{K}X\mathcal{K} \to M$ is equal to $\overline{\varphi}$ on the generating set X and hence they are equal on $\mathcal{K}X\mathcal{K}$. This means that $\overline{\varphi}$ is uniquely determined if it exists.

By Theorem 2.41, the $\mathcal{K}$-bimodule M can be viewed as a left $\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}}$-module. In addition, by Definition 2.42 and using the universal property of free modules, there exists a unique $(\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}})$-module homomorphism $\tilde{\varphi}\colon (\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}})X \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & (\mathcal{K} \otimes_{\mathbb{Z}} \mathcal{K}^{\mathrm{op}})X \\ & \underset{\varphi}{\searrow} & \downarrow \tilde{\varphi} \\ & & M \end{array}$$

In particular, $\tilde{\varphi}\big(\sum_{i\in I}\sum_{j=0}^{p_i}(k_{i,j} \otimes k'_{i,j})x_i\big) = \sum_{i\in I}\sum_{j=0}^{p_i}(k_{i,j} \otimes k'_{i,j})\varphi(x_i)$. By Theorem 2.41, the map $\tilde{\varphi}$ induces a $\mathcal{K}$-bimodule homomorphism. □

2.3 Tensor rings and differential rings

In this section, we explain two important algebraic structures which we use frequently in the following chapters.

2.3.1 Tensor rings

A *tensor ring* over an arbitrary ring is a generalization of the tensor algebra over a commutative ring. We start this subsection by explaining the notion of a *$\mathcal{K}$-ring* and its basic properties.

Definition 2.44. *A ring $\mathcal{R}$ which is also a $\mathcal{K}$-bimodule, is called a* $\mathcal{K}$-ring *if the associativity property*

$$(xy)z = x(yz)$$

holds for all x, y, z in $\mathcal{R}$ or $\mathcal{K}$ where two of them are in $\mathcal{R}$ and one is in $\mathcal{K}$.

Remark 2.45. *The notion of a $\mathcal{K}$-ring is more general than the notion of $\mathcal{K}$-algebra, since the action of $\mathcal{K}$ does not necessarily centralize the ring even if $\mathcal{K}$ is commutative, i.e., we do not require $kr = rk$ for any $k \in \mathcal{K}$ and $r \in \mathcal{R}$. Therefore, we can describe the difference by saying that whereas a $\mathcal{K}$-algebra ($\mathcal{K}$ commutative) is a ring $\mathcal{R}$ with a homomorphism from $\mathcal{K}$ to the center of $\mathcal{R}$, a $\mathcal{K}$-ring is a ring $\mathcal{R}$ with a homomorphism $\rho\colon \mathcal{K} \to \mathcal{R}$.*

Lemma 2.46. *Given a ring $\mathcal{R}$ and a homomorphism $\rho\colon \mathcal{K} \to \mathcal{R}$, we can always make $\mathcal{R}$ a $\mathcal{K}$-bimodule by defining the scalar multiplications*

$$k \cdot r = \rho(k)r \quad \textit{and} \quad r \cdot k = r\rho(k),$$

where the associativity property holds for all elements of $\mathcal{R}$ or $\mathcal{K}$. Conversely, given a ring $\mathcal{R}$ which is also a $\mathcal{K}$-bimodule and satisfies the associativity property, we can define a ring homomorphism $\rho\colon \mathcal{K} \to \mathcal{R}$ by $\rho(k) = k \cdot 1$.

Proof. Let $\mathcal{R}$ be a ring and let $\rho\colon \mathcal{K} \to \mathcal{R}$ be a ring homomorphism. We consider the scalar multiplications defined above and show that $\mathcal{R}$ is a $\mathcal{K}$-bimodule which satisfies the associativity property. Verifying that $\mathcal{R}$ is a left and right $\mathcal{K}$-module is straightforward. Then it becomes a $\mathcal{K}$-bimodule since

$$\begin{aligned}(k \cdot r) \cdot k' = (\rho(k)r) \cdot k' &= (\rho(k)r)\rho(k') \\ &= \rho(k)(r\rho(k')) = \rho(k)(r \cdot k') = k \cdot (r \cdot k')\end{aligned}$$

for all $k, k' \in \mathcal{K}$ and $r \in \mathcal{R}$. Moreover, we have

$$k \cdot (rr') = \rho(k)(rr') = (\rho(k)r)r' = (k \cdot r)r',$$

$$(rr') \cdot k = (rr')\rho(k) = r(r'\rho(k)) = r(r' \cdot k),$$

$$(r \cdot k)r' = r\rho(k)r' = r(\rho(k)r') = r(k \cdot r')$$

for any $r, r' \in \mathcal{R}$ and $k, k' \in \mathcal{K}$ and hence the associative property holds. Conversely, let $\mathcal{R}$ be $\mathcal{K}$-bimodule together with the scalar multiplications $\cdot\colon \mathcal{K} \times R \to R$ and $\cdot\colon R \times \mathcal{K} \to \mathcal{R}$. Then the map $\rho(k) = k \cdot 1$ is a ring homomorphism, since $\rho(1_{\mathcal{K}}) = 1_{\mathcal{K}} \cdot 1 = 1$,

$$\rho(k + k') = (k + k') \cdot 1 = k \cdot 1 + k' \cdot 1 = \rho(k) + \rho(k'),$$

$$\begin{aligned}\rho(k)\rho(k') = (k \cdot 1)\rho(k') = k \cdot (1\rho(k')) &= k \cdot \rho(k') \\ &= k \cdot (k' \cdot 1) = (kk') \cdot 1 = \rho(kk')\end{aligned}$$

for all $k, k' \in \mathcal{K}$. □

Definition 2.47. *Let $\mathcal{R}$ and $\mathcal{S}$ be two $\mathcal{K}$-rings. We call a ring homomorphism $\psi\colon \mathcal{R} \to \mathcal{S}$ a* $\mathcal{K}$-ring homomorphism *if it is also a $\mathcal{K}$-bimodule homomorphism.*

Lemma 2.48. *Let $\mathcal{R}$ and $\mathcal{S}$ be two $\mathcal{K}$-rings. Then a ring homomorphism $\psi\colon \mathcal{R} \to \mathcal{S}$ is a $\mathcal{K}$-ring homomorphism if and only if the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\rho} & \mathcal{R} \\ & {}_{\rho'}\searrow & \downarrow{\scriptstyle\psi} \\ & & \mathcal{S} \end{array}$$

Proof. If the diagram is commutative, then we have

$$\begin{aligned}\psi(k\cdot r\cdot k') = \psi(\rho(k)r\rho(k')) &= \psi(\rho(k))\psi(r)\psi(\rho(k'))\\ &= \rho'(k)\psi(r)\rho'(k') = k\cdot\psi(r)\cdot k',\end{aligned}$$

for all $k, k' \in \mathcal{K}$ and $r \in \mathcal{R}$. Conversely, if ψ is a $\mathcal{K}$-bimodule homomorphism then

$$\psi(\rho(k)) = \psi(\rho(k)1) = \psi(k\cdot 1) = k\cdot\psi(1) = k\cdot 1 = \rho'(k),$$

for all $k \in \mathcal{K}$. Hence the diagram commutes. □

Definition 2.49. *A $\mathcal{K}$-*subring *of a $\mathcal{K}$-ring $\mathcal{R}$ is a subring $\mathcal{S}$ of $\mathcal{R}$ that is a $\mathcal{K}$-bimodule as well.*

Definition 2.50. *A* left ideal *of a $\mathcal{K}$-ring $\mathcal{R}$ is a left $\mathcal{R}$-submodule. Similarly we define a* right ideal *as a right $\mathcal{R}$-submodule. A* two-sided ideal *of $\mathcal{R}$ is a left ideal which is a right ideal as well.*

Remark 2.51. *Any ideal I of a $\mathcal{K}$-ring $\mathcal{R}$ is also a $\mathcal{K}$-subbimodule of $\mathcal{R}$: the ring $\mathcal{R}$ is a $\mathcal{K}$-bimodule with scalar multiplications*

$$k\cdot r = \rho(k)r \quad \textit{and} \quad r\cdot k = r\rho(k).$$

Since the ideal I is closed under the scalar multiplication in $\mathcal{R}$, we conclude that I with the scalar multiplications defined above is a $\mathcal{K}$-subbimodule of $\mathcal{R}$.

Lemma 2.52. *Let $\mathcal{R}$ be a $\mathcal{K}$-ring and I be a* two-sided ideal *of $\mathcal{R}$. Then the quotient module*

$$\mathcal{R}/I = \{r + I \mid r \in \mathcal{R}\},$$

together with the following multiplication

$$(r_1 + I)(r_2 + I) = (r_1 r_2 + I) \quad (\textit{for all } r_1, r_2 \in \mathcal{R}),$$

is a $\mathcal{K}$-ring which is called the quotient *ring $\mathcal{R}/I$.*

Proof. Since I is an ideal of the $\mathcal{K}$-ring $\mathcal{R}$, by Remark 2.51 the ring $\mathcal{R}/I$ is a $\mathcal{K}$-bimodule as well. Moreover, one can easily check that the associativity property $(xy)z = x(yz)$ holds whenever two of them are in $\mathcal{R}/I$ and one is in $\mathcal{K}$. For instance,

$$\begin{aligned}((r_1 + I)(r_2 + I))k = (r_1 r_2 + I)k &= r_1 r_2 k + I\\ &= (r_1 + I)(r_2 k + I) = (r_1 + I)((r_2 + I)k).\end{aligned}$$

Hence, by Definition 2.44 it is a $\mathcal{K}$-ring. □

For the definition of the tensor ring, we need to consider the tensor product of finitely many bimodules.

Definition 2.53. *Let $M_1, \dots, M_n$ be $\mathcal{K}$-bimodules. Given an abelian group A, a map $\varphi\colon M_1 \times \cdots \times M_n \to A$ is called* balanced *if it is multiadditive and satisfies the property*

$$\varphi(m_1, \dots, m_i k, m_{i+1}, \dots, m_n) = \varphi(m_1, \dots, m_i, k m_{i+1}, \dots, m_n)$$

for all $k \in \mathcal{K}$, $m_j \in M_j$, where $i = 1, \dots, n-1$ and $j = 1, \dots, n$.

Theorem 2.54. *(Universal property of the tensor product of bimodules) Let $M_1, \dots, M_n$, where $n \geq 1$, be $\mathcal{K}$-bimodules and let $(M_1 \otimes \cdots \otimes M_n, \otimes)$ be the tensor product of $M_1, \dots, M_n$ together with the balanced map $\otimes$. Then for any balanced map $\varphi\colon M_1 \times \cdots \times M_n \to L$, where L is an abelian group, there exists a unique group homomorphism $\overline{\varphi}\colon M_1 \otimes \cdots \otimes M_n \to L$ such that the following diagram is commutative:*

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{\otimes} & M_1 \otimes \cdots \otimes M_n \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \overline{\varphi}} \\ & & L \end{array}$$

In particular, $\overline{\varphi}$ satisfies $\overline{\varphi}(\sum_{i=1}^{p} m_{i,1} \otimes \cdots \otimes m_{i,n}) = \sum_{i=1}^{p} \varphi(m_{i,1}, \dots, m_{i,n})$.

Proof. Similar to the proof of Theorem 2.37. □

As a $\mathcal{K}$-bimodule, $M_1 \otimes \cdots \otimes M_n$ is generated by the following set

$$\{m_1 \otimes \cdots \otimes m_n \mid m_i \in M_i \text{ for } i = 1, \dots, n\},$$

and each element of this generating set is called a *pure tensor*. Every element $t \in M_1 \otimes \cdots \otimes M_n$ can be written as a finite sum of pure tensors, i.e,

$$t = \sum_{i=1}^{p} m_{i,1} \otimes \cdots \otimes m_{i,n}.$$

Note that if $M_1, \dots, M_n$ are $\mathcal{K}$-bimodules, then $M_1 \otimes \cdots \otimes M_n$ is again a $\mathcal{K}$-bimodule with scalar multiplications

$$k(m_1 \otimes \cdots \otimes m_n) = k m_1 \otimes \cdots \otimes m_n \quad \text{and} \quad (m_1 \otimes \cdots \otimes m_n) k = m_1 \otimes \cdots \otimes m_n k.$$

We denote the n-fold tensor product of a $\mathcal{K}$-module M with itself over $\mathcal{K}$ by $M^{\otimes n}$ (n factors). In addition, we define $M^{\otimes 1} = M$ and $M^{\otimes 0}$ as the free $\mathcal{K}$-module $\mathcal{K}\epsilon$, where ϵ denotes the *empty tensor*.

Definition 2.55. *The* $\mathcal{K}$-tensor ring *on a* $\mathcal{K}$*-bimodule* M*, is the* $\mathcal{K}$*-bimodule*

$$\mathcal{K}\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

together with the ring multiplication $M^{\otimes p} \times M^{\otimes q} \to M^{\otimes(p+q)}$ *defined by*

$$(m_1 \otimes \cdots \otimes m_p, \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_q) \mapsto m_1 \otimes \cdots \otimes m_p \otimes \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_q$$

which can be extended to $\mathcal{K}\langle M\rangle$ *by biadditivity. With this multiplication,* $\mathcal{K}\langle M\rangle$ *is a* $\mathcal{K}$*-ring with* ϵ *being its identity element. Every element* $t \in \mathcal{K}\langle M\rangle$ *is unique of the form* $t = \sum_{n\in\mathbb{N}} t_n$*.*

Theorem 2.56. *(Universal property of tensor rings). Let* M *be a* $\mathcal{K}$*-bimodule and let* $\mathcal{R}$ *be a* $\mathcal{K}$*-ring. Let* $\mathcal{K}\langle M\rangle$ *be the* $\mathcal{K}$*-tensor ring on* M *and* $\iota\colon M \hookrightarrow \mathcal{K}\langle M\rangle$ *be the inclusion map. Then for every* $\mathcal{K}$*-bimodule homomorphism* $\varphi\colon M \to \mathcal{R}$*, there exists a unique* $\mathcal{K}$*-ring homomorphism* $\overline{\varphi}\colon \mathcal{K}\langle M\rangle \to \mathcal{R}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & \mathcal{K}\langle M\rangle \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\overline{\varphi}} \\ & & \mathcal{R} \end{array}$$

In particular, $\overline{\varphi}$ *satisfies*

$$\overline{\varphi}\big(k\epsilon + \sum_{i=1}^{p} m_{i,1} \otimes \cdots \otimes m_{i,n_i}\big) = k1 + \sum_{i=1}^{p} \varphi(m_{i,1}) \cdots \varphi(m_{i,n_i}).$$

Proof. Since the $\mathcal{K}$-ring $\mathcal{K}\langle M\rangle$ is generated by the set M, for a $\mathcal{K}$-bimodule homomorphism $\varphi\colon M \to \mathcal{R}$, the $\mathcal{K}$-ring homomorphism $\overline{\varphi}$ is uniquely determined if it exists. For each $n \geq 1$, the map φ_n from $M \times \cdots \times M$ (n times) to $\mathcal{R}$ defined by $\varphi_n(m_1, \ldots, m_n) = \varphi(m_1) \cdots \varphi(m_n)$ is balanced since

$$\begin{aligned} &\varphi_n(m_1, \ldots, m_i + m_i'k, m_{i+1}, \ldots, m_n) = \varphi(m_1) \cdots \varphi(m_i + m_i'k) \cdots \varphi(m_n) \\ &= \varphi(m_1) \cdots \varphi(m_i)\varphi(m_{i+1}) \cdots \varphi(m_n) + \varphi(m_1) \cdots \varphi(m_i')\varphi(km_{i+1}) \cdots \varphi(m_n) \\ &= \varphi_n(m_1, \ldots, m_i, m_{i+1}, \ldots, m_n) + \varphi_n(m_1, \ldots, m_i', km_{i+1}, \ldots, m_n), \end{aligned}$$

for all $m_j, m_j' \in M$, where $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, and $k \in \mathcal{K}$. Hence, by the universal property of the tensor product of bimodules, the map φ_n

induces a unique $\mathcal{K}$-bimodule homomorphism $\tilde{\varphi}_n$ from $M \otimes \cdots \otimes M$ (n times) to $\mathcal{R}$ such that the following diagram is commutative:

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{\otimes} & M \otimes \cdots \otimes M \\ & \searrow^{\varphi_n} & \downarrow \tilde{\varphi}_n \\ & & \mathcal{R} \end{array}$$

In addition, the $\mathcal{K}$-bimodule homomorphism $\tilde{\varphi}_0 \colon \mathcal{K}\epsilon \to \mathcal{R}$ is defined by $\tilde{\varphi}_0(\epsilon) = 1$. On putting the maps $\tilde{\varphi}_n$ together, by the universal property of the direct sum of bimodules, we obtain a unique $\mathcal{K}$-bimodule homomorphism $\overline{\varphi} \colon \mathcal{K}\langle M \rangle \to \mathcal{R}$. Finally, since $\overline{\varphi}$ is additive and

$$\begin{aligned} \overline{\varphi}(m\tilde{m}) &= \overline{\varphi}(m_1 \otimes \cdots \otimes m_r \otimes \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_s) \\ &= \varphi(m_1) \cdots \varphi(m_r)\varphi(\tilde{m}_1) \cdots \varphi(\tilde{m}_s) \\ &= \overline{\varphi}(m_1 \otimes \cdots \otimes m_r)\overline{\varphi}(\tilde{m}_1 \otimes \cdots \otimes \tilde{m}_s) = \overline{\varphi}(m)\overline{\varphi}(\tilde{m}), \end{aligned}$$

for all $m = m_1 \otimes \cdots \otimes m_r \in M^{\otimes r}$, $\tilde{m} = \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_s \in M^{\otimes s}$, it is a $\mathcal{K}$-ring homomorphism. $\square$

Definition 2.57. *We define the* free $\mathcal{K}$-ring $\mathcal{K}\langle X \rangle$ *on a set X as the $\mathcal{K}$-tensor ring $\mathcal{K}\langle M \rangle$ over the free $\mathcal{K}$-bimodule $\mathcal{K}X\mathcal{K}$ on the set X.*

Remark 2.58. *Note that as a $\mathcal{K}$-bimodule, the free $\mathcal{K}$-ring $\mathcal{K}\langle X \rangle$ is generated by the set*

$$Y = \{x_1 \otimes k_2 x_2 \otimes \cdots \otimes k_n x_n \mid n \in \mathbb{N}, k_i \in \mathcal{K}, \text{ and } x_j \in X \text{ for all } 2 \leq i \leq n \text{ and } 1 \leq j \leq n\}.$$

Theorem 2.59. *(Universal property of free $\mathcal{K}$-rings) Let $\mathcal{K}\langle X \rangle$ be the free $\mathcal{K}$-ring on a set $X = \{x_i \mid i \in I\}$, indexed by a set I, and let $\mathcal{R}$ be a $\mathcal{K}$-ring. Then for each map $\varphi \colon X \to \mathcal{R}$, there exists a unique $\mathcal{K}$-ring homomorphism $\overline{\varphi} \colon \mathcal{K}\langle X \rangle \to \mathcal{R}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{K}\langle X \rangle \\ & \searrow^{\varphi} & \downarrow \overline{\varphi} \\ & & \mathcal{R} \end{array}$$

Proof. Let $\iota_1 \colon X \hookrightarrow \mathcal{K}X\mathcal{K}$ be the canonical embedding. By the universal property of the free bimodules, for any map $\varphi \colon X \to \mathcal{R}$, there exists a unique $\mathcal{K}$-bimodule homomorphism $\tilde{\varphi} \colon \mathcal{K}X\mathcal{K} \to \mathcal{R}$ such that the left side of the following diagram is commutative. Let $\iota_2 \colon \mathcal{K}X\mathcal{K} \hookrightarrow \mathcal{K}\langle X \rangle$ be the canonical embedding and let $\mathcal{K}\langle X \rangle = \mathcal{K}\langle M \rangle$ be the free $\mathcal{K}$-ring on the set X. By the universal property of the tensor rings, the map $\tilde{\varphi}$ induces a unique $\mathcal{K}$-ring homomorphism $\overline{\varphi} \colon \mathcal{K}\langle X \rangle \to \mathcal{R}$ such that the right side of the diagram

$$\begin{array}{ccccc} X & \overset{\iota_1}{\hookrightarrow} & \mathcal{K}X\mathcal{K} & \overset{\iota_2}{\hookrightarrow} & \mathcal{K}\langle X\rangle \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} & \swarrow^{\overline{\varphi}} & \\ & & \mathcal{R} & & \end{array}$$

is commutative and this completes the proof. □

2.3.2 Differential rings

In the following we recall the definition of differential rings which is used in Chapter 4 to define integro-differential rings and integro-differential rings with linear substitutions.

Definition 2.60. *Let $\mathcal{R}$ be a ring and let $\partial\colon \mathcal{R} \to \mathcal{R}$ be an additive map satisfying the* Leibniz *rule*

$$\partial rs = (\partial r)s + r\partial s$$

for all $r, s \in \mathcal{R}$. Then, $(\mathcal{R}, \partial)$ is called a differential ring *and its* ring of constants *is given by*

$$\{c \in \mathcal{R} \mid \partial c = 0\}.$$

Note that in a differential ring $(\mathcal{R}, \partial)$ the ring $\mathcal{R}$ and the ring of constants $\mathcal{K}$ are not necessarily commutative rings. The ring $\mathcal{R}$ has both a left and a right $\mathcal{K}$-module structure which satisfy

$$(c_1 r)c_2 = c_1(rc_2)$$

for all $c_1, c_2 \in \mathcal{K}$ and for all $r \in \mathcal{R}$. Hence, it is a $\mathcal{K}$-bimodule. We also have

$$\partial cr = (\partial c)r + c\partial r = c\partial r \quad \text{and} \quad \partial rc = (\partial r)c + r\partial c = (\partial r)c,$$

for all $c \in \mathcal{K}$ and for all $r \in \mathcal{R}$. This means ∂ is both left and right linear over the ring of constants $\mathcal{K}$ and thus ∂ is a $\mathcal{K}$-bimodule endomorphism. The following example shows for a given commutative differential ring, it is always possible to construct a differential ring of matrices with entries in this ring.

Example 2.61. *Let $(\mathcal{S}, \partial)$ be a commutative differential ring and let*

$$\mathcal{R} = M_n(\mathcal{S})$$

be the ring of $n \times n$ matrices over the ring $\mathcal{S}$. We define a map $\partial\colon \mathcal{R} \to \mathcal{R}$ as follows: for any $A = (a_{ij}) \in \mathcal{R}$, where $i, j = 1, \ldots, n$, we have $\partial A = (\partial a_{ij})$.

The map ∂ satisfies the Leibniz rule: let $A = (a_{ij})$ and $B = (b_{ij})$ be elements of $\mathcal{R}$. If $AB = (c_{ij})$ then $c_{ij} = \sum\limits_{k=1}^{n} a_{ik}b_{kj}$ and

$$\partial c_{ij} = \sum_{k=1}^{n} \partial a_{ik}b_{kj} = \sum_{k=1}^{n} \big((\partial a_{ik})b_{kj} + a_{ik}\partial b_{kj}\big) = \sum_{k=1}^{n}(\partial a_{ik})b_{kj} + \sum_{k=1}^{n} a_{ik}\partial b_{kj}$$

for $i, j = 1, \ldots, n$. This implies that $\partial AB = (\partial A)B + A\partial B$ and thus $(\mathcal{R}, \partial)$ is a differential ring with ring of constants given by matrices with constant entries, i.e.,

$$\{C = (c_{ij}) \in \mathcal{R} \mid \partial c_{ij} = 0\}.$$

2.4 Basics of term rewriting

Term rewriting is a simple computational method which is based on the repeated application of simplification rules. It is particularly suited for tasks such as symbolic computation, program analysis and program transformation. Exploiting term rewriting helps to solve these tasks in a very effective manner. Intuitively, a reduction is analogous to any step by step activity and mathematically this means we are simply talking about binary relations.

Definition 2.62. *An* abstract reduction system *is a pair $(A, \rightarrow)$, where the reduction $\rightarrow$ is a binary relation on the set A, i.e. $\rightarrow \subseteq A \times A$. Instead of $(a, b) \in \rightarrow$ we write $a \rightarrow b$.*

The term "reduction" has been chosen due to the fact that in many applications something decreases with each reduction step, but cannot decrease forever. Recall that for the sets A, B, and C and the relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the *composition* of the relations is defined by

$$R \circ S = \{(x, z) \in A \times C \mid \exists\, y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Based on this definition, we introduce some symbols and their notions.

Definition 2.63. *Composing a reduction with itself, we define the following notions:*

(i)	$\stackrel{0}{\to}$	$:= \{(x,x) \mid x \in A\}$	identity
(ii)	$\stackrel{i+1}{\longrightarrow}$	$:= \stackrel{i}{\to} \circ \to$	$(i+1)$-fold composition, $i \geq 0$
(iii)	$\stackrel{+}{\to}$	$:= \bigcup_{i>0} \stackrel{i}{\to}$	transitive closure
(iv)	$\stackrel{*}{\to}$	$:= \stackrel{+}{\to} \cup \stackrel{0}{\to}$	reflexive transitive closure
(v)	$\stackrel{=}{\to}$	$:= \to \cup \stackrel{0}{\to}$	reflexive closure
(vi)	$\leftarrow$	$:= \{(y,x) \mid x \to y\}$	inverse
(vii)	$\leftrightarrow$	$:= \to \cup \leftarrow$	symmetric closure
$(viii)$	$\stackrel{*}{\leftrightarrow}$	$:= (\leftrightarrow)^*$	reflexive transitive symmetric closure

Remark 2.64. *Notations like $\stackrel{*}{\to}$ and $\leftarrow$ only work for arrow-like symbols. We also write $x \stackrel{*}{\to} y$ if there exists some finite path from x to y.*

Note that the P *closure* of a relation R is the least set with property P which contains R. For example, $\stackrel{*}{\to}$, the reflexive transitive closure of $\to$, is the least reflexive and transitive relation which contains $\to$. It is easy to show that $\stackrel{*}{\to}$ is the least equivalence relation containing $\to$. We also denote by $\stackrel{*}{\leftrightarrow}$ the reflexive transitive symmetric closure of $\to$. The reader may notice that for arbitrary P and R, the P closure of R need not exist, but in the above cases they always do because reflexivity, transitivity and symmetry are closed under arbitrary intersections. In such cases the P closure of R can be defined directly as the intersection of all sets with property P which contain R. In the following we add some terminology to this notation.

1. x is *reducible* iff there is a y such that $x \to y$.
2. x is *in normal form* (irreducible) iff it is not reducible.
3. y is *a normal form* of x iff $x \stackrel{*}{\to} y$ and y is in normal form. If x has a uniquely determined normal form, the latter is denoted by $\downarrow$.
4. x and y are *joinable* iff there is a z such that $x \stackrel{*}{\to} z \stackrel{*}{\leftarrow} y$, in which case we write $x \downarrow y$.

Example 2.65. *Let $A := \mathbb{N} \setminus \{0,1\}$ and $\to := \{(m,n) \mid m > n$ and $n \mid m\}$. Then*

(i) m is in normal form iff m is prime.

(ii) p is a normal form of m iff p is a prime factor of m.

(iii) $m \downarrow n$ iff m and n are not relatively prime.

(iv) $\stackrel{+}{\to} = \to$ because $>$ and "divides" are already transitive.

(v) $\overset{*}{\leftrightarrow} = A \times A$.

Example 2.66. *Let* $A := \langle\{a, b\}\rangle$ *(the set of words over the alphabet* $\{a, b\}$*) and* $\rightarrow := \{(ubav, uabv) \mid u, v \in A\}$*. Then*

(i) *w is in normal form iff w is sorted, i.e. of the form* a^*b^**.*

(ii) *Every w has a unique normal form* $w\downarrow$*, the result of sorting w.*

(iii) $w_1 \downarrow w_2$ *iff* $w_1 \overset{*}{\leftrightarrow} w_2$ *iff* w_1 *and* w_2 *contain the same number of as and bs.*

As an important application of rewriting systems one can point to the *word problem* for sets of identities which can be described as follows: let E be a set of identities and let $x, y \in A$. Is it possible to transform x into y, using the identities in E as rewrite rules that can be applied in both directions?

In order to answer this question, one possibility is to consider the identities as uni-directional rewrite rules. Two elements x and y are called equivalent if they can be transformed into each other by applying identities in both directions. For checking the equivalence between x and y, we reduce x to a normal form x_1 and y to a normal form y_1. Then we check whether x_1 and y_1 are syntactically equal.

While exploiting this method for the word problem above, two problems arise that must be considered. First is that equivalent terms can have distinct normal forms and second is normal forms need not exist since the process of reducing a term may lead to an infinite chain of rule applications. In order to ensure existence and uniqueness of normal forms we will introduce the notions of termination and confluence as important properties for a reduction system. In the following definition we list some of the most important properties of a reduction system.

Definition 2.67. *A reduction* $\rightarrow$ *is called*

Church-Rosser	*iff*	$x \overset{*}{\leftrightarrow} y$ *implies* $x \downarrow y$ (see Figure 2.1).
confluent	*iff*	$y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2$ *implies* $y_1 \downarrow y_2$ (see Figure 2.1).
terminating	*iff*	*there is no infinite descending chain* $a_0 \rightarrow a_1 \rightarrow \cdots$
normalizing	*iff*	*every element has a normal form.*
convergent	*iff*	*it is both confluent and terminating.*

Remark 2.68. *Each diagram in Figure 2.1 has an exact meaning. Solid arrows represent universal and dashed arrows existential quantification; The whole diagram is an implication of the form* $\forall \overline{x}\colon P(\overline{x}) \Rightarrow \exists \overline{y}\colon Q(\overline{x}, \overline{y})$*. For instance, the confluence diagram becomes*

$$\forall x, y_1, y_2 . y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2 \Rightarrow \exists z\colon y_1 \overset{*}{\rightarrow} z \overset{*}{\leftarrow} y_2.$$

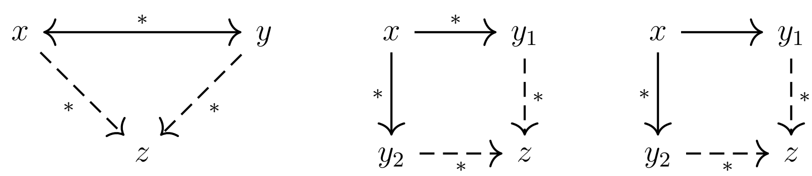

Figure 2.1: Church-Rosser property, confluence and semi-confluence.

Note that the given reductions in Examples 2.65 and 2.66 are terminating, but only the second one is Church-Rosser and confluent. For proving equivalence between the Church-Rosser and confluent properties we need to define an intermediate property:

Definition 2.69. *A relation* $\to$ *is* semi-confluent (see Figure 2.1) *iff*

$$y_1 \leftarrow x \overset{*}{\to} y_2 \Rightarrow y_1 \downarrow y_2.$$

One may think that semi-confluence is weaker than confluence, but in fact they are equivalent:

Theorem 2.70. *The following conditions are equivalent:*

(*i*) $\to$ *has the Church-Rosser property.*

(*ii*) $\to$ *is confluent.*

(*iii*) $\to$ *is semi-confluent.*

In addition, this theorem has the following consequences.

Corollary 2.71. *If* $\to$ *is confluent and* $x \overset{*}{\leftrightarrow} y$ *then*

(*i*) $x \to y$ *if* y *is in normal form, and*

(*ii*) $x = y$ *if both* x *and* y *are in normal form.*

Therefore, for confluent relations, two elements are equivalent iff they are joinable. Of course the test for joinability can be a difficult task (and even undecidable) if the relation does not terminate. Given two elements which are not joinable, when should we stop the search for a common successor in case of an infinite reduction starting from one of the two elements, as in the following example?

$$a_0 \to a_1 \to a_2 \to \dots,$$
$$b_0 \to b_1 \to b_2 \to \dots.$$

It turns out for determining joinability we only need to check normalization. For understanding this better, let us investigate the relationship between termination, normalization, confluence, and the uniqueness of normal forms.

Remark 2.72. *If $\to$ is confluent then every element has at most one normal form. Since every element has at least one normal form if $\to$ is normalizing, it follows that for confluent and normalizing relations every element x has exactly one normal form which we write $x\downarrow$:*

Lemma 2.73. *If $\to$ is normalizing and confluent, every element has a unique normal form.*

Having established under what conditions the notation $x\downarrow$ is well-defined, we can conclude the following theorem:

Theorem 2.74. *If $\to$ is normalizing and confluent then $x \overset{*}{\leftrightarrow} y \Leftrightarrow x\downarrow = y\downarrow$.*

Proof. The $\Leftarrow$ direction is trivial. Conversely, if $x \overset{*}{\leftrightarrow} y$ then $x\downarrow \overset{*}{\leftrightarrow} y\downarrow$ and hence $x\downarrow = y\downarrow$ by Corollary 2.71. □

Hence we have eventually arrived at a very goal-directed equivalence test: simply check if the normal forms of both elements are identical. If normal forms are computable and identity is decidable then we may conclude that equivalence becomes decidable as well. Proving confluence can be a difficult task since one has to consider forks $y_1 \overset{*}{\leftarrow} x \overset{*}{\to} y_2$ of arbitrary length. Let us look at different ways of localizing the confluence test to single-step forks $y_1 \leftarrow x \to y_2$.

Definition 2.75. *A relation $\to$ is* locally confluent (see Figure 2.2) *iff*

$$y_1 \leftarrow x \to y_2 \Rightarrow y_1 \downarrow y_2.$$

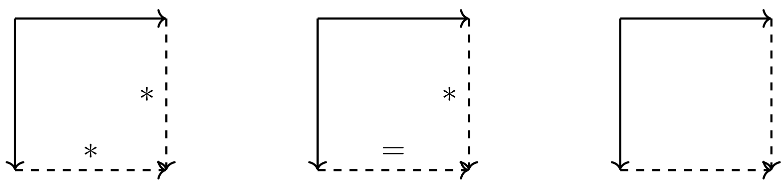

Figure 2.2: Local confluence, strong confluence, and the diamond property.

Local confluence is strictly weaker than confluence. As an easy example we look at Figure 2.3: although both local forks $a \leftarrow 0 \to 1$ and $0 \leftarrow 1 \to b$ can be closed, but the reduction is not confluent. Still one might think that the cycle between 0 and 1 makes it confluent, but the second example in Figure 2.3 (only an initial segment of the infinite graph generated by $2n \to a, 2n+1 \to b$ and $n \to n+1$, is shown) proves that this is not the case.

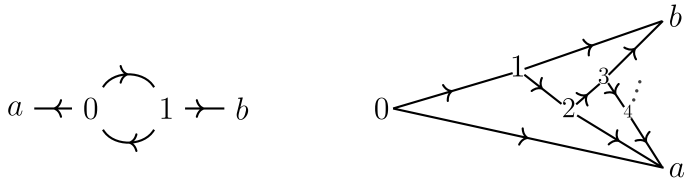

Figure 2.3: Local confluence does not imply confluence.

Even for acyclic relations (i.e. there is no element a such that $a \xrightarrow{+} a$), local confluence does not imply confluence. Both example are nonterminating. This is a consequence of ***Newman's Lemma***:

Theorem 2.76. *A terminating relation is confluent if it is locally confluent.*

Termination enables us to check confluence through local confluence in a simple way. It is still possible to localize the confluence test when relations are nonterminating if we restrict the notion of closedness for forks.

Definition 2.77. *A relation* $\to$ *is* strongly confluent (see Figure 2.2) *iff*

$$y_1 \leftarrow x \to y_2 \Rightarrow \exists z \colon y_1 \xrightarrow{*} z \xleftarrow{=} y_2.$$

One should be aware of the symmetry in this definition: $y_1 \leftarrow x \to y_2$ must imply both $y_1 \xrightarrow{*} z_1 \xleftarrow{=} y_2$ and $y_1 \xrightarrow{*} z_2 \xleftarrow{=} y_2$ for suitable z_1 and z_2. Hence, neither of the relations in Figure 2.3 are strongly confluent.

Lemma 2.78. *Any strongly confluent relation is confluent.*

In order to benefit from this strong property, we do not apply Lemma 2.78 directly to the real object of interest $\to$. Rather we define a strongly confluent relation $\to_s$ such that $\xrightarrow{*} = \xrightarrow{*}_s$. Now Lemma 2.78 yields confluence of $\to_s$ which carries over to $\to$ using the following observation:

Remark 2.79. *If* $\xrightarrow{*}_1 = \xrightarrow{*}_2$ *then* $\to_1$ *is confluent iff* $\to_2$ *is confluent.*

The following lemma facilitates the application of this fact:

Lemma 2.80. *If* $\to_1 \subseteq \to_2 \subseteq \xrightarrow{*}_1$ *then* $\xrightarrow{*}_1 = \xrightarrow{*}_2$.

Proof. Since the reflexive transitive closure is a monotone and idempotent operation, $\to_1 \subseteq \to_2 \subseteq \xrightarrow{*}_1$ implies $\xrightarrow{*}_1 \subseteq \xrightarrow{*}_2 \subseteq (\xrightarrow{*}_1)^* = \xrightarrow{*}_1$ and thus $\xrightarrow{*}_1 = \xrightarrow{*}_2$. □

Putting Lemma 2.78, Remark 2.79, and Lemma 2.80 together we obtain

Corollary 2.81. *If* $\rightarrow_1 \subseteq \rightarrow_2 \subseteq \overset{*}{\rightarrow}_1$ *and* $\rightarrow_2$ *is strongly confluent, then* $\rightarrow_1$ *is confluent.*

In practice we are able to work with a yet stronger property:

Definition 2.82. *A relation* $\rightarrow$ *has the* diamond property (see Figure 2.2) *if and only if*

$$y_1 \leftarrow x \rightarrow y_2 \Rightarrow \exists z \colon y_1 \rightarrow z \leftarrow y_2.$$

The diamond property implies trivially strong confluence. Hence Corollary 2.81 also holds if $\rightarrow_2$ has the diamond property. Moreover, the reduction system $\rightarrow$ is confluent iff $\overset{*}{\rightarrow}$ has the diamond property.

2.5 Linear functional systems

In this section, we briefly recall characterization of the transformations, which map solutions of a linear functional system (e.g., differential, time-delay, difference, ...) to solutions of another one. This characterization relies on the so-called *algebraic analysis approach* which provides a unified mathematical framework for studying linear systems of functional equations, by methods of module theory, homological algebra and sheaf theory. For more details, see [45] and the references therein. Within this approach, we define a rectangular system of q linear functional equations in p unknown functions by means of a $q \times p$ matrix with entries in a noncommutative ring $\mathcal{D}$ of functional operators. If $\mathcal{F}$ is a left $\mathcal{D}$-module, e.g., a functional space which is closed under the left action of $\mathcal{D}$, then a linear system, also called *behavior*, can be defined as

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

A transformation between systems defined by $R \in \mathcal{D}^{q \times p}$ and $R' \in \mathcal{D}^{q' \times p'}$ maps a solution $\eta' \in \ker_{\mathcal{F}}(R'.)$ to a solution $\eta \in \ker_{\mathcal{F}}(R.)$. If we can find a matrix $P \in \mathcal{D}^{p \times p'}$ for which there exists a matrix $Q \in \mathcal{D}^{q \times q'}$ satisfying

$$R\,P = Q\,R', \tag{2.1}$$

then the matrix P induces such a transformation by $\eta = P\eta'$, since for all $\eta' \in \ker_{\mathcal{F}}(R'.)$ we easily see that

$$R\,\eta = R\,(P\,\eta') = Q\,(R'\,\eta') = 0.$$

We exploit equation (2.1) later in Sections 5.2.2 and 5.2.3. The rest of this section briefly summarizes the algebraic background of the algebraic analysis approach. A behavior is actually the solution space of an underlying system of linear equations and depends on $\mathcal{F}$. Algebraically, the intrinsic object of linear system $R\eta = 0$ is its associated module

$$M := \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q}R),$$

which is a left $\mathcal{D}$-module finitely presented by R, i.e., both numerator and denominator are finitely generated $\mathcal{D}$-modules. This module M defines a linear system of equations: let $\{f_j\}_{j=1,\dots,p}$ be the standard basis of the free left $\mathcal{D}$-module $\mathcal{D}^{1\times p}$, let $\pi\colon \mathcal{D}^{1\times p} \to M$ be the canonical projection onto M, and let $y_j := \pi(f_j)$ for $j = 1,\dots,p$. Then, it is easy to check that $\{y_j\}_{j=1,\dots,p}$ is a set of generators of M. Let us denote by $R_{i\bullet}$ the i^{th} row of the matrix R. The set of generators $\{y_j\}_{j=1,\dots,p}$ of M satisfies the $\mathcal{D}$-linear relations

$$\sum_{j=1}^{p} R_{ij}y_j = \sum_{j=1}^{p} R_{ij}\pi(f_j) = \pi(\sum_{j=1}^{p} R_{ij}f_j) = \pi(R_{i\bullet}) = 0,$$

for $i = 1,\dots,q$, since $R_{i\bullet} \in \mathcal{D}^{1\times q}R$. Hence, if we note $y := (y_1,\dots,y_p)^T \in M^p$, then y satisfies $Ry = 0$. Let us define the abelian group

$$\hom_{\mathcal{D}}(M,\mathcal{F}) = \{\phi\colon M \to \mathcal{F} \mid \phi \text{ is a left } \mathcal{D}\text{-linear map}\}.$$

The next result shows that the behavior $\ker_F(R.)$ can intrinsically be interpreted as the dual $\hom_{\mathcal{D}}(M,\mathcal{F})$ of M.

Theorem 2.83. *With the above notations, we have the following isomorphism of abelian groups:*

$$\begin{aligned}\chi\colon\ \hom_{\mathcal{D}}(M,\mathcal{F}) &\to \ker_{\mathcal{F}}(R.)\\ \phi &\mapsto \eta := (\phi(y_1),\dots,\phi(y_P))^T,\end{aligned}$$

whose inverse χ^{-1} is defined by $\chi^{-1}(\eta) = \phi_\eta$, where $\phi_\eta(\pi(\lambda)) := \lambda\eta$ for all $\lambda \in \mathcal{D}^{1\times p}$ and $\eta \in \ker_F(R.)$.

In the following, we just prove that the map χ is bijective. Clearly, for $\eta = \chi(\phi)$ we can prove that $\eta \in \ker_{\mathcal{F}}(R.)$. Moreover, ϕ_η is a well-defined left $\mathcal{D}$-linear map which implies $\phi_\eta \in \hom_{\mathcal{D}}(M,\mathcal{F})$. We note $\sigma(\eta) := \phi_\eta$ where $\eta \in \ker_{\mathcal{F}}(R.)$. Then, we get $(\sigma\circ\chi)(\phi) = \phi_{(\phi(y_1),\dots,\phi(y_p))^T}$, i.e.

$$(\sigma\circ\chi)(\phi)(\pi(\lambda)) = \sum_{j=1}^{p} \lambda_j\phi(y_j) = \phi\big(\sum_{j=1}^{p} \lambda_j\pi(f_j)\big) = \phi(\pi(\lambda)),$$

and thus $\sigma \circ \chi = \mathrm{id}_{\hom_{\mathcal{D}}(M,\mathcal{F})}$. Finally, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we have

$$(\chi \circ \sigma)(\eta) = (\phi_\eta(y_1), \ldots, \phi_\eta(y_p))^T,$$

where $\phi_\eta(y_j) = f_j \eta = \eta_j$, i.e., $\chi \circ \sigma = \mathrm{id}_{\ker_{\mathcal{F}}(R.)}$.

For two behaviors $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ as above, we consider the left $\mathcal{D}$-modules M and M' finitely presented by $R \in \mathcal{D}^{q\times p}$ and $R' \in \mathcal{D}^{q'\times p'}$, respectively. The following theorem shows that every left $\mathcal{D}$-homomorphism $\phi : M \longrightarrow M'$ induces an abelian group homomorphism $\phi^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$, i.e., a transformation which sends a solution of $R'\eta' = 0$ to a solution of $R\eta = 0$.

Theorem 2.84. *(Cluzeau and Quadrat [14]). We have:*

(i) Any $\phi \in \hom_{\mathcal{D}}(M, M')$ is defined by

$$\phi(\pi(\lambda)) = \pi'(\lambda P),$$

for all $\lambda \in \mathcal{D}^{1\times p}$, where $P \in \mathcal{D}^{p\times p'}$ satisfies $\mathcal{D}^{1\times q}(RP) \subseteq \mathcal{D}^{1\times q'} R'$, i.e., P is such that there exists $Q \in \mathcal{D}^{q\times q'}$ satisfying:

$$RP = QR'. \tag{2.2}$$

(ii) ϕ induces the following homomorphism of abelian groups:

$$\begin{aligned} \phi^*\colon\ \ker_{\mathcal{F}}(R'.) &\to \ker_{\mathcal{F}}(R.) \\ \eta' &\mapsto \eta := P\eta'. \end{aligned}$$

Chapter 3

Tensor reduction systems

The main goal of this chapter is to elaborate generalized Bergman's basis-free approach for tensor reduction systems. The relevant references can be found in Section 1.3. The framework provided uses quotients of tensor rings over two-sided ideals for constructing rings of operators. A motivating example of such rings, given in Section 3.1, is the ring of differential operators over a noncommutative differential ring, as generalization of the algebra of differential operators over a commutative ring.

One main contribution to this chapter, presented in Section 3.2, is providing a detailed proof for the Bergman's diamond lemma in tensor setting by including a stronger notion for resolvability of ambiguities. The diamond lemma provides a heuristic approach for computer-assisted construction of a confluent reduction system starting from a given reduction system. The process is analogous to Buchberger's algorithm and Knuth-Bendix completion [32] as well. A confluent tensor reduction system enables effective computations based on normal forms.

Another contribution to generalize Bergman's tensor setting is made in Section 3.3 by introducing the concept of specialization. This new setting is based on constructing tensor rings over free modules that can be split into further submodules. It allows to define reduction homomorphisms over bigger domains, which leads to reduce number of ambiguities significantly. Therefore, it makes the algorithmic verification of the confluence criterion more efficient.

3.1 Introductory example

For a short discussion on several approaches for modelling rings of operators, we use the well-known example of differential operators. Recall that differential operators with polynomial coefficients (Weyl algebra) over a field $\mathcal{K} \supseteq \mathbb{Q}$ can be defined as the quotient algebra

$$\mathcal{K}\langle X, D\rangle/(DX - XD - 1)$$

of the free polynomial algebra $\mathcal{K}\langle X, D\rangle$ by a two-sided ideal; see for example [18]. Now, we consider a commutative differential ring $(\mathcal{R}, \partial)$ with ring of constants $\mathcal{K}$. For the case where $\mathcal{R}$ is a finitely presented $\mathcal{K}$-algebra it is true that the differential operators $\mathcal{R}\langle\partial\rangle$ are a finitely presented $\mathcal{K}$-algebra as well, analogous to the Weyl algebra.

Skew polynomials are a well-established approach that only introduces finitely many rules for differential operators over arbitrary differential rings $\mathcal{R}$ (e.g. rational functions): they are represented by defining a multiplication on normal forms $\sum f_i\partial^i$ based on the commutation rule

$$\partial \cdot f = f\partial + \partial f.$$

Viewed as construction by generators and relations, this amounts to (potentially) infinitely many relations, one for each generator of $\mathcal{R}$.

In the following, we motivate and illustrate informally tensor reduction systems. For a commutative differential ring, the construction leads to a quotient of the tensor algebra as in [28]. The commutation rule for skew polynomials above corresponds to a reduction homomorphism for tensors below. In contrast to the case of skew polynomials, where the ring $\mathcal{R}$ is regarded as the coefficient ring, in the tensor construction below $\mathcal{R}$ is just assumed to be a $\mathcal{K}$-module and we tensor over the ring $\mathcal{K}$ only. Consequently, for the multiplication in $\mathcal{R}$, we need to introduce an additional reduction homomorphism for tensors.

Example 3.1. *Let $(\mathcal{R}, \partial)$ be a commutative differential ring with ring of constants $\mathcal{K}$. Recall from Subsection 2.3.2, by the Leibniz rule, that the derivation $\partial\colon \mathcal{R} \to \mathcal{R}$ is a $\mathcal{K}$-module homomorphism. In addition, commutativity of the ring $\mathcal{R}$ implies the multiplication operators induced by $f \in \mathcal{R}$ mapping $g \mapsto fg$ are $\mathcal{K}$-module homomorphisms as well.*

Now consider the $\mathcal{K}$-tensor algebra $\mathcal{K}\langle M\rangle$ on the $\mathcal{K}$-module $M = \mathcal{R} \oplus \mathcal{K}\partial$ where $M_{\mathsf{D}} = \mathcal{K}\partial$ denotes the free left $\mathcal{K}$-module generated by the symbol ∂. The identities in $\mathcal{K}\langle M\rangle$ reflect the identities coming from the $\mathcal{K}$-linearity of the operators and their compositions, where we interpret $\otimes$ as composition of operators. Using reduction rules defined by $\mathcal{K}$-module homomorphisms on certain submodules of the tensor algebra, we are able to incorporate the additional identities. We consider two homomorphisms defined by

$$f \otimes g \mapsto fg \quad \text{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f,$$

corresponding to the composition of multiplication operators and the Leibniz rule. These two reduction rules induce the two-sided ideal

$$J = (f \otimes g - fg, \partial \otimes f - f \otimes \partial - \partial f \mid f, g \in \mathcal{R})$$

which is used for defining the $\mathcal{K}$-algebra of differential operators as the quotient algebra

$$\mathcal{R}\langle\partial\rangle = \mathcal{K}\langle M\rangle / J.$$

Our goal is to obtain unique normal forms in the quotient by applying the reduction rules above. A tensor of the form

$$\partial \otimes f \otimes g$$

corresponds to an overlap ambiguity of these two rules, since we can reduce it by the homomorphisms in different ways to obtain either

$$(f \otimes \partial + \partial f) \otimes g \quad \text{or} \quad \partial \otimes (fg).$$

For checking resolvability of the ambiguity the S-polynomial formed by the difference of these alternatives should be reducible to zero. In the present case, it reduces to zero because of the Leibniz rule in $\mathcal{R}$. Another ambiguity is expressed by tensors of the form $f \otimes g \otimes h$ and is resolvable as well. Since all ambiguities are resolvable, we obtain normal forms in terms of irreducible tensors

$$\partial^{\otimes j} \quad \text{and} \quad f \otimes \partial^{\otimes j}.$$

For differential operators with matrix coefficients, we assume $\mathcal{R}$ to be a ring of matrices over some (commutative) differential ring. As a concrete case, one may think of the matrix ring $M_n(C^\infty(\mathbb{R}))$ where its ring of constants $\mathcal{K}$ is the ring of constant matrices in $M_n(C^\infty(\mathbb{R}))$. Then, not only is $\mathcal{R}$ a noncommutative differential ring but also $\mathcal{K}$ is no longer commutative and elements of $\mathcal{K}$ do not commute with elements of $\mathcal{R}$. Consequently, $\mathcal{R}$ is not a $\mathcal{K}$-algebra anymore and rather it is a $\mathcal{K}$-ring. More generally, we consider an arbitrary differential ring $\mathcal{R}$. It is a bimodule over its ring of constants $\mathcal{K}$ and tensoring over $\mathcal{K}$ leads to a construction of the differential operators as a quotient of the tensor ring instead of the tensor algebra.

Example 3.2. *Let $(\mathcal{R}, \partial)$ be an arbitrary differential ring with ring of constants $\mathcal{K}$. In contrast to ∂ which is a $\mathcal{K}$-bimodule homomorphism of $\mathcal{R}$, multiplication operators $B \mapsto AB$ in general are only right $\mathcal{K}$-module homomorphisms. We consider the $\mathcal{K}$-tensor ring $\mathcal{K}\langle M\rangle$ on the $\mathcal{K}$-bimodule*

$$M = \mathcal{R} \oplus M_{\mathsf{D}}$$

where by M_{D} we denote the $\mathcal{K}$-bimodule non-freely generated by ∂ subject to the identity

$$K_1\partial \cdot K_2 = K_1K_2\partial,$$

which is used to define the right scalar multiplication on M_{D}. Therefore, we are required to consider one more relation on M_{D} and hence it is a non-free $\mathcal{K}$-bimodule. The identities in the tensor ring $\mathcal{K}\langle M\rangle$ reflect the identities coming from additivity of the operators and their compositions. Reduction rules are $\mathcal{K}$-bimodule homomorphisms defined by the same formulae as above. For further details we ask the reader to see Example 3.17.

3.2 Bergman's setting

In this section, we explain analogs of Gröbner bases in tensor rings following [5, Sec. 6], using standard notation for rewriting systems from [3] and Section 2.4. First, we outline the construction and some properties of the $\mathcal{K}$-tensor ring $\mathcal{K}\langle M\rangle$ on a $\mathcal{K}$-bimodule M over an arbitrary ring $\mathcal{K}$ with unit element. If $\mathcal{K}$ is commutative and the left and right scalar multiplication on M agree, then $\mathcal{K}\langle M\rangle$ is the tensor algebra on M, which is a generalization of the noncommutative polynomial algebra on a set of indeterminates. In contrast to the noncommutative polynomials, in the tensor ring the "coefficients" in $\mathcal{K}$ do not commute with the "indeterminates".

3.2.1 Diamond lemma in tensor rings

We first consider a family of $\mathcal{K}$-bimodules $(M_x)_{x\in X}$ indexed by a set X. These modules M_x play the role of the indeterminates in the noncommutative polynomial algebra. Recall that the word monoid on X is denoted by $\langle X\rangle$. For a word $W = x_1\cdots x_n \in \langle X\rangle$, we denote the tensor product of the corresponding bimodules by

$$M_W := M_{x_1} \otimes \cdots \otimes M_{x_n}.$$

In particular, for the empty word/tensor ϵ we have $M_\epsilon = \mathcal{K}\epsilon$. Here, the pure tensors $m_1 \otimes \cdots \otimes m_n \in M_W$ with $m_i \in M_{x_i}$ play the role of the monomials in the noncommutative polynomial algebra. Now, we consider the direct sum

$$M := \bigoplus_{x\in X} M_x \tag{3.1}$$

and the $\mathcal{K}$-tensor ring on M, which can be written as

$$\mathcal{K}\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n} = \bigoplus_{W\in\langle X\rangle} M_W = \bigoplus_{W\in\langle X\rangle} (\bigotimes_{i=1}^{|W|} M_{x_i}). \tag{3.2}$$

We have already seen in Section 2.3 that every tensor $t \in \mathcal{K}\langle M\rangle$ can be written as a sum of pure tensors. However, in contrast to linear combinations of monomials in the noncommutative polynomial algebra, this representation is not unique. This is due to the fact that $M^{\otimes n}$ is not freely generated as a $\mathcal{K}$-bimodule by the pure tensors, e.g. in $M^{\otimes 2}$ for $m_3 = m_1 + m_2 \in M$ we have

$$m_1 \otimes m_3 + m_2 \otimes m_1 = m_3 \otimes m_1 + m_1 \otimes m_2.$$

However, using bimodule homomorphisms, one can still define reductions analogous to polynomial reduction for (non-)commutative Gröbner bases.

Definition 3.3. *Let M be given by* (3.1). *A* reduction rule *for $\mathcal{K}\langle M\rangle$ is given by a pair (W, h) of a word $W \in \langle X\rangle$ and a $\mathcal{K}$-bimodule homomorphism $h\colon M_W \to \mathcal{K}\langle M\rangle$. For a reduction rule $r = (W, h)$ and words $A, B \in \langle X\rangle$, we define a* reduction *as the $\mathcal{K}$-bimodule homomorphism*

$$h_{A,r,B}\colon \mathcal{K}\langle M\rangle \to \mathcal{K}\langle M\rangle$$

acting as $\mathrm{id}_A \otimes h \otimes \mathrm{id}_B$ on M_{AWB} and the identity on all other M_V with $V \in \langle X\rangle$ and $V \neq AWB$.

Thus, for a pure tensor $a \otimes w \otimes b \in M_{AWB}$ with $a \in M_A$, $w \in M_W$, and $b \in M_B$, the reduction $h_{A,r,B}$ is given by

$$a \otimes w \otimes b \mapsto a \otimes h(w) \otimes b.$$

Therefore, as for polynomial reduction, we "replace" the "leading monomial" w by the "tail" $h(w)$ given by the homomorphism h.

In the following, we explain some terminology for reduction systems over tensor rings, see also Section 2.4. Let $t \in \mathcal{K}\langle M\rangle$. A reduction $h_{A,r,B}$ acts trivially on t, i.e. $h_{A,r,B}(t) = t$, if the summand of t in M_{AWB} is zero, see (3.2). A reduction rule $r = (W, h)$ *reduces* t to $s \in \mathcal{K}\langle M\rangle$ if a reduction $h_{A,r,B}$ for some $A, B \in \langle X\rangle$ acts non-trivially on t and $h_{A,r,B}(t) = s$, and we write $t \to_r s$.

A *reduction system* for $\mathcal{K}\langle M\rangle$ is a set Σ of reduction rules. Every reduction system Σ induces a *reduction relation* $\to_\Sigma$ on tensors by defining $t \to_\Sigma s$ for $t, s \in \mathcal{K}\langle M\rangle$ if $t \to_r s$ for some reduction rule $r \in \Sigma$. Fixing a reduction system Σ, we say that $t \in \mathcal{K}\langle M\rangle$ *can be reduced* to $s \in \mathcal{K}\langle M\rangle$ by Σ if $t = s$ or there exists a finite sequence of reduction rules $r_1, \ldots, r_n$ in Σ such that

$$t \to_{r_1} t_1 \to_{r_2} \cdots \to_{r_{n-1}} t_{n-1} \to_{r_n} s$$

and we write $t \xrightarrow{*}_\Sigma s$. In other words, $\xrightarrow{*}_\Sigma$ denotes the reflexive transitive closure of the reduction relation $\to_\Sigma$. Note that we can represent any finite chain of reductions by a composition of $\mathcal{K}$-bimodule homomorphisms: we let $\rho_i := h_{A_i,r_i,B_i}$, for $i = 1, \ldots, n$ with suitable $A_i, B_i \in \langle X \rangle$, and by ρ we denote the $\mathcal{K}$-bimodule homomorphism $\rho_n \cdots \rho_1$. This allows us to represent the chain above as $\rho(t) = s$.

The set of *irreducible words* $\langle X \rangle_{\mathrm{irr}} \subseteq \langle X \rangle$ consists of those words having no subwords from the set $\{W \mid (W, h) \in \Sigma\}$. We define the $\mathcal{K}$-subbimodule of *irreducible tensors* as

$$\mathcal{K}\langle M \rangle_{\mathrm{irr}} = \bigoplus_{W \in \langle X \rangle_{\mathrm{irr}}} M_W = \bigoplus_{W \in \langle X \rangle_{\mathrm{irr}}} (\bigotimes_{i=1}^{|W|} M_{x_i}). \tag{3.3}$$

In order to guarantee that for a given reduction system Σ all sequences of reductions are finite, we consider partial orders on $\langle X \rangle$ with some specific properties.

Definition 3.4. *A partial order $\leq$ on $\langle X \rangle$ is called a* semigroup partial order *on $\langle X \rangle$ if the condition*

$$B < \tilde{B} \Rightarrow ABC < A\tilde{B}C,$$

holds for all $A, B, \tilde{B}, C \in \langle X \rangle$. If in addition $\epsilon \leq A$ for all $A \in \langle X \rangle$, then it is called a monoid partial order*. It is called* Noetherian *if there are no infinite descending chains.*

Remark 3.5. *Note that a lexicographic order on $\langle X \rangle$ is not a semigroup order if $|X| > 1$: consider a lexicographic order on $X = \{x_1, x_2\}$ with $\epsilon < x_1 < x_2$. Then, by definition of semigroup order, from $\epsilon < x_1$ we have $x_2 < x_1 x_2$ which contradicts with definition of the lexicographic order above. However, a (weighted) degree-lexicographic order of the words is a semigroup (total) order on $\langle X \rangle$, and it is Noetherian if X is finite. Given a semigroup S with a semigroup partial order $\preceq$ on it and a semigroup homomorphism $\varphi\colon \langle X \rangle \to S$, we can define the induced semigroup partial order on $\langle X \rangle$ by*

$$V \leq W :\Leftrightarrow V = W \text{ or } \varphi(V) \prec \varphi(W).$$

For example, for $S = \mathbb{N}$ with the usual order and the homomorphism given by $\varphi(x_0) = 1$ for $x_0 \in X$ and $\varphi(x) = 0$ for $x \in X \setminus \{x_0\}$, the induced partial order just compares the degree in x_0. Given two semigroups S_1 and S_2 with corresponding semigroup partial orders $\leq_1$ and $\leq_2$ respectively, we can combine them lexicographically to obtain a semigroup partial order on $S = S_1 \times S_2$ by

$$(a_1, a_2) \leq (b_1, b_2) :\Leftrightarrow a_1 <_1 b_1 \text{ or } a_1 = b_1 \text{ and } a_2 \leq_2 b_2.$$

In the following lemmas, for a ring $\mathcal{K}$, we let $M = \bigoplus_{x\in X} M_x$ where $(M_x)_{x\in X}$ is a fixed family of $\mathcal{K}$-bimodules indexed by a set X. We also let Σ be a fixed reduction system on $\mathcal{K}\langle M\rangle$. For the rest of this chapter, we fix the notation

$$M_{<W} := \bigoplus_{\substack{V\in\langle X\rangle \\ V<W}} M_V$$

where W is a word in $\langle X\rangle$. Note that $M_{<W}$ is a $\mathcal{K}$-subbimodule of $\mathcal{K}\langle M\rangle$.

Definition 3.6. *A semigroup partial order* $\leq$ *is called* compatible *with a reduction system* Σ *if for all reduction rules* $(W,h)\in\Sigma$*,*

$$h(M_W) \subseteq M_{<W}.$$

If a compatible semigroup partial order is Noetherian, then there do not exist infinite sequences of reductions in Σ. In other words, the reduction relation $\to_\Sigma$ is *terminating* or *Noetherian.* So, in that case, every $t\in\mathcal{K}\langle M\rangle$ can be reduced in finitely many steps to an irreducible tensor

$$t \xrightarrow{*}_\Sigma s \in \mathcal{K}\langle M\rangle_{\mathrm{irr}}$$

and such an s is called a *normal form* of t. In general, a tensor can have different normal forms. If the element $t\in\mathcal{K}\langle M\rangle$ has a *unique normal form*, we denote it by $t{\downarrow_\Sigma}$.

In order to guarantee that normal forms for reduction systems on tensor rings are unique, we state below Bergman's analog of Buchberger's criterion for Gröbner bases [7]. In the context of Gröbner-Shirshov bases for various algebraic structures this is also referred to as the Composition-Diamond Lemma; see e.g. the survey [6].

Let Σ be a reduction system. We study the cases when two different reductions act non-trivially on tensors in M_W for $W\in\langle X\rangle$.

Definition 3.7. *An* overlap ambiguity *is given by two* (*not necessarily distinct*) *reduction rules* $(W,h),(\tilde{W},\tilde{h})\in\Sigma$ *and nonempty words* $A,B,C\in\langle X\rangle$ *such that*

$$W = AB \quad \text{and} \quad \tilde{W} = BC.$$

It is called resolvable *if for all pure tensors* $a\in M_A$*,* $b\in M_B$*, and* $c\in M_C$ *the* S-polynomial *can be reduced to zero:*

$$h(a\otimes b)\otimes c - a\otimes\tilde{h}(b\otimes c) \xrightarrow{*}_\Sigma 0.$$

An inclusion ambiguity *is given by distinct reduction rules* $(W,h),(\tilde{W},\tilde{h})\in\Sigma$ *and words* $A,B,C\in\langle X\rangle$ *with* $W=B$ *and* $\tilde{W}=ABC$*. It is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ the* S-polynomial *can be reduced to zero:*

$$a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \xrightarrow{*}_{\Sigma} 0.$$

With slight abuse of notation, we refer to S-polynomials of an overlap or inclusion ambiguity, respectively, by

$$\mathrm{SP}(A\underline{B}, \underline{B}C) \quad \text{or} \quad \mathrm{SP}(\underline{B}, A\underline{B}C).$$

A reduction system Σ induces the two-sided *reduction ideal*

$$I_\Sigma := (t - h(t) \mid (W, h) \in \Sigma \text{ and } t \in M_W) \subseteq \mathcal{K}\langle M \rangle. \tag{3.4}$$

Note that the definition of resolvability above differs from the definition used by Bergman. Actually, he uses two different notions for resolvability of ambiguities, which we briefly describe below. Both of them are weaker than Definition 3.7 in general. However, we show in Theorem 3.15 that if every tensor has a unique normal form, then all three definitions of resolvability are equivalent. We express one slightly weaker notion as follows.

Definition 3.8. *An overlap ambiguity given by (not necessarily distinct) reduction rules (W, h) and $(\tilde{W}, \tilde{h})$ in Σ and words $A, B, C \in \langle X \rangle$ with $W = AB$ and $\tilde{W} = BC$ is called* j-resolvable*, if for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ there exists a tensor $t \in \mathcal{K}\langle M \rangle$ such that*

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} a \otimes \tilde{h}(b \otimes c).$$

Similarly, an inclusion ambiguity given by two distinct reduction rules r and $\tilde{r}$ in Σ and words $A, B, C \in \langle X \rangle$ with $W = B$ and $\tilde{W} = ABC$ is called j-resolvable*, if for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$, there exists a tensor $t \in \mathcal{K}\langle M \rangle$ such that*

$$a \otimes h(b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} \tilde{h}(a \otimes b \otimes c).$$

In other words, in both cases, the two different results of the reductions of $a \otimes b \otimes c$ are joinable.

Example 3.9. *Consider the $\mathcal{K}$-tensor ring $\mathcal{K}\langle M \rangle$ for $M = M_A \oplus M_B \oplus M_C$ on the alphabet $X = \{A, B, C\}$. Let $\Sigma = \{r_1, r_2, r_3, r_4\}$ be a reduction system on X with*

$$r_1 = (AB, a \otimes b \mapsto a), \quad r_2 = (BC, b \otimes c \mapsto a + c),$$
$$r_3 = (AC, a \otimes c \mapsto 0), \quad r_4 = (AC, a \otimes c \mapsto -a \otimes a).$$

It is easy to see that any degree-lexicographic order with $C > A$ is compatible with Σ. The overlap ambiguity between the reduction rules r_1 and r_2 is j-resolvable, since for all pure tensors $a \in M_A$, $b \in M_B$ and $c \in M_C$ we have

$$h_1(a \otimes b) \otimes c = a \otimes c \rightarrow_{r_3} 0,$$

and

$$a \otimes h_2(b \otimes c) = a \otimes a + a \otimes c \rightarrow_{r_4} a \otimes a - a \otimes a = 0.$$

However, this ambiguity is not resolvable:

$$h_1(a \otimes b) \otimes c - a \otimes h_2(b \otimes c) = -a \otimes a.$$

Before describing the other notion of resolvability, which is even weaker and depends on semigroup partial order $\leq$, we require more notation: for a semigroup partial order $\leq$ on $\langle X \rangle$ compatible with the reduction system Σ and any word W in $\langle X \rangle$, we denote by $I_{\Sigma,W}$ the $\mathcal{K}$-bimodule generated by

$$\bigcup_{\substack{V \in \langle X \rangle \\ V < W}} \{t - s \mid t \in M_V \text{ and } t \rightarrow_\Sigma s \in \mathcal{K}\langle M \rangle\}.$$

Definition 3.10. *An overlap or inclusion ambiguity with words $A, B, C \in \langle X \rangle$ is called* $\leq$-resolvable *if and only if all its S-polynomials are contained in the $\mathcal{K}$-bimodule $I_{\Sigma,ABC}$.*

Remark 3.11. *If the semigroup partial order $\leq$ is compatible with Σ, then $I_{\Sigma,W}$ is contained in a "truncation" $I_\Sigma \cap M_{<W}$ of the reduction ideal I_Σ.*

In order to determine whether a given tensor $t \in \mathcal{K}\langle M \rangle$ belongs to $I_{\Sigma,W}$ or not, one possibility is to work with some reduction of t under Σ, as the following result shows:

Lemma 3.12. *Let $\leq$ be a semigroup partial order compatible with Σ. Let ρ be a finite composition of reductions. If $t \in M_{<W}$, then*

$$t - \rho(t) \in I_{\Sigma,W}.$$

As a consequence, for $t \in M_{<W}$, we see that $t \in I_{\Sigma,W}$ if and only if $\rho(t) \in I_{\Sigma,W}$.

Proof. We prove the lemma by induction: first assume that ρ represents only one reduction rule $r_1 = (W_1, h_1)$, say $\rho := h_{A_1,r_1,B_1}$ for $A_1, B_1 \in \langle X \rangle$. Since $t \in M_{<W}$, we get $t = \sum_{i=1}^{p} a_i \otimes w_i \otimes b_i + t_1$ where $a_i \in M_{A_1}$, $w_i \in M_{W_1}$, $b_i \in M_{B_1}$ for $i = 1, \ldots, p$, and $t_1 \in M_{<W}$. If $A_1 W_1 B_1 \not< W$, we get $\rho(t) = t$ which implies the trivial result $t - \rho(t) = t - t = 0 \in I_{\Sigma,W}$. If $A_1W_1B_1 < W$, then it follows that

$$t - \rho(t) = a_1 \otimes (w_1 - h_1(w_1)) \otimes b_1 \in I_{\Sigma,W},$$

and $\rho(t) \in M_{<W}$. Now let $\rho = \rho_n \cdots \rho_1$ and $t' = \rho_{n-1} \cdots \rho_1(t)$. By the induction hypothesis, $t - t' \in I_{\Sigma,W}$ and $t' \in M_{<W}$. Moreover, we have proved that $t' - \rho_n(t') \in I_{\Sigma,W}$. Therefore,

$$t - \rho(t) = t - \rho_n(t') = t - t' + t' - \rho_n(t') \in I_{\Sigma,W}$$

as required. □

For the rest of this section, we denote by $\mathcal{K}\langle M\rangle_{\text{un}}$ the set of all elements with unique normal forms in $\mathcal{K}\langle M\rangle$. In addition, we define the map

$$\begin{aligned} \pi_\Sigma \colon \mathcal{K}\langle M\rangle_{\text{un}} &\to \mathcal{K}\langle M\rangle_{\text{irr}} \\ t &\mapsto t\downarrow_\Sigma . \end{aligned}$$

Lemma 3.13. *Let $\leq$ be a Noetherian semigroup partial order on $\langle X\rangle$ that is compatible with Σ. Then*

(*i*) *The set $\mathcal{K}\langle M\rangle_{\text{un}}$ is a $\mathcal{K}$-subbimodule, and the map π_Σ is a $\mathcal{K}$-bimodule homomorphism.*

(*ii*) *Suppose that $a, b, c \in \mathcal{K}\langle M\rangle$ are such that for all homogeneous components a_A, b_B, c_C of a, b, c in any of the summands M_A, M_B, M_C of $\mathcal{K}\langle M\rangle$, respectively, the product $a_A \otimes b_B \otimes c_C$ has a unique normal form. (In particular this implies that $a \otimes b \otimes c$ has a unique normal form.) Let ρ be a finite composition of reductions. Then $a \otimes \rho(b) \otimes c$ has unique normal form,*

$$\pi_\Sigma(a \otimes \rho(b) \otimes c) = \pi_\Sigma(a \otimes b \otimes c).$$

Proof. (*i*) Suppose that $t, t' \in \mathcal{K}\langle M\rangle_{\text{un}}$ and $k, k' \in \mathcal{K}$. Since $\leq$ is Noetherian, the element $kt + t'k' \in \mathcal{K}\langle M\rangle$ can be reduced in finitely many steps to an irreducible tensor in $\mathcal{K}\langle M\rangle_{\text{irr}}$. Let ρ be any finite composition of reductions such that $\rho(kt + t'k') = t''$ where $t'' \in \mathcal{K}\langle M\rangle_{\text{irr}}$. Since t has a unique normal form, we can always find a composition of reductions ρ' such that $\rho'\rho(t) = \pi_\Sigma(t)$. Similarly, there is a composition of reductions ρ'' such that $\rho''\rho'\rho(t') = \pi_\Sigma(t')$. Now since t'' is irreducible and reductions are $\mathcal{K}$-bimodule homomorphisms, we can conclude

$$t'' = \rho''\rho'\rho(kt + t'k') = k\rho''\rho'\rho(t) + \rho''\rho'\rho(t')k' = k\pi_\Sigma(t) + \pi_\Sigma(t')k',$$

as required.

(ii) By (i) and the way (ii) is formulated, it clearly suffices to prove (ii) in the case where a, b, c are elements from M_A, M_B, M_C, respectively, and a single reduction $h_{D,r,E}$. In this case, $a \otimes h_{D,r,E}(b) \otimes c = h_{AD,r,EC}(a \otimes b \otimes c)$, which is the image of $a \otimes b \otimes c \in M_{ABC}$ under a reduction. Hence, it has a unique normal form if $a \otimes b \otimes c$ does, with the same reduced form. □

Lemma 3.14. *Let $\leq$ be a Noetherian semigroup partial order on $\langle X \rangle$ that is compatible with the reduction system Σ. If all ambiguities of Σ are $\leq$-resolvable then every $t \in \mathcal{K}\langle M \rangle$ has a unique normal form $t \downarrow_\Sigma$.*

Proof. It will suffice to prove that for all $D \in \langle X \rangle$, all elements $d \in M_D$ have unique normal forms, since by Lemma 3.13 the set $\mathcal{K}\langle M \rangle_{\mathrm{un}}$ forms a $\mathcal{K}$-subbimodule. We proceed by induction: let $D \in \langle X \rangle$ be minimal w.r.t. $\leq$ or $M_{\mathcal{D}} \subseteq \mathcal{K}\langle M \rangle_{\mathrm{irr}}$. Then any $d \in M_D$ has a unique normal form, since there is no reduction acts non-trivially on it. Now, let $D \in \langle X \rangle$ be non-minimal w.r.t. $\leq$ and assume that for all $E < D$ all $e \in M_E$ have unique normal form. Thus, the domain of π_Σ contains the $\mathcal{K}$-subbimodule $M_{<D}$, so the kernel of π_Σ contains $I_{\Sigma,D}$. We prove that all pure tensors $d \in M_D$ have unique normal forms and hence by Lemma 3.13 all elements in M_D have unique normal forms. We must show that given any two reductions $h_{L,r,R'}(d)$ and $h_{L',\tilde{r},R}(d)$, each acting non-trivially on $d \in M_{\mathsf{D}}$ (and hence each sending d to an element of $M_{<D}$), we will have $\pi_\Sigma(h_{L,r,R'}(d)) = \pi_\Sigma(h_{L',\tilde{r},R}(d))$ for all $d \in M_{\mathsf{D}}$. There are three cases, according to the relative location of the subwords W and $\tilde{W}$ in D. Of course, we can assume without loss of generality that $|L| \leq |L'|$, i.e., the indicated copy of W in D begins no later than the indicated copy of $\tilde{W}$.

Case 1. The subwords W and $\tilde{W}$ overlap in D, but neither contains the other. Then $D = LABCR$, where $(r, \tilde{r}, A, B, C)$ is an overlap ambiguity of Σ. Suppose that $d = l \otimes a \otimes b \otimes c \otimes r \in M_D$ where $l \in M_L, a \in M_A, b \in M_B, c \in M_C, r \in M_R$ and hence

$$h_{L,r,R'}(d) - h_{L',\tilde{r},R}(d) = l \otimes (h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c)) \otimes r.$$

By assumption, we have $h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \in I_{\Sigma,ABC}$ and hence

$$l \otimes (h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c)) \otimes r \in I_{\Sigma,LABCR} (= I_{\Sigma,D}),$$

which is annihilated by π_Σ. Thus $\pi_\Sigma(\rho_1(d)) - \pi_\Sigma(\rho_1'(d)) = 0$, as needed.

Case 2. One of the subwords $W, \tilde{W}$ of D is contained in the other. Suppose that $W = ABC, \tilde{W} = B$ and $d = l \otimes a \otimes b \otimes c \otimes r \in M_D$, then

$$h_{L,r,R'}(d) - h_{L',\tilde{r},R}(d) = l \otimes (h(a \otimes b \otimes c) - a \otimes \tilde{h}(b) \otimes c) \otimes r.$$

By resolvability of inclusion ambiguities we have $h(a\otimes b\otimes c) - a\otimes \tilde{h}(b) \in I_{\Sigma,ABC}$. Consequently,

$$l \otimes (h(a \otimes b \otimes c) - a \otimes \tilde{h}(b) \otimes c) \otimes r \in I_{\Sigma,LABCR} (= I_{\Sigma,D}).$$

The case $W = B$ and $\tilde{W} = ABC$ is analogous.

Case 3. W and $\tilde{W}$ are disjoint subwords of D. Then $D = LWN\tilde{W}R$ and $d = l \otimes w \otimes n \otimes \tilde{w} \otimes r$ where $l \in M_L,\ w \in M_W,\ n \in M_N,\ \tilde{w} \in M_{\tilde{W}}$, and $r \in M_R$. The elements we must prove equal are

$$\pi_\Sigma(l \otimes h(w) \otimes n \otimes \tilde{w} \otimes r) \quad \text{and} \quad \pi_\Sigma(l \otimes w \otimes n \otimes \tilde{h}(\tilde{w}) \otimes r).$$

By Lemma 3.13 (ii) we have

$$\pi_\Sigma(l \otimes h(w) \otimes n \otimes \tilde{w} \otimes r) = \pi_\Sigma(l \otimes h(w) \otimes n \otimes \tilde{h}(\tilde{w}) \otimes r),$$

and

$$\pi_\Sigma(l \otimes w \otimes n \otimes \tilde{h}(\tilde{w}) \otimes r) = \pi_\Sigma(l \otimes h(w) \otimes n \otimes \tilde{h}(\tilde{w}) \otimes r),$$

which completes the proof. □

For studying operator algebras, we want to compute in the factor ring $\mathcal{K}\langle M\rangle / I_\Sigma$. If all ambiguities are resolvable, then we can do this effectively using reductions in $\mathcal{K}\langle M\rangle$ and the corresponding normal forms with respect to $\to_\Sigma$. This is the *confluence criterion* (condition (i) below) that we will check algorithmically.

Theorem 3.15. *Let $\mathcal{K}$ be a ring, let $(M_x)_{x\in X}$ be a family of $\mathcal{K}$-bimodules indexed by a set X, and let $M = \bigoplus_{x\in X} M_x$. Let Σ be a reduction system on $\mathcal{K}\langle M\rangle$ and let $\leq$ be a Noetherian semigroup partial order on $\langle X\rangle$ that is compatible with Σ. Then the following are equivalent:*

- (i) *All ambiguities of Σ are resolvable.*
- (ii) *All ambiguities of Σ are j-resolvable.*
- (iii) *All ambiguities of Σ are $\leq$-resolvable.*
- (iv) *Every $t \in \mathcal{K}\langle M\rangle$ has a unique normal form $t\downarrow_\Sigma$.*
- (v) *$\mathcal{K}\langle M\rangle / I_\Sigma$ and $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ are isomorphic as $\mathcal{K}$-bimodules.*

If these conditions hold, then we can define a multiplication on $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ by $s \cdot t := (s \otimes t)\downarrow_\Sigma$ so that $\mathcal{K}\langle M\rangle / I_\Sigma$ and $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ are isomorphic as $\mathcal{K}$-rings.

Proof. Recall that by Noetherianity of $\leq$ every $t \in \mathcal{K}\langle M \rangle$ can be reduced in finitely many steps to an irreducible tensor $t \xrightarrow{*}_{\Sigma} s \in \mathcal{K}\langle M \rangle_{\mathrm{irr}}$. We start by proving the equivalence $(iv) \Leftrightarrow (v)$. Equivalent to (v) we have $\mathcal{K}\langle M \rangle = \mathcal{K}\langle M \rangle_{\mathrm{irr}} \oplus I_{\Sigma}$. Suppose that (iv) holds, then the map $\pi_{\Sigma} \colon \mathcal{K}\langle M \rangle \to \mathcal{K}\langle M \rangle_{\mathrm{irr}}$ is a projection; if $t \in \ker \pi_{\Sigma}$ then $t \downarrow_{\Sigma} = 0$. This implies that $t \in I_{\Sigma}$ and $\ker \pi_{\Sigma} \subseteq I_{\Sigma}$. In addition, we have $\ker \pi_{\Sigma} \supseteq I_{\Sigma}$, since for any $a \in M_A$, $b \in M_B$, $w \in M_W$, and $r = (W, h) \in \Sigma$ by use of Lemma 3.13 we have

$$\pi_{\Sigma}(a \otimes (w - h(w)) \otimes b) = \pi_{\Sigma}(a \otimes w \otimes b) - \pi_{\Sigma}(a \otimes h(w) \otimes b) = 0$$

which proves (v). Conversely, assume (v) and suppose $t \in \mathcal{K}\langle M \rangle$ is reduced to either of $s, s' \in \mathcal{K}\langle M \rangle_{\mathrm{irr}}$. Therefore, we get $s - s' \in \mathcal{K}\langle M \rangle_{\mathrm{irr}} \cap I_{\Sigma} = \{0\}$ which proves (iv).

The final comment in the statement of the theorem is clear, and the implications of $(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii)$ are easy: assuming (iv), every $d \in \mathcal{K}\langle M \rangle$ has a unique normal form, say $d \downarrow_{\Sigma}$. This means for all overlap ambiguities $r = (W, h), \tilde{r} = (\tilde{W}, \tilde{h})$ with $W = AB$ and $\tilde{W} = BC$ and for all elements $d = a \otimes b \otimes c \in M_{ABC}$,

$$\pi_{\Sigma}(h_{\epsilon,r,C}(d) = \pi_{\Sigma}(d) = \pi_{\Sigma}(h_{A,\tilde{r},\epsilon}(d)),$$

which implies by Lemma 3.13 that

$$(h_{\epsilon,r,C}(d)) - h_{A,\tilde{r},\epsilon}(d)) \xrightarrow{*}_{\Sigma} 0.$$

Similarly, for all inclusion ambiguities $r = (W, h), \tilde{r} = (\tilde{W}, \tilde{h})$ with $W = B$ and $\tilde{W} = ABC$ and for all elements $d = a \otimes b \otimes c \in M_{ABC}$,

$$\pi_{\Sigma}(h_{A,r,C}(d)) = \pi_{\Sigma}(d) = \pi_{\Sigma}(h_{\epsilon,\tilde{r},\epsilon}(d)),$$

which implies by the same lemma that

$$(h_{A,r,C}(d) - h_{\epsilon,\tilde{r},\epsilon}(d)) \xrightarrow{*}_{\Sigma} 0.$$

For proving the implication $(i) \Rightarrow (ii)$, consider the overlap ambiguity given by $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and nonempty words $A, B, C \in \langle X \rangle$ such that

$$W = AB \quad \text{and} \quad \tilde{W} = BC.$$

By (i), for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ the corresponding S-polynomial can be reduced to zero:

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_{\Sigma} 0.$$

In other words, there is a composition ρ of reductions such that

$$\rho(h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c)) = 0.$$

Therefore, we have

$$\rho(h(a \otimes b) \otimes c) = \rho(a \otimes \tilde{h}(b \otimes c)) =: t$$

which allows us to conclude

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} a \otimes \tilde{h}(b \otimes c).$$

The proof for inclusion ambiguities of $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ with $W = B$ and $\tilde{W} = ABC$ is analogous. Now let us prove the implication $(ii) \Rightarrow (iii)$. By (ii), for the overlap ambiguity $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and nonempty words $A, B, C \in \langle X \rangle$ such that

$$W = AB \quad \text{and} \quad \tilde{W} = BC,$$

and for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ we have

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} a \otimes \tilde{h}(b \otimes c).$$

This means there are two compositions ρ and ρ' of reductions such that

$$t = \rho(h(a \otimes b) \otimes c) = \rho'(a \otimes \tilde{h}(b \otimes c)).$$

The corresponding S-polynomial can be written as

$$\big(h(a \otimes b) \otimes c - \rho(h(a \otimes b) \otimes c)\big) - \big(a \otimes \tilde{h}(b \otimes c) - \rho'(a \otimes \tilde{h}(b \otimes c)\big).$$

Here, $h(a \otimes b) \otimes c, a \otimes \tilde{h}(b \otimes c) \in M_{<ABC}$. Moreover, by Lemma 3.12, both

$$h(a \otimes b) \otimes c - \rho(h(a \otimes b) \otimes c) \quad \text{and} \quad a \otimes \tilde{h}(b \otimes c) - \rho'(a \otimes \tilde{h}(b \otimes c))$$

live in $I_{\Sigma,ABC}$. Therefore, the corresponding S-polynomial is contained in $I_{\Sigma,ABC}$. Analogously, for inclusion ambiguities, we can prove the implication $(ii) \Rightarrow (iii)$. Finally, by Lemma 3.14 the implication $(iii) \Rightarrow (iv)$ holds and this completes the proof. □

Remark 3.16. *In order to do computations in the quotient ring $\mathcal{K}\langle M \rangle / I_\Sigma$, we are interested in finding a system of representatives. According to Theorem 3.15, the irreducible tensors $\mathcal{K}\langle M \rangle_{\mathrm{irr}}$ are such a system when the tensor reduction system is confluent. Therefore, if the given reduction system is not confluent, we try to complete it to a confluent one such that it generates the same reduction ideal of* (3.4).

In the following example we revisit Example 3.2 for studying it formally in the tensor ring setting.

Example 3.17. *Let $(\mathcal{R}, \partial)$ be a differential ring and let $\mathcal{K}$ denote its ring of constants. We consider the $\mathcal{K}$-bimodule $M_{\mathsf{R}} = \mathcal{R}$ (indexed by the letter R). In addition, we consider the free left $\mathcal{K}$-module $M_{\mathsf{D}} = \mathcal{K}\partial$ generated by ∂ (indexed by the letter D), which we view as a $\mathcal{K}$-bimodule with right multiplication*

$$K_1\partial \cdot K_2 = K_1K_2\partial,$$

for all $K_1, K_2 \in \mathcal{K}$. This definition is based on left $\mathcal{K}$-linearity of the operation ∂ on $\mathcal{R}$:

$$(K_1\partial K_2)A = K_1\partial(K_2A) = (K_1K_2\partial)A.$$

Let $M = M_{\mathsf{R}} \oplus M_{\mathsf{D}}$ be the module of basic operators. Then words on the alphabet $X = \{\mathsf{R}, \mathsf{D}\}$ index the direct summands of the $\mathcal{K}$-tensor ring $\mathcal{K}\langle M\rangle$.

We interpret elements $A \in \mathcal{R}$ as multiplication operators, ∂ as the derivation on $\mathcal{R}$, and the tensor product $\otimes$ as the composition of operators. So, we consider the reduction system $\Sigma = \{r_{\mathsf{RR}}, r_{\mathsf{DR}}\}$ with the reduction rules

$$r_{\mathsf{RR}} = (\mathsf{RR}, A \otimes B \mapsto AB) \quad \text{and} \quad r_{\mathsf{DR}} = (\mathsf{DR}, \partial \otimes A \mapsto A \otimes \partial + \partial A)$$

corresponding to the composition of multiplication operators and the Leibniz rule. These two reduction rules induce the two-sided ideal

$$I_\Sigma = (A \otimes B - AB, \partial \otimes A - A \otimes \partial - \partial A \mid A, B \in \mathcal{R})$$

which we use to define the $\mathcal{K}$-ring of differential operators as the quotient ring

$$\mathcal{R}\langle\partial\rangle = \mathcal{K}\langle M\rangle / I_\Sigma$$

of the tensor ring by the two-sided reduction ideal. The informal definition of the reduction homomorphisms above can be made formal in the following way. First, since

$$\begin{aligned} M_{\mathsf{R}} \times M_{\mathsf{R}} &\to M_{\mathsf{R}} \\ (A, B) &\mapsto AB \end{aligned}$$

is a balanced map, it induces a well-defined homomorphism $M_{\mathsf{RR}} \to M_{\mathsf{R}}$ of abelian groups. This homomorphism is a $\mathcal{K}$-bimodule homomorphism, which we use to define r_{RR}. Extending the definition

$$\beta(\partial, A) := A \otimes \partial + \partial A$$

by

$$\beta(K_1\partial, A) := \beta(\partial, K_1A),$$

we obtain a balanced map $\beta\colon M_{\mathsf{D}} \times M_{\mathsf{R}} \to M_{\mathsf{RD}} \oplus M_{\mathsf{R}}$*, since*

$$\beta(K_1\partial \cdot K_2, A) = \beta(K_1K_2\partial, A) = \beta(\partial, K_1K_2A) = \beta(K_1\partial, K_2A).$$

Like above, β *induces a* $\mathcal{K}$*-bimodule homomorphism* $M_{\mathsf{DR}} \to M_{\mathsf{RD}} \oplus M_{\mathsf{R}}$ *constituting* r_{DR}.

Any semigroup partial order $\leq$ *on* $\langle X\rangle$ *with* $\mathsf{RR} > \mathsf{R}$*, as well as* $\mathsf{DR} > \mathsf{RD}$ *and* $\mathsf{DR} > \mathsf{R}$ *is compatible with* Σ*, e.g. the degree-lexicographic order with* $\mathsf{D} > \mathsf{R}$*. There are two overlap ambiguities. The S-polynomials of the first ambiguity reduce to zero in two steps:*

$$\mathrm{SP}(\mathsf{R\underline{R}}, \mathsf{\underline{R}R}) = (AB) \otimes C - A \otimes (BC) \to_{r_{\mathsf{RR}}} (AB)C - A(BC) = 0,$$

and the S-polynomials of the second ambiguity reduce to zero by applying the Leibniz rule.

$$\begin{aligned}\mathrm{SP}(\mathsf{D\underline{R}}, \mathsf{\underline{R}R}) &= (A \otimes \partial + \partial A) \otimes B - \partial \otimes (AB) \\ &\to_{r_{\mathsf{DR}}} A \otimes B \otimes \partial + A \otimes \partial B + (\partial A)B - AB \otimes \partial - \partial(AB) \\ &\to_{r_{\mathsf{RR}}} AB \otimes \partial + A\partial B + (\partial A)B - AB \otimes \partial - \partial(AB) \\ &= A\partial B + (\partial A)B - \partial(AB) = 0.\end{aligned}$$

Now since $\langle X\rangle_{\mathrm{irr}} = \{\epsilon, \mathsf{R}, \mathsf{D}, \mathsf{RD}, \mathsf{D}^2, \mathsf{RD}^2, \ldots\}$*, by Theorem 3.15 we conclude that every* $t \in \mathcal{K}\langle M\rangle$ *has a unique normal form* $t\downarrow_\Sigma$ *in* $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$*, where*

$$\mathcal{K}\langle M\rangle_{\mathrm{irr}} = \mathcal{K}\epsilon \oplus M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus (M_{\mathsf{R}} \otimes M_{\mathsf{D}}) \oplus M_{\mathsf{D}}^{\otimes 2} \oplus (M_{\mathsf{R}} \otimes M_{\mathsf{D}}^{\otimes 2}) \oplus \ldots.$$

In other words, we can write $t\downarrow_\Sigma$ *as a sum of pure tensors of the form* $\epsilon,\ A,\ \partial,\ A \otimes \partial,\ \partial \otimes \partial,\ A \otimes \partial \otimes \partial, \ldots$ *and recover the well-known normal forms of differential operators.*

Remark 3.18. *If some* $\alpha \in M_x$ *corresponds to a left* $\mathcal{K}$*-linear operator, like* $\partial \in M_{\mathsf{D}}$ *above, then for the right scalar multiplication of left multiples of* α*, we always have*

$$K_1\alpha \cdot K_2 = K_1K_2\alpha$$

with $K_1, K_2 \in \mathcal{K}$*. As soon as such an operator is present, the ring over which the tensors are formed has to contain* $\mathcal{K}$ *in order to incorporate the corresponding relations directly into the tensor ring.*

3.2.2 Computational aspects

We investigate the algorithmic aspects of Theorem 3.15 by assuming a finite reduction system Σ on a finite alphabet X. In addition, we should consider a compatible Noetherian semigroup partial order.

For obtaining the set of ambiguities, it is enough to work in the word monoid $\langle X \rangle$. Similarly, determining the set of irreducible words $\langle X \rangle_{\mathrm{irr}}$ is a purely combinatorial problem on words as well, cf. the proofs of Theorems 4.14, 4.22, and 6.9. In order to check resolvability of ambiguities, it is sufficient to work with S-polynomials which are constructed from general elements of the basic bimodules M_x. The result of a reduction step, i.e. the application of a homomorphism from the reduction system, needs to be simplified in the tensor ring. This involves application of properties of the tensor product and of identities in the bimodules, like the Leibniz rule in Example 3.17. In practice, the reduction to zero often can be detected heuristically without having a canonical simplifier in the bimodules.

The Mathematica package `TenReS` supports verification of the confluence criterion and completion based on S-polynomial computations. We can exploit it for generating all ambiguities and corresponding S-polynomials of a reduction system given by the user. The package itself contains routines for computing in the tensor ring, but still we have to implement identities needed for computing in the bimodules of equation (3.1) in each concrete case.

In contrast to specifying new identities in the polynomial resp. term algebra, already the constructive specification of reduction homomorphisms in the tensor setting is not clear in general.

3.3 Tensor setting with specialization

In order to exploit Bergman's tensor setting directly, the sum in (3.1) has to be direct. Consequently, in a reduction system overlaps between domains of reduction rules cannot occur, in fact even their tensor factors cannot overlap. Considering this, for the case we have overlapping domains (or factors), reduction rules must be split into several smaller parts so that domains of those smaller rules do not overlap. Therefore, in practice computations with such reduction systems can be inconvenient and inefficient, as the smaller rules technically are just individual rules that need to be applied separately. In addition, this causes some redundancy in the investigation of ambiguities and S-polynomials. Sticking to the above definition of reduction systems for tensor rings, this situation cannot be avoided.

Example 3.19. *Note that in Example 3.17 irreducible tensors still have some relations among them when they act as operators. For instance, $K\epsilon \in M^{\otimes 0}$ and $K \in M$ both act by multiplying with $K \in \mathcal{K}$. Therefore, we require an additional reduction rule reducing $K \in M$ to $K\epsilon \in M^{\otimes 0}$ for $K \in \mathcal{K}$. Fixing a direct complement $\mathcal{R} = \mathcal{K} \oplus \tilde{\mathcal{R}}$ in $\mathcal{R}$ for defining the reduction rule*

$$r_{\mathsf{K}} = (\mathsf{K}, 1 \mapsto \epsilon),$$

would cause the splitting of the reduction rule r_{RR} into four reduction rules $r_{\mathsf{KK}}, r_{\mathsf{K\tilde{R}}}, r_{\mathsf{\tilde{R}K}}, r_{\mathsf{\tilde{R}\tilde{R}}}$ and similarly r_{DR} would split into two reduction rules. The aim of this section is to introduce a framework that allows coexistence of the reduction rule r_{K} with the reduction rules r_{RR} and r_{DR}.

To resolve this situation, in this section we introduce a more flexible tensor setting where the definable reduction systems are much more general. Although the induced reduction relations in the new system are also more general, the corresponding reduction ideals are not, however.

Definition 3.20. *Let M be a $\mathcal{K}$-bimodule and let Z be an alphabet. A family $(M_z)_{z \in Z}$ of $\mathcal{K}$-subbimodules of M is called a* decomposition with specialization*, if $M = \sum_{z \in Z} M_z$ and there exists a subset $X \subseteq Z$ such that*

(*i*) *we have the direct sum decomposition $M = \bigoplus_{x \in X} M_x$ and*

(*ii*) *for every $z \in Z$ the corresponding module M_z satisfies*

$$M_z = \bigoplus_{x \in S(z)} M_x \tag{3.5}$$

where $S(z) := \{x \in X \mid M_x \subseteq M_z\}$ is the set of specializations *of z.*

Following this definition we conclude that $S(x) = \{x\}$ for $x \in X$. In the following, we define a framework for tensor reduction systems that are based on such a decomposition with specialization. To this end, we fix a $\mathcal{K}$-bimodule M, alphabets $X \subseteq Z$, and a decomposition $(M_z)_{z \in Z}$ of M with specialization.

For words $W = w_1 \dots w_n \in \langle Z \rangle$, the corresponding subbimodule of $\mathcal{K}\langle M \rangle$ is defined as before by $M_W := M_{w_1} \otimes \cdots \otimes M_{w_n}$. Because of (3.5), any M_W is a direct sum of certain M_V, for $V \in \langle X \rangle$. For an exact statement, we can extend the notion of specialization from the alphabet Z to the whole word monoid $\langle Z \rangle$ by the definition below such that we have the following generalization of (3.5):

$$M_W = \bigoplus_{V \in S(W)} M_V. \tag{3.6}$$

Definition 3.21. *For a word $W = w_1 \dots w_n \in \langle Z \rangle$ we define the set of* specializations *of W by*

$$S(W) := \{v_1 \dots v_n \in \langle X \rangle \mid \forall i : v_i \in S(w_i)\}.$$

Remark 3.22. *The reader should note that for any $V \in \langle X \rangle$ and $W \in \langle Z \rangle$ we have either $M_V \cap M_W = \{0\}$ or $M_V \subseteq M_W$. Furthermore, the specializations of $W \in \langle Z \rangle$ are also given by*

$$S(W) = \{V \in \langle X \rangle \mid M_V \subseteq M_W\}.$$

Definition 3.3 carries over by replacing X with Z. For such a reduction system Σ on Z we define the reduction ideal I_Σ by (3.4) and we define $\langle X \rangle_{\text{irr}}$ as the set of words from $\langle X \rangle$ containing no subwords from the set

$$\bigcup_{(W,h) \in \Sigma} S(W).$$

Based on $\langle X \rangle_{\text{irr}}$ we define $\mathcal{K}\langle M \rangle_{\text{irr}}$ as in (3.3). Furthermore, for every reduction system Σ on Z we call its reformulation as a reduction system on X the *refined reduction system* Σ_X, which is given by

$$\Sigma_X := \bigcup_{(W,h) \in \Sigma} \{(V, h|_{M_V}) \mid V \in S(W)\}. \tag{3.7}$$

Lemma 3.23. *Let Σ be a reduction system on Z and let Σ_X be its refinement on X. Then the reduction ideals and the irreducible words are the same for Σ and for Σ_X. Moreover, also $\mathcal{K}\langle M \rangle_{\text{irr}}$ stays the same.*

Proof. Follows immediately from the definitions. □

Remark 3.24. *In the refined reduction relation $\to_{\Sigma_X}$ every reduction $h_{A,r,B}$ is given by a reduction rule r and the words $A, B \in \langle X \rangle$. Consequently, the refined reduction system Σ_X does not define the same reduction relation induced by the reduction system Σ, even when Σ and Σ_X are equal. In general, we neither have $\to_\Sigma \subseteq \to_{\Sigma_X}$ nor $\to_{\Sigma_X} \subseteq \to_\Sigma$, but $\to_\Sigma \subseteq \overset{*}{\to}_{\Sigma_X}$ holds.*

Example 3.25. *In Example 3.17, by fixing a direct complement $\mathcal{R} = \mathcal{K} \oplus \tilde{\mathcal{R}}$ in $\mathcal{R}$, we can define the $\mathcal{K}$-bimodule*

$$M_{\mathsf{R}} := M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}}$$

where $M_{\mathsf{K}} := \mathcal{K}$ and $M_{\tilde{\mathsf{R}}} := \tilde{R}$. We define two alphabets

$$X = \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{D}\}, \quad Z = X \cup \{\mathsf{R}\},$$

and the $\mathcal{K}$*-bimodule*

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{D}}.$$

On the alphabets above, we consider the reduction systems

$$\Sigma_X = \{r_{\mathsf{K}}, r_{\mathsf{KK}}, r_{\mathsf{K}\tilde{\mathsf{R}}}, r_{\tilde{\mathsf{R}}\mathsf{K}}, r_{\tilde{\mathsf{R}}\tilde{\mathsf{R}}}, r_{\mathsf{DK}}, r_{\mathsf{D}\tilde{\mathsf{R}}}\},\ \Sigma = \{r_{\mathsf{K}}, r_{\mathsf{RR}}, r_{\mathsf{DR}}\},$$

respectively. It is easy to check $\rightarrow_{\Sigma} \not\subseteq \rightarrow_{\Sigma_X}$*. For instance, for the reduction rule* r_{K} *in* Σ*,* $K \in M_{\mathsf{K}}$*, and* $B \in M_{\mathsf{R}}$ *we have* $h_{\epsilon, r_{\mathsf{K}}, \mathsf{R}}(K \otimes B) = KB$*, whereas there does not exist any reduction rule in* Σ_X *with the same result. Moreover, for the reduction rule* $r_{\tilde{\mathsf{R}}\tilde{\mathsf{R}}}$ *in* Σ_X *and* $A = K_1 + \tilde{A}, B = K_2 + \tilde{B}$ *with* $K_1, K_2 \in M_{\mathsf{K}}$ *and* $\tilde{A}, \tilde{B} \in M_{\tilde{\mathsf{R}}}$*, the reduction* $h_{\epsilon, r_{\tilde{\mathsf{R}}\tilde{\mathsf{R}}}, \epsilon}$ *reduces*

$$A \otimes B = K_1 \otimes K_2 + K_1 \otimes \tilde{B} + \tilde{A} \otimes K_2 + \tilde{A} \otimes \tilde{B}$$

to the tensor

$$K_1 \otimes K_2 + K_1 \otimes \tilde{B} + \tilde{A} \otimes K_2 + \tilde{A}\tilde{B},$$

which is not obtained by applying any rule in Σ *and thus* $\rightarrow_{\Sigma_X} \not\subseteq \rightarrow_{\Sigma}$*.*

Example 3.26. *Assume that in Example 3.25 we have the reduction systems* $\Sigma = \Sigma_X = \{r_{\mathsf{K}}\}$*. Then for the reduction rule* r_{K}*,* $K \in M_{\mathsf{K}}$*, and* $B \in M_{\mathsf{R}}$ *we have* $h_{\epsilon, r_{\mathsf{K}}, \mathsf{R}}(K \otimes B) = KB$*, which can not be obtained by applying any reduction rule in* Σ_X*. Therefore, we conclude that* $\rightarrow_{\Sigma_X} \neq \rightarrow_{\Sigma}$*.*

Definition 3.27. *A partial order* $\leq$ *on* $\langle Z \rangle$ *is called* consistent with specialization *if for all words* $V, W \in \langle Z \rangle$ *with* $V < W$ *we also have* $\tilde{V} < \tilde{W}$ *for all specializations* $\tilde{V} \in S(V)$ *and* $\tilde{W} \in S(W)$*.*

As a consequence of the above definition a word W is incomparable to all elements in $S(W)$, except possibly W itself, which can be viewed by considering the two cases $V \in S(W)$ and $W \in S(V)$ in the definition.

A semigroup partial order $\leq$ on $\langle Z \rangle$ is called *compatible* with a reduction system Σ on Z if for all $(W, h) \in \Sigma$ we have

$$h(M_W) \subseteq \sum_{\substack{V \in \langle Z \rangle \\ V < W}} M_V.$$

If $\leq$ is consistent with specialization, then for any $\tilde{W} \in S(W)$ we have

$$\sum_{\substack{V \in \langle Z \rangle \\ V < W}} M_V \subseteq M_{<\tilde{W}}.$$

Lemma 3.28. *Let Σ be a reduction system on Z and let $\leq$ be a semigroup partial order on $\langle Z \rangle$ consistent with specialization and compatible with Σ. Then the restricted order $\leq$ on $\langle X \rangle$ is compatible with Σ_X.*

Proof. By definition of Σ_X, for each reduction rule $(\tilde{W}, \tilde{h}) \in \Sigma_X$ there exists $(W, h) \in \Sigma$ such that $\tilde{W} \in S(W)$ and $\tilde{h} = h|_{M_{\tilde{W}}}$. Therefore, by the assumptions, we have

$$\tilde{h}(M_{\tilde{W}}) = h(M_{\tilde{W}}) \subseteq h(M_W) \subseteq \sum_{\substack{V \in \langle Z \rangle \\ V < W}} M_V \subseteq M_{<\tilde{W}}. \qquad \square$$

In fact the sum $\mathcal{K}\langle M \rangle = \sum_{W \in \langle Z \rangle} M_W$ is not necessarily direct anymore. Taking this into account, we require to generalize the notion of ambiguities.

Definition 3.29. *Let $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ be two (not necessarily distinct) reduction rules and let $A, B_1, B_2, C \in \langle Z \rangle$ be nonempty words with*

$$W = AB_1, \quad \tilde{W} = B_2C, \quad \text{and} \quad S(B_1) \cap S(B_2) \neq \emptyset,$$

then we call this an overlap ambiguity. *An overlap ambiguity is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ the S-polynomial can be reduced to zero:*

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_{\Sigma} 0.$$

Similarly, an inclusion ambiguity *is given by two distinct reduction rules $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and words $A, B_1, B_2, C \in \langle Z \rangle$ with $W = B_1$, $\tilde{W} = AB_2C$, and $S(B_1) \cap S(B_2) \neq \emptyset$. An inclusion ambiguity is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ the S-polynomial can be reduced to zero: $a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \xrightarrow{*}_{\Sigma} 0$.*

If $B_1 \neq B_2$ for an overlap or inclusion ambiguity, then we say that the ambiguity is with specialization.

Again, we use $\mathrm{SP}(A\underline{B_1}, \underline{B_2}C)$ or $\mathrm{SP}(\underline{B_1}, A\underline{B_2}C)$, respectively, to refer to S-polynomials of an overlap or inclusion ambiguity.

Remark 3.30. *The reader should note that in total there now can be four types of ambiguities: in addition to the two types of ambiguities (without specialization) of Definition 3.7 there are also corresponding versions with specialization as defined above.*

Corresponding to Definition 3.8 for the word monoid $\langle Z \rangle$, we have the following definition.

Definition 3.31. *An overlap ambiguity (with specialization) given by (not necessarily distinct) reduction rules* (W, h) *and* $(\tilde{W}, \tilde{h})$ *in* Σ *and nonempty words* $A, B_1, B_2, C \in \langle Z \rangle$*, with* $W = AB_1$*,* $\tilde{W} = B_2C$*, and* $S(B_1) \cap S(B_2) \neq \emptyset$ *is called* j-resolvable *if for all pure tensors* $a \in M_A$*,* $b \in M_{B_1} \cap M_{B_2}$*, and* $c \in M_C$ *there exists a tensor* $t \in \mathcal{K}\langle M \rangle$ *such that*

$$h(a \otimes b) \otimes c \xrightarrow{*}_\Sigma t \xleftarrow{*}_\Sigma a \otimes \tilde{h}(b \otimes c).$$

Similarly, an inclusion ambiguity (with specialization) given by reduction rules (W, h) *and* $(\tilde{W}, \tilde{h})$ *in* Σ *and nonempty words* $A, B_1, B_2, C \in \langle Z \rangle$*, with* $W = B_1$*,* $\tilde{W} = AB_2C$*, and* $S(B_1) \cap S(B_2) \neq \emptyset$ *is called* j-resolvable *if for all pure tensors* $a \in M_A$*,* $b \in M_{B_1} \cap M_{B_2}$*, and* $c \in M_C$ *there exists a tensor* $t \in \mathcal{K}\langle M \rangle$ *such that*

$$a \otimes h(b) \otimes c \xrightarrow{*}_\Sigma t \xleftarrow{*}_\Sigma \tilde{h}(a \otimes b \otimes c).$$

Considering the definition above, one can prove the following generalization of Bergman's result. For demonstrating properties of the reduction system Σ on Z, we apply Bergman's result (Theorem 3.15) to the refined reduction system Σ_X on X.

Theorem 3.32. *Let* M *be a* $\mathcal{K}$*-bimodule and let* $(M_z)_{z \in Z}$ *be a decomposition with specialization. Let* Σ *be a reduction system on* Z *over* $\mathcal{K}\langle M \rangle$ *and let* $\leq$ *be a Noetherian semigroup partial order on* $\langle Z \rangle$ *consistent with specialization and compatible with* Σ*. Then the following are equivalent:*

(*i*) *All ambiguities of* Σ *are resolvable.*

(*ii*) *All ambiguities of* Σ *are j-resolvable.*

(*iii*) *Every* $t \in \mathcal{K}\langle M \rangle$ *has a unique normal form* $t \downarrow_\Sigma$*.*

(*iv*) $\mathcal{K}\langle M \rangle / I_\Sigma$ *and* $\mathcal{K}\langle M \rangle_{\text{irr}}$ *are isomorphic as* $\mathcal{K}$*-bimodules.*

Moreover, if these conditions are satisfied, then we can define a multiplication on $\mathcal{K}\langle M \rangle_{\text{irr}}$ *by* $s \cdot t := (s \otimes t)\downarrow_\Sigma$ *so that* $\mathcal{K}\langle M \rangle / I_\Sigma$ *and* $\mathcal{K}\langle M \rangle_{\text{irr}}$ *are isomorphic as* $\mathcal{K}$*-rings.*

Proof. First we prove the implication (*iii*) $\Rightarrow$ (*i*). Any S-polynomial of an ambiguity of Σ is of the form $h(t) - \tilde{h}(t)$ for some pure tensor $t \in \mathcal{K}\langle M \rangle$ and reductions h and $\tilde{h}$ of Σ. Let ρ and ρ' be compositions of reductions of Σ such that $\rho(h(t)) \in \mathcal{K}\langle M \rangle_{\text{irr}}$ and $\rho'(\rho(\tilde{h}(t))) \in \mathcal{K}\langle M \rangle_{\text{irr}}$. Since t has a unique normal form w.r.t. Σ then $\rho'(\rho(h(t))) = \rho(h(t)) = \rho'(\rho(\tilde{h}(t)))$ and thus

$$\rho'(\rho(h(t) - \tilde{h}(t))) = \rho'(\rho(h(t))) - \rho'(\rho(\tilde{h}(t))) = 0.$$

For proving the implication $(i) \Rightarrow (ii)$, consider the overlap ambiguity given by $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and nonempty words $A, B_1, B_2, C \in \langle Z \rangle$ where

$$W = AB_1, \quad \tilde{W} = B_2C, \quad \text{and} \quad S(B_1) \cap S(B_2) \neq \emptyset.$$

By (i), for all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ the corresponding S-polynomial can be reduced to zero:

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_{\Sigma} 0.$$

In other words, there is a $\mathcal{K}$-bimodule homomorphism ρ inducing a chain of reduction rules in Σ such that

$$\rho(h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c)) = 0.$$

Therefore, we have

$$\rho(h(a \otimes b) \otimes c) = \rho(a \otimes \tilde{h}(b \otimes c)) =: t$$

which allows us to conclude

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} a \otimes \tilde{h}(b \otimes c).$$

The idea of proof for inclusion ambiguities $W = B_1$ and $\tilde{W} = AB_2C$ is analogous.

The rest of the proof is reduced to Theorem 3.15 via properties of the refined reduction system Σ_X. Lemma 3.23 shows that we can replace the reduction system Σ by its refinement Σ_X without changing the reduction ideal or $\mathcal{K}\langle M \rangle_{\mathrm{irr}}$, hence statement (iv) holds for Σ if and only if it holds for Σ_X. Furthermore, we note that every S-polynomial of Σ_X is also an S-polynomial of Σ and that $\xrightarrow{*}_{\Sigma} \subseteq \xrightarrow{*}_{\Sigma_X}$, hence statement (i) holds for Σ_X if it holds for Σ. Analogously, if for the words $A, B_1, B_2, C \in \langle Z \rangle$ and all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ there exists a tensor $t \in \mathcal{K}\langle M \rangle$ such that

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} a \otimes \tilde{h}(b \otimes c) \quad \text{or} \quad a \otimes h(b) \otimes c \xrightarrow{*}_{\Sigma} t \xleftarrow{*}_{\Sigma} \tilde{h}(a \otimes b \otimes c)$$

then by the fact that $\xrightarrow{*}_{\Sigma} \subseteq \xrightarrow{*}_{\Sigma_X}$, we conclude that

$$h(a \otimes b) \otimes c \xrightarrow{*}_{\Sigma_X} t \xleftarrow{*}_{\Sigma_X} a \otimes \tilde{h}(b \otimes c) \quad \text{or} \quad a \otimes h(b) \otimes c \xrightarrow{*}_{\Sigma_X} t \xleftarrow{*}_{\Sigma_X} \tilde{h}(a \otimes b \otimes c)$$

Hence statement (ii) holds for Σ_X if it holds for Σ.

If statement (iii) holds for Σ_X, then by $\xrightarrow{*}_{\Sigma} \subseteq \xrightarrow{*}_{\Sigma_X}$ and the fact that $\mathcal{K}\langle M \rangle_{\mathrm{irr}}$ does not change it also holds for Σ. Finally, Lemma 3.28 implies that Σ_X and the restriction of $\leq$ to $\langle X \rangle$ satisfy the assumptions of Theorem 3.15, which concludes the proof. $\square$

Remark 3.33. *Note that for ambiguities with specializations in $\langle Z \rangle$ it is not clear how to define notion of resolvability with respect to an order $\leq$, since in this case we do not know what is the corresponding $\mathcal{K}$-bimodule to $I_{\Sigma,ABC}$ in Definition 3.10, such that we can claim that resolvability with respect to the order holds for Σ_X if it holds for Σ. Therefore, in this thesis, we do not phrase in Theorem 3.32 analogue of statement (iii) in Theorem 3.15.*

Note that for $W, \tilde{W} \in \langle Z \rangle$ having a common specialization, i.e. $S(W) \cap S(\tilde{W}) \neq \emptyset$, there does not necessarily exist $V \in \langle Z \rangle$ such that $S(V) = S(W) \cap S(\tilde{W})$. In general, the intersection of two modules is given by

$$M_W \cap M_{\tilde{W}} = \bigoplus_{V \in S(W) \cap S(\tilde{W})} M_V = \bigotimes_{k=1}^{n} \bigoplus_{x \in S(w_k) \cap S(\tilde{w}_k)} M_x,$$

where $W = w_1 \ldots w_n$ and $\tilde{W} = \tilde{w}_1 \ldots \tilde{w}_n$.

Example 3.34. *Consider alphabets $X = \{x_1, x_2, x_3\}$ and $Z = X \cup \{y_1, y_2\}$ with bimodules $M_{y_1} = M_{x_1} \oplus M_{x_3}$ and $M_{y_2} = M_{x_2} \oplus M_{x_3}$. The words $W = x_1 y_2 y_1$ and $\tilde{W} = y_1 y_2 y_2$ in $\langle Z \rangle$ satisfy $S(W) \cap S(\tilde{W}) = \{x_1 x_2 x_3, x_1 x_3 x_3\} \neq \emptyset$. We have $M_W \cap M_{\tilde{W}} = M_{x_1} \otimes M_{y_2} \otimes M_{x_3}$. So, in this case, there even exists a word $V = x_1 y_2 x_3$ that satisfies $S(V) = S(W) \cap S(\tilde{W})$ and $M_V = M_W \cap M_{\tilde{W}}$.* □

Example 3.35. *Consider alphabets $X = \{x_1, x_2, x_3, x_4\}$ and $Z = X \cup \{y_1, y_2\}$ with $S(y_i) = X \setminus \{x_{5-i}\}$. The words $W = y_1$ and $\tilde{W} = y_2$ satisfy*

$$S(W) \cap S(\tilde{W}) = \{x_1, x_2\} \neq \emptyset$$

and there is no word V with $S(V) = S(W) \cap S(\tilde{W})$. □

In order to describe the intersection of modules in terms of words again it will be convenient to also consider another partial order $\preceq$ on $\langle Z \rangle$, which is induced by the natural partial order, given by set inclusion, on all sets of the form $S(W) \subseteq \langle X \rangle$. In other words, we have $V \preceq W$ in $\langle Z \rangle$ if and only if $S(V) \subseteq S(W)$, which holds if and only if M_V is contained in M_W.

In addition, for a set $S \subseteq \langle Z \rangle$ we define the $\mathcal{K}$-bimodule

$$M_S := \sum_{W \in S} M_W \subseteq \mathcal{K}\langle M \rangle \tag{3.8}$$

with M_S being the trivial bimodule $\{0\}$ if S is empty. We also define

$$lb(S) := \{V \in \langle Z \rangle \mid V \preceq W \text{ for all } W \in S\}$$

as the set of all lower bounds of S with respect to the partial order $\preceq$. Note that this implies

$$\bigcap_{W\in S} M_W = M_{lb(S)} = M_{lb(S)\cap\langle X\rangle}$$

where we have $lb(S)\cap\langle X\rangle = \bigcap_{W\in S} S(W)$. If $\preceq$ satisfies the ascending chain condition, it is enough to consider only maximal elements of $lb(S)$ for $\bigcap_{W\in S} M_W = M_{lb(S)}$.

Example 3.36. *Consider alphabets $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Z = X \cup \{y_1, y_2, y_3, z_1, z_2\}$ with $S(y_i) = \{x_i, x_{i+1}\}$ and $S(z_i) = X \setminus \{x_{7-i}\}$. The words $W = z_1$ and $\tilde{W} = z_2$ satisfy $S(W) \cap S(\tilde{W}) = \{x_1, x_2, x_3, x_4\} \neq \emptyset$ and there is no word V with $S(V) = S(W)\cap S(\tilde{W})$. We have $lb(W, \tilde{W}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$ and the maximal elements of $lb(W, \tilde{W})$ are y_1, y_2, y_3. As explained above, we have $M_W \cap M_{\tilde{W}} = M_{lb(W,\tilde{W})} = M_{lb(W,\tilde{W})\cap\langle X\rangle} = M_{\{y_1,y_2,y_3\}}$. In this example, we can even find words such that the intersection is a direct sum of as few modules as possible: $M_W \cap M_{\tilde{W}} = M_{y_1} \oplus M_{y_3}$.* □

Remark 3.37. *As a special case of the above tensor setting, we can briefly explain the multi-level tensor setting presented in [30, Subsec. 3.1] as follows. Consider a family of alphabets $(X_i)_{i\in I}$ such that each corresponds to a direct sum decomposition $M = \bigoplus_{x\in X_i} M_x$, the "levels". On the index set I we can define a partial order $\preceq$ such that $i \preceq j$ if and only if $(M_x)_{x\in X_i}$ is a refinement of $(M_x)_{x\in X_j}$, i.e. there exists a partition $(X_{x_j})_{x_j\in X_j}$ of X_i such that*

(i) $X_{x_j} = \{x_i\}$ for all $x_j \in X_i \cap X_j$ and

(ii) $M_{x_j} = \bigoplus_{x_i\in X_{x_j}} M_{x_i}$ for all $x_j \in X_j$.

The set I is required to have a least element $0 \in I$ w.r.t. $\preceq$, i.e. there exists a finest level that is a refinement of all levels. Defining $X := X_0$ and $Z := \bigcup_{i\in I} X_i$ we easily recognize this as a special case of the above tensor setting with specialization. The setting is worthwhile because of the following property. If $\preceq$ is a total order on I, i.e. if all levels are nested, then for any $W, \tilde{W} \in \langle Z\rangle$ with $S(W)\cap S(\tilde{W}) \neq \emptyset$, there exists (at least one) $V \in \langle Z\rangle$ such that $S(V) = S(W) \cap S(\tilde{W})$, i.e. $M_V = M_W \cap M_{\tilde{W}}$.

Remark 3.38. *Analogous to Buchberger's algorithm [7] and Knuth-Bendix completion [32], the completion process in our setting is done by adding new rules corresponding to non-resolvable ambiguities (S-polynomials resp. critical pairs); see also [8]. Deciding existence of finite Gröbner bases and the undecidability of the word problem are obstructions for general algorithms inherited from the noncommutative polynomial algebra case [39].*

Remark 3.39. *The method we follow for completing tensor reduction systems involves also non-algorithmic steps, in addition to semi-decision algorithms used in noncommutative Gröbner basis computations and Knuth-Bendix completion. One of the problematic cases is to define a new reduction homomorphism based on the S-polynomials of a non-resolvable ambiguity. As for verification of confluence, a compatible semigroup partial order is sufficient, as one can also start the completion process with a compatible semigroup partial order instead of a total one. Extending this order in a compatible way may not always be possible.*

3.3.1 Computational aspects and implementation

Many properties that we discussed for Bergman's tensor setting also hold for the tensor setting with specialization we introduced above. For instance, determining ambiguities and irreducible words is done just on the level of words. In the following, we discuss the differences of the two settings.

The main computational benefit of Theorem 3.32 compared to Theorem 3.15 lies in the fact that for the confluence criterion, we only need to check ambiguities of Σ on the alphabet Z and no computations with Σ_X are needed. Computing with the refined reduction system on X instead, would generally lead to a higher number of ambiguities, since one reduction rule in Σ can give rise to many reduction rules in Σ_X. Only for determination of irreducible words we restrict to $\langle X \rangle$.

If we formulate our reduction system Σ on the alphabet Z, instead of using some $\tilde{\Sigma}$ on the smaller alphabet X for the same reduction ideals $I_{\tilde{\Sigma}} = I_\Sigma$, we may be able to considerably reduce the size of the reduction system. This may happen in two different ways. Firstly, assume a partition of X such that some homomorphisms in $\tilde{\Sigma}$ are defined by the same formula and the homomorphisms differ only by the choice of their domain and the corresponding words are obtained as specializations from some template. Then the corresponding reduction rules from $\tilde{\Sigma}$ could be merged into one reduction rule in Σ. This is exactly what happens for $\tilde{\Sigma} = \Sigma_X$. Secondly, also extending the domain of some homomorphism from $\tilde{\Sigma}$ may contribute to obtaining a smaller reduction system Σ. So usually we will have $\tilde{\Sigma} \subset \Sigma_X$.

By means of the Mathematica package `TenReS` we can discover all overlap and inclusion ambiguities with specialization together with their corresponding S-polynomials. Therefore, it supports completing a given reduction system to a confluent one, based on S-polynomial computations. In addition, it can be used for other purposes such as verification of the confluence criterion, computing normal forms for a given ring of operators, proving operator identities, and so on.

Chapter 4

Integro-differential operators

Our tensor setting with specialization, described in Section 3.3, is flexible enough to model integro-differential operators where constants are not commutative, see Section 4.1. Moreover, working with tensors we do not need to fix a basis for the coefficient ring. Integro-differential rings over a field of constants have been introduced in [51], where they are used for multiplying and factoring linear boundary problems for ordinary differential equations. In fact, one of the main applications of integro-differential operators is that they describe the differential equation, boundary conditions and the solution operator (Green's operator) of a linear boundary problem in a uniform language.

In order to illustrate computations in these rings, we verify algebraically the variation of constants method. For finding solutions of differential equations and linear boundary problems, we need to apply integro-differential operators to functions from the left. More precisely, we require a left module over the ring of integro-differential operators. In Section 4.2, we show that any integro-differential ring is a left module over the corresponding ring of operators.

For obtaining normal forms in the ring of integro-differential operators, we complete the defining reduction system to a confluent one, see Section 4.3. Computations with normal forms can be used to partly automatize solving operator equations. In many applications, for finding solutions of differential systems, we have to apply operators to vectors of functions. In Section 4.4, we discuss vector-valued "functions" as a left module over a ring of integro-differential operators, where coefficients are matrices of "functions".

Another application of our tensor setting, explained in in Section 4.5, is the ring of integro-differential operators with linear substitutions. This ring provides an algebraic framework for solving time-delay differential equations. Like in the previous example, we discover a confluent reduction system and the corresponding normal forms for this ring. We illustrate how, by elementary computations in this framework, results like the method of steps can be found and proven in an automated way. For further references on the rings of IDO and IDOLS, see also Section 1.1.

4.1 Integro-differential rings and operators

Bergman's tensor setting can be used for defining the ring of integro-differential operators (IDO) over an arbitrary integro-differential ring. It is an algebraic structure with (matrices of) coefficients and the operations differentiation, integration, and evaluation applying on them; recall that by the fundamental theorem of calculus, for matrix-valued function $A(t)$, we have

$$\frac{d}{dt}\int_{t_0}^{t} A(s)\,ds = A(t),$$

and the evaluation at t_0 can be expressed in terms of differentiation and integration as follows:

$$A(t_0) = A(t) - \int_{t_0}^{t} A'(s)\,ds.$$

Moreover, the evaluation at t_0 of a product is the product of the individual evaluations. In other words, evaluation at t_0 is a multiplicative operation. Based on these properties, we define an integro-differential ring analogous to the definition of an integro-differential algebra in [52, 23].

Definition 4.1. *Let $(\mathcal{R}, \partial)$ be a differential ring with ring of constants $\mathcal{K}$ such that $\partial\mathcal{R} = \mathcal{R}$. Moreover, let $\textstyle\int : \mathcal{R} \to \mathcal{R}$ be a $\mathcal{K}$-bimodule homomorphism satisfying the identity*

$$\partial \textstyle\int A = A, \tag{4.1}$$

for all $A \in \mathcal{R}$. We call $(\mathcal{R}, \partial, \textstyle\int)$ an integro-differential ring *if the* evaluation

$$\mathrm{E}A := A - \textstyle\int \partial A \tag{4.2}$$

is multiplicative, i.e. for all $A, B \in \mathcal{R}$ we have

$$\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B.$$

Analogous to Example 2.61, where we obtained a differential ring from a commutative differential ring, we can always construct an integro-differential ring whose coefficients are matrices with entries in a commutative integro-differential ring.

Example 4.2. *Let* $(\mathcal{S}, \partial, \int)$ *be a commutative integro-differential ring and let* $\mathcal{R} = M_n(\mathcal{S})$*. Recall from Example 2.61 that the ring* $\mathcal{R}$ *together with the map* $\partial\colon \mathcal{R} \to \mathcal{R}$*, defined by* $\partial A = (\partial a_{ij})$ *for any* $A = (a_{ij}) \in \mathcal{R}$ *and* $i, j = 1, \ldots, n$*, is a differential ring with ring of constants*

$$\mathcal{K} = \{(a_{ij}) \mid a_{ij} \in \mathcal{S} \text{ and } \partial(a_{ij}) = 0\}.$$

We define a map $\int\colon \mathcal{R} \to \mathcal{R}$ *componentwise by*

$$\textstyle\int A = (\int a_{ij}).$$

Then the map $\int$ *satisfies* (4.1)*: let* $A = (a_{ij})$ *be an arbitrary element of* $\mathcal{R}$*. Since for any* $a_{i,j} \in \mathcal{S}$*, we have* $\partial \int a_{ij} = a_{ij}$ *then* $\partial \int A = A$*. Moreover, the map* $\mathrm{E}\colon \mathcal{R} \to \mathcal{K}$ *defined by*

$$\mathrm{E}A = (\mathrm{E}a_{ij})$$

is multiplicative: for $A = (a_{ij})$ *and* $B = (b_{ij})$*, if* $AB = (c_{ij})$ *then* $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ *and hence*

$$\mathrm{E}c_{ij} = \mathrm{E}(\sum_{k=1}^{n} a_{ik}b_{kj}) = \sum_{k=1}^{n} \mathrm{E}a_{ik}b_{kj} = \sum_{k=1}^{n} (\mathrm{E}a_{ik})\mathrm{E}b_{kj}.$$

This implies $\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B$ *and thus* $(\mathcal{R}, \partial, \int)$ *is an integro-differential ring over its ring of constants.*

Remark 4.3. *In practice, most of the time, it is enough to compute in the free integro-differential ring that is generated by the expressions occurring in the systems under consideration. In addition, identifying more specific relations among the generators, enable us to compute modulo additional identities taken from the integro-differential ideal generated by those relations. This is the approach taken in all our examples and in our package. Formally, the free ring of IDO is constructed by considering the term algebra on the set of generators modulo the identities that hold in any integro-differential ring. See, for example, [3, Ch. 3] or [16, Ch. 1] for details on the general construction of free algebraic structures in universal algebra.*

Example 4.4. *To model a fundamental system of the equation*

$$\dot{x}(t) - A(t)x(t) = 0,$$

we assume $\Phi \in \mathcal{R}$ *is invertible and satisfies* $\partial\Phi - A\Phi = 0$*. In order to compute with this* Φ*, we consider the free integro-differential ring generated by the* *symbols A, Φ, Φ^{-1}. We consider a factor ring of this free ring by computing modulo the additional identities*

$$\partial\Phi - A\Phi = 0, \quad \Phi\Phi^{-1} = 1, \quad \Phi^{-1}\Phi = 1, \quad \partial(\Phi^{-1}) = -\Phi^{-1}(\partial\Phi)\Phi^{-1}.$$

Hence, we can compute $\partial(\Phi^{-1}) = -\Phi^{-1}A$. So, this identity also holds in $\mathcal{R}$.

The next lemma shows that in any integro-differential ring, the evaluation E acts as the identity on the constants, in particular, it is also a homomorphism of rings with unity. In addition, we can decompose the ring $\mathcal{R}$ as the direct sum of constant and non-constant "functions".

Lemma 4.5. *Let $(\mathcal{R}, \partial, \textstyle\int)$ be an integro-differential ring with ring of constants $\mathcal{K}$. Then, we have $\mathrm{E}1 = 1$, $\mathrm{E}A \in \mathcal{K}$ for all $A \in \mathcal{R}$, and*

$$\mathcal{R} = \mathcal{K} \oplus \textstyle\int\mathcal{R},$$

as direct sum of $\mathcal{K}$-bimodules.

Proof. We first compute $\mathrm{E}1 = 1 - \int\partial 1 = 1$. Then, for all $A \in \mathcal{R}$, we see that

$$\partial\mathrm{E}A = \partial(A - \textstyle\int\partial A) = \partial A - \partial A = 0.$$

For proving the second assertion, we have

$$A = \mathrm{E}A + A - \mathrm{E}A = \mathrm{E}A + \textstyle\int\partial A,$$

for any $A \in \mathcal{R}$ and hence $\mathcal{R} = \mathcal{K} + \int\mathcal{R}$. Finally, let $A \in \mathcal{K} \cap \int\mathcal{R}$ and $B \in \mathcal{R}$ such that $A = \int B$. Then $0 = \partial A = \partial\int B = B$, which implies $A = 0$. □

For the rest of this section, we fix an arbitrary integro-differential ring $(\mathcal{R}, \partial, \int)$ with ring of constants $\mathcal{K}$. By an operator, we understand in the following a $\mathcal{K}$-bimodule homomorphism from $\mathcal{R}$ to $\mathcal{R}$. For instance, the operations $\partial, \int, \mathrm{E}$ can be viewed as operators.

Following Lemma 4.5, we consider the direct sum decomposition $\mathcal{R} = \mathcal{K} \oplus \int\mathcal{R}$ and the corresponding $\mathcal{K}$-bimodules

$$M_{\mathsf{K}} = \mathcal{K} \quad \text{and} \quad M_{\tilde{\mathsf{R}}} = \textstyle\int\mathcal{R} \tag{4.3}$$

(indexed by the letters K and $\tilde{\mathsf{R}}$). One should note we do not interpret the elements of M_{K} and $M_{\tilde{\mathsf{R}}}$ as functions but as left multiplication operators $B \mapsto AB$ induced by those functions. For studying linear boundary problems algebraically, we are also required to deal with other multiplicative "functionals" on $\mathcal{R}$ with the same properties as E. We consider the set

$$\Phi := \{\varphi\colon \mathcal{R} \to \mathcal{K} \mid \varphi \text{ is a } \mathcal{K}\text{-bimodule homomorphism with } \varphi AB = (\varphi A)\varphi B \text{ and } \varphi 1 = 1\}. \tag{4.4}$$

It is also possible to define Φ as a proper subset (containing E) of the full set defined above which leads us to work with a smaller ring of operators later. For the operators ∂, $\int$, E, and $\varphi \in \tilde{\Phi}$ with $\tilde{\Phi} = \Phi \setminus \{\mathrm{E}\}$, we consider the free left $\mathcal{K}$-modules

$$M_{\mathsf{D}} = \mathcal{K}\partial, \quad M_{\mathsf{I}} = \mathcal{K}{\textstyle\int}, \quad M_{\mathsf{E}} = \mathcal{K}\mathrm{E}, \quad M_{\tilde{\mathsf{\Phi}}} = \mathcal{K}\tilde{\Phi} \tag{4.5}$$

generated by them (indexed by the letters D, I, E, and $\tilde{\mathsf{\Phi}}$). These modules together with right multiplication defined by

$$K_1\alpha \cdot K_2 = K_1K_2\alpha$$

where $\alpha \in \{\partial, \int, \mathrm{E}\} \cup \tilde{\Phi}$ and $K_1, K_2 \in \mathcal{K}$ are viewed as $\mathcal{K}$-bimodules, because of correspondence between the generators of these modules and left $\mathcal{K}$-linear operators. We define two alphabets

$$X = \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{D}, \mathsf{I}, \mathsf{E}, \tilde{\mathsf{\Phi}}\} \quad \text{and} \quad Z = X \cup \{\mathsf{R}, \mathsf{\Phi}\}, \tag{4.6}$$

with the $\mathcal{K}$-bimodules $(M_x)_{x\in X}$ defined in (4.3) and (4.5) as well as

$$M_{\mathsf{R}} = M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}} \quad \text{and} \quad M_{\mathsf{\Phi}} = M_{\mathsf{E}} \oplus M_{\tilde{\mathsf{\Phi}}}. \tag{4.7}$$

Now, we define the module M by

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus M_{\mathsf{I}} \oplus M_{\mathsf{\Phi}}, \tag{4.8}$$

which turns $(M_z)_{z\in Z}$ into a decomposition with specialization.

Definition 4.6. *Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring with ring of constants $\mathcal{K}$. We call*

$$\mathcal{R}\langle \partial, {\textstyle\int}, \Phi\rangle := \mathcal{K}\langle M\rangle / I_{\Sigma_0}$$

the ring of integro-differential operators*, where I_{Σ_0} is the two-sided reduction ideal induced by the reduction system*

$$\begin{aligned}\Sigma_0 = \{&(\mathsf{K}, 1 \mapsto \epsilon), (\mathsf{RR}, A \otimes B \mapsto AB), (\mathsf{\Phi R}, \varphi \otimes A \mapsto (\varphi A)\varphi),\\ &(\mathsf{\Phi\Phi}, \psi \otimes \varphi \mapsto \varphi), (\mathsf{DR}, \partial \otimes A \mapsto A \otimes \partial + \partial A), (\mathsf{D\Phi}, \partial \otimes \varphi \mapsto 0),\\ &(\mathsf{DI}, \partial \otimes {\textstyle\int} \mapsto \epsilon), (\mathsf{ID}, {\textstyle\int} \otimes \partial \mapsto \epsilon - \mathrm{E})\}.\end{aligned}$$

Remark 4.7. *According to* (3.4)*, the two sided reduction ideal I_{Σ_0} is generated by union of the set*

$$\{1 - \epsilon, \partial \otimes {\textstyle\int} - \epsilon, {\textstyle\int} \otimes \partial - \epsilon + \mathrm{E}\}$$

with the set of families

$$\begin{aligned}\{A \otimes B - AB, \varphi \otimes A - (\varphi A)\varphi,\ \psi \otimes \varphi - \varphi,\\ \partial \otimes A - A \otimes \partial - \partial A,\ \partial \otimes \varphi \mid A, B \in M_{\mathsf{R}},\ \varphi, \psi \in M_{\mathsf{\Phi}}\}.\end{aligned}$$

Remark 4.8. *In analogy to the definition of reduction homomorphisms in Section 3.2, we should make the informal definitions of reduction rules in* Σ_0 *and their consequences given in Definition 4.6 formal. For example,*

$$\beta_{\mathsf{ID}}(\textstyle\int, \partial) := \epsilon - \mathrm{E}$$

is extended to a balanced map on $M_{\mathsf{I}} \times M_{\mathsf{D}}$ *via*

$$\beta_{\mathsf{ID}}(K\textstyle\int, L\partial) := KL\beta_{\mathsf{ID}}(\textstyle\int, \partial)$$

and similarly

$$\beta_{\mathsf{IR\Phi}}(\textstyle\int, A, \varphi) := \textstyle\int A \otimes \varphi$$

with $\varphi \in \Phi$ *extended to a balanced map on* $M_{\mathsf{I}} \times M_{\mathsf{R}} \times M_{\mathsf{\Phi}}$ *by*

$$\beta_{\mathsf{IR\Phi}}(K\textstyle\int, A, \sum_i K_i\varphi_i) := \sum_i \beta_{\mathsf{IR\Phi}}(\textstyle\int, KAK_i, \varphi_i).$$

Even without introducing normal forms for the ring of IDO, it is possible to do computations by means of reduction rules in Σ_0. In the following, we provide an algebraic proof for the variation of constants method for matrices of generic size.

Example 4.9. *Consider the differential system*

$$\dot{x}(t) - A(t)x(t) = f(t)$$

where A *is a matrix in* $\mathcal{R} = C^\infty(\mathbb{R})^{n\times n}$*. The system corresponds to the operator* $L = \partial - A \in \mathcal{R}\langle\partial, \int, \Phi\rangle$*. Let* $\Phi \in \mathcal{R}$ *be an invertible solution of* $Ly = 0$*. Then the operator*

$$H := \Phi \otimes \textstyle\int \otimes \Phi^{-1}$$

is a right inverse of L *since independent of the size* n *we have*

$$\begin{aligned} L \otimes H = (\partial - A) \otimes \Phi \otimes \textstyle\int \otimes \Phi^{-1} &\rightarrow_{r_{\mathsf{RR}}} \partial \otimes \Phi \otimes \textstyle\int \otimes \Phi^{-1} - A\Phi \otimes \textstyle\int \otimes \Phi^{-1} \\ &\rightarrow_{r_{\mathsf{DR}}} \Phi \otimes \partial \otimes \textstyle\int \otimes \Phi^{-1} \rightarrow_{r_{\mathsf{DI}}} \Phi \otimes \Phi^{-1} \rightarrow_{r_{\mathsf{RR}}} \Phi\Phi^{-1} \rightarrow_{r_{\mathsf{K}}} \varepsilon. \end{aligned}$$

This is exactly the formula $x = Hf$ for a particular solution of $Lx = f$ that is obtained from a fundamental matrix by variation of constants:

$$x(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s)\, ds.$$

4.2 Integro-differential ring as an IDO-module

For every integro-differential ring $(\mathcal{R}, \partial, \textstyle\int)$, the ring $\mathcal{R}$ is a module over the corresponding ring of integro-differential operators. In the following, we elaborate on the structure of this module.

Let $\mathcal{R}\langle \partial, \textstyle\int, \Phi\rangle$ be the ring of integro-differential operators over an integro-differential ring $(\mathcal{R}, \partial, \textstyle\int)$ with ring of constants $\mathcal{K}$. The abelian group

$$\operatorname{End}(\mathcal{R}) = \{\psi \colon \mathcal{R} \to \mathcal{R} \mid \psi \text{ is an additive map}\}$$

together with composition as multiplication is a ring. We define a map $\mathcal{K} \mapsto \operatorname{End}(\mathcal{R})$ by $K \mapsto (B \mapsto KB)$. Since in particular $\mathcal{R}$ is a left $\mathcal{K}$-module, then for all $B \in \mathcal{R}$ and $K_1, K_2 \in \mathcal{K}$ we have

$$1B = B, \quad (K_1 + K_2)B = K_1B + K_2B, \quad (K_1K_2)B = K_1(K_2B)$$

and hence the map is a ring homomorphism. Therefore, by Lemma 2.46, we conclude that $\operatorname{End}(\mathcal{R})$ is a $\mathcal{K}$-ring. In the following, we define the maps $\mathcal{R} \to \operatorname{End}(\mathcal{R})$ by $A \mapsto (B \mapsto AB)$ and $\{\partial, \textstyle\int\} \cup \Phi \to \operatorname{End}(\mathcal{R})$ by

$$\partial \mapsto \partial, \quad \textstyle\int \mapsto \textstyle\int, \quad \varphi \mapsto \varphi.$$

Let $\{\partial, \textstyle\int\} \cup \Phi \to \mathcal{K}\partial \oplus \mathcal{K}\textstyle\int \oplus \mathcal{K}\Phi$ be the inclusion map. By Theorem 2.5, there exists a unique $\mathcal{K}$-module homomorphism $\mathcal{K}\partial \oplus \mathcal{K}\textstyle\int \oplus \mathcal{K}\Phi \to \operatorname{End}(\mathcal{R})$. Recall that we can view the left $\mathcal{K}$-modules $\mathcal{K}\partial$, $\mathcal{K}\textstyle\int$, and $\mathcal{K}\Phi$ as $\mathcal{K}$-bimodules with the right multiplication defined by

$$K_1\alpha \cdot K_2 = K_1K_2\alpha$$

where $K_1, K_2 \in \mathcal{K}$ and $\alpha \in \{\partial, \textstyle\int\} \cup \Phi$, since the generators of these modules correspond to left $\mathcal{K}$-linear operators. For the $\mathcal{K}$-bimodule

$$M = \mathcal{R} \oplus \mathcal{K}\partial \oplus \mathcal{K}\textstyle\int \oplus \mathcal{K}\Phi$$

let $M \to \mathcal{K}\langle M\rangle$ be the inclusion map. By Theorem 2.56, the $\mathcal{K}$-bimodule homomorphism $\tilde{\theta} \colon M \to \operatorname{End}(\mathcal{R})$ can be extended uniquely to a $\mathcal{K}$-ring homomorphism $\theta \colon \mathcal{K}\langle M\rangle \to \operatorname{End}(\mathcal{R})$ such that the following diagram is commutative.

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & \mathcal{K}\langle M\rangle \\ & {\scriptstyle\tilde{\theta}}\searrow & \downarrow{\scriptstyle\theta} \\ & & \operatorname{End}(\mathcal{R}) \end{array}$$

Since any reduction rule in Σ_0 corresponds to an identity in $\mathcal{R}$, we conclude that $I_{\Sigma_0} \subseteq \operatorname{Ker}(\theta)$. For instance, for any $A \in \mathcal{R}$ and $\varphi \in \Phi$ we have

$$\theta(\varphi \otimes A - (\varphi A)\varphi) = \varphi \circ A - (\varphi A)\varphi = 0,$$

since $(\varphi \circ A)B = \varphi(AB) = (\varphi A)\varphi B$. As another example, we have

$$\theta({\textstyle\int} \otimes \partial - \epsilon + \mathrm{E}) = {\textstyle\int} \circ \partial - \epsilon + \mathrm{E} = 0,$$

since for any $B \in \mathcal{R}$ we have $({\textstyle\int} \circ \partial)B = {\textstyle\int}\partial B = B - \mathrm{E}B$. Now, let $\pi \colon \mathcal{K}\langle M\rangle \to \mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle$ be the canonical map. By the factor theorem, there exists a unique $\mathcal{K}$-ring homomorphism $\overline{\theta} \colon \mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle \to \operatorname{End}(\mathcal{R})$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{K}\langle M\rangle & \xrightarrow{\pi} & \mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle \\ & {\scriptstyle\theta}\searrow & \downarrow{\scriptstyle\overline{\theta}} \\ & & \operatorname{End}(\mathcal{R}) \end{array}$$

Hence, the ring $\mathcal{R}$ is an $\mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle$-module.

Proposition 4.10. *The ring $\mathcal{R}$ together with the scalar multiplication*

$$\begin{aligned} \bullet \colon \mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle \times \mathcal{R} &\to \mathcal{R} \\ (\overline{L}, A) &\mapsto \overline{\theta}(\overline{L})A \end{aligned}$$

is an $\mathcal{R}\langle\partial, {\textstyle\int}, \Phi\rangle$-module.

4.3 Completion of tensor reduction system for the ring of IDO

In the following, we describe the completion process for the ring of IDO. Consider the reduction system Σ_0 given in Definition 4.6. We collect the conditions that any compatible partial order on $\langle Z\rangle$ in (4.6) has to satisfy. For simplicity, we restrict ourselves to monoid partial order where we have the condition $\epsilon < A$, for any $A \in \langle Z\rangle$. We also require to consider the additional properties $\mathsf{DR} > \mathsf{RD}$ and $\mathsf{ID} > \mathsf{E}$. Then, in order to obtain the minimal partial order which is consistent with specialization, we also have to consider $\mathsf{DK} > \mathsf{KD}$, $\mathsf{DK} > \tilde{\mathsf{R}}\mathsf{D}$, $\mathsf{D}\tilde{\mathsf{R}} > \mathsf{KD}$, and $\mathsf{D}\tilde{\mathsf{R}} > \tilde{\mathsf{R}}\mathsf{D}$.

The reduction rules r_{DI} and r_{ID} have two overlap ambiguities with each other, one is resolvable and one is not. The latter has S-polynomial

$$\mathrm{SP}(\mathsf{I}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{I}) = (\epsilon - \mathrm{E}) \otimes {\textstyle\int} - {\textstyle\int} \otimes \epsilon = -\mathrm{E} \otimes {\textstyle\int}.$$

This trivially gives rise to the new rule

$$(\mathsf{EI}, \mathrm{E} \otimes \textstyle\int \mapsto 0).$$

Remark 4.11. *In general, each time a new rule is discovered, we also add the corresponding identity in the coefficient ring $\mathcal{R}$. For instance, by applying the operator identity induced by the rule r_{EI} on any $B \in \mathcal{R}$, we have*

$$(\mathrm{E} \cdot \textstyle\int) \bullet B = 0.$$

This allows us to consider the following identity for the coefficient ring $\mathcal{R}$.

$$\mathrm{E}\textstyle\int B = 0$$

For the reduction rules r_{ID} and r_{DR}, we get a non-resolvable overlap ambiguity with S-polynomials

$$\begin{aligned} \mathrm{SP}(\mathsf{I\underline{D}}, \mathsf{\underline{D}R}) = (\epsilon - \mathrm{E}) \otimes A - \textstyle\int \otimes (A \otimes \partial + \partial A) \to_{r_{\Phi\mathsf{R}}} \\ A - (\mathrm{E}A)\mathrm{E} - \textstyle\int \otimes A \otimes \partial - \textstyle\int \otimes \partial A. \end{aligned}$$

While we could reduce further, by using r_{K} for example, we will not be able to reduce to zero for all $A \in \mathcal{R}$. Based on the expression above, however, we can introduce a new rule

$$(\mathsf{IRD}, \textstyle\int \otimes A \otimes \partial \mapsto A - (\mathrm{E}A)\mathrm{E} - \textstyle\int \otimes \partial A)$$

that allows us to reduce all the S-polynomials of the overlap ambiguity of r_{ID} and r_{DR} to zero. We should also add $\mathsf{IRD} > \mathsf{E}$ to the conditions we need for compatible partial orders. Analogous to the previous case, by applying the operator identity induced by this rule on any $B \in \mathcal{R}$, we observe that

$$(\textstyle\int \cdot A \cdot \partial) \bullet B = A \bullet B - (\mathrm{E}A)\mathrm{E} \bullet B - (\textstyle\int \cdot \partial A) \bullet B.$$

This allows us to consider the following identity for the coefficient ring $\mathcal{R}$.

$$\textstyle\int A \partial B = AB - (\mathrm{E}A)\mathrm{E}B - \textstyle\int \partial AB$$

The rule r_{IRD} gives rise to a non-resolvable overlap ambiguity with r_{DI} among others. The corresponding S-polynomials can be reduced to

$$\begin{aligned} \mathrm{SP}(\mathsf{IR\underline{D}}, \mathsf{\underline{D}I}) &= (A - (\mathrm{E}A)\mathrm{E} - \textstyle\int \otimes \partial A) \otimes \textstyle\int - \textstyle\int \otimes A \otimes \epsilon \\ &\to_{r_{\mathsf{EI}}} A \otimes \textstyle\int - \textstyle\int \otimes \partial A \otimes \textstyle\int - \textstyle\int \otimes A. \end{aligned}$$

We define a new reduction homomorphism on M_{IRI} that reduces $\int \otimes \partial A \otimes \int$ to $A \otimes \int - \int \otimes A$. We replace A by $\int A$ and then arrive at the rule

$$(\mathsf{IRI}, \textstyle\int \otimes A \otimes \int \mapsto \int A \otimes \int - \int \otimes \int A).$$

Similarly, the overlap ambiguity of r_{IRD} and $r_{\mathsf{D\Phi}}$ with S-polynomials

$$\begin{aligned} \mathrm{SP}(\mathsf{IR\underline{D}}, \mathsf{\underline{D}\Phi}) &= (A - (\mathrm{E}A)\mathrm{E} - \textstyle\int \otimes \partial A) \otimes \varphi - \int \otimes A \otimes 0 \\ &\to_{r_{\Phi\Phi}} A \otimes \varphi - (\mathrm{E}A)\varphi - \textstyle\int \otimes \partial A \otimes \varphi. \end{aligned}$$

Replacing A with $\int A$ in the S-polynomials above and then using $\mathrm{E}\int A = 0$ we obtain the rule

$$(\mathsf{IR\Phi}, \textstyle\int \otimes A \otimes \varphi \mapsto \int A \otimes \varphi)$$

We consider the inclusion ambiguity (with specialization) of the new rule r_{IRI} with r_{K}, which has irreducible S-polynomial

$$\begin{aligned} \mathrm{SP}(\mathsf{\underline{K}}, \mathsf{I\underline{R}I}) &= \textstyle\int \otimes \epsilon \otimes \int - (\int 1 \otimes \int - \int \otimes \int 1) \\ &= \textstyle\int \otimes \int - \int 1 \otimes \int + \int \otimes \int 1. \end{aligned}$$

At this point, the leading term is not determined by our partial order above. Hence, we add $\mathsf{I} > \tilde{\mathsf{R}}$ to the conditions we had for the partial orders, since we want to have the new rule

$$(\mathsf{II}, \textstyle\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1).$$

Analogously, the overlap ambiguity of $r_{\mathsf{IR\Phi}}$ and $r_{\mathsf{D\Phi}}$ has an inclusion ambiguity with r_{K}. It leads to the irreducible S-polynomials

$$\mathrm{SP}(\mathsf{\underline{K}}, \mathsf{I\underline{R}\Phi}) = \textstyle\int \otimes \epsilon \otimes \varphi - \int 1 \otimes \varphi = \int \otimes \varphi - \int 1 \otimes \varphi,$$

which gives rise to the new rule

$$(\mathsf{I\Phi}, \textstyle\int \otimes \varphi \mapsto \int 1 \otimes \varphi).$$

Thereby, we obtain the confluent reduction system Σ_{IDO} given in Table 4.1.

Reduction rules in Σ_0	
K	$1 \mapsto \epsilon$
RR	$A \otimes B \mapsto AB$
ΦR	$\varphi \otimes A \mapsto (\varphi A)\varphi$
ΦΦ	$\psi \otimes \varphi \mapsto \varphi$
DR	$\partial \otimes A \mapsto A \otimes \partial + \partial A$
DΦ	$\partial \otimes \varphi \mapsto 0$
DI	$\partial \otimes \int \mapsto \epsilon$
ID	$\int \otimes \partial \mapsto \epsilon - \mathrm{E}$
Consequences of reduction rules in Σ_0	
EI	$\mathrm{E} \otimes \int \mapsto 0$
IΦ	$\int \otimes \varphi \mapsto \int 1 \otimes \varphi$
II	$\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1$
IRΦ	$\int \otimes A \otimes \varphi \mapsto \int A \otimes \varphi$
IRD	$\int \otimes A \otimes \partial \mapsto A - \int \otimes \partial A - (\mathrm{E}A)\mathrm{E}$
IRI	$\int \otimes A \otimes \int \mapsto \int A \otimes \int - \int \otimes \int A$

Table 4.1: Reduction rules for IDO

The whole completion process for Table 4.1 can be found in the example file of the `TenReS` package. The following table presents identities in the coefficient ring $\mathcal{R}$ corresponding to the reduction rules in Σ_0 and their consequences obtained by the completion process.

Identities in $\mathcal{R}$ corresponding to reduction rules in Σ_0	
$\varphi AB = (\varphi A)\varphi B$	$\partial \int B = B$
$\psi\varphi B = \varphi B$	$\int \partial B = B - \mathrm{E}B$
$\partial AB = A\partial B + (\partial A)B$	$\partial \varphi B = 0$
Identities in $\mathcal{R}$ corresponding to consequences of reduction rules in Σ_0	
$\mathrm{E}\int B = 0$	$\int A\partial B = AB - \int (\partial A)B - (\mathrm{E}A)\mathrm{E}B$
$\int A\varphi B = (\int A)\varphi B$	$\int A \int B = (\int A)\int B - \int (\int A)B$

Table 4.2: Identities in $\mathcal{R}$ corresponding to reduction system Σ_{IDO}

Remark 4.12. *Among the identities listed in Table 4.2,* $\mathrm{E}\smallint B = 0$*, integration by parts*

$$\smallint A\partial B = AB - \smallint(\partial A)B - (\mathrm{E}A)\mathrm{E}B,$$

and the Rota-Baxter identity

$$\smallint A\smallint B = (\smallint A)\smallint B - \smallint(\smallint A)B$$

for the integral can either be verified directly or are consequences of S-polynomial computations as explained in Remark 4.11. The remaining identities follow immediately from the definitions (like multiplicativity of functionals, $\mathcal{K}$*-linearity, and the Leibniz rule).*

Example 4.13. *We continue Example 4.9 by doing computations using the confluent reduction system* Σ_{IDO}*. The equation* $Lx = f$ *is equivalent to*

$$(H \otimes L)x = Hf.$$

We easily find the irreducible form

$$\begin{aligned} H \otimes L &= \Phi \otimes \smallint \otimes \Phi^{-1} \otimes (\partial - A) \\ &\to_{r_{\mathsf{IRD}}} \Phi \otimes (\Phi^{-1} - \smallint \otimes \partial(\Phi^{-1}) - (\mathrm{E}\Phi^{-1})\mathrm{E}) - \Phi \otimes \smallint \otimes \Phi^{-1}A \\ &= 1 - \Phi\mathrm{E}\Phi^{-1} \otimes \mathrm{E}, \end{aligned}$$

where we used the identity $\partial(\Phi^{-1}) + \Phi^{-1}A = 0$ *obtained in Example 4.4. Defining the projector* $P = \Phi\mathrm{E}\Phi^{-1} \otimes \mathrm{E}$ *allows us to write* $(H \otimes L)x = Hf$ *as* $x = Px + Hf$*, which yields the general solution obtained by variation of constants:*

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)f(s)\,ds.$$

With the aim of doing computations in the ring $\mathcal{R}\langle\partial, \smallint, \Phi\rangle$, we analyse the reduction system defined by Table 4.1 according to Theorem 3.32 and determine normal forms of tensors. Following the definition in (3.7), the refined reduction system Σ_X is obtained, according to (4.7), by splitting rules whose words contain R or $\mathsf{\Phi}$ into "smaller" rules using $S(\mathsf{R}) = \{\mathsf{K}, \tilde{\mathsf{R}}\}$ and $S(\mathsf{\Phi}) = \{\mathsf{E}, \tilde{\mathsf{\Phi}}\}$. For instance, the reduction rule $(\mathsf{\Phi R}, h) \in \Sigma_{\mathrm{IDO}}$ is split into the rules $(W, h|_{M_W}) \in \Sigma_X$ where $W \in S(\mathsf{\Phi R}) = \{\mathsf{EK}, \mathsf{E}\tilde{\mathsf{R}}, \tilde{\mathsf{\Phi}}\mathsf{K}, \tilde{\mathsf{\Phi}}\tilde{\mathsf{R}}\}$.

Theorem 4.14. *Let* $(\mathcal{R}, \partial, \smallint)$ *be an integro-differential ring with constants* $\mathcal{K}$ *and let* Φ *be the set of multiplicative* $\mathcal{K}$*-bimodule homomorphisms given by* (4.4)*. Let* M *be defined by* (4.7) *and* (4.8) *and let the reduction system*

Σ_{IDO} *be defined by Table 4.1. Then every* $t \in \mathcal{K}\langle M\rangle$ *has a unique normal form* $t{\downarrow_{\Sigma_{\mathrm{IDO}}}}$*, which is given by a sum of pure tensors of the form*

$$A \otimes \varphi \otimes \partial^j \quad \text{or} \quad A \otimes \varphi \otimes \textstyle\int \otimes B$$

where $j \in \mathbb{N}_0$*, each of* $A, B \in M_{\tilde{\mathsf{R}}}$ *and* $\varphi \in \Phi$ *may be absent, and* $\varphi \otimes \int$ *does not specialize to* $\mathrm{E} \otimes \int$*. Moreover, as* $\mathcal{K}$*-rings we have*

$$\mathcal{R}\langle \partial, \textstyle\int, \Phi\rangle \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}$$

where the multiplication on $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ *is defined by* $s \cdot t := (s \otimes t){\downarrow_{\Sigma_{\mathrm{IDO}}}}$*.*

Proof. We consider the alphabets X and Z given by (4.6). This turns $(M_z)_{z\in Z}$ into a decomposition with specialization for the module M, see Definition 3.20. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ_{IDO}, it is sufficient to require the order to satisfy

$$\mathsf{DR} > \mathsf{RD},\ \mathsf{IRD} > \mathsf{E},\ \mathsf{ID} > \mathsf{E},\ \mathsf{I} > \tilde{\mathsf{R}}.$$

For instance, we could use a degree-lexicographic order with $\mathsf{I} > \mathsf{D} > \mathsf{\Phi} > \mathsf{R}$ on $\langle\{\mathsf{R}, \mathsf{D}, \mathsf{I}, \mathsf{\Phi}\}\rangle \subseteq \langle Z\rangle$ or other degree-lexicographic orders with $\mathsf{D} > \mathsf{R}$ and $\mathsf{I} > \mathsf{R}$. We extend it to a monoid partial order on $\langle Z\rangle$ based on Definition 3.27 in order to make it consistent with specialization. Then, by the package `TenReS`, we verify that all ambiguities of Σ_{IDO} are resolvable, see Subsection 4.3.1. Hence, by Theorem 3.32 every element of $\mathcal{K}\langle M\rangle$ has a unique normal form and $\mathcal{K}\langle M\rangle/I_{\Sigma_{\mathrm{IDO}}} \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}$ as $\mathcal{K}$-rings.

It remains to determine the explicit form of elements in $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\mathrm{irr}}$ in $\langle X\rangle$. Irreducible words containing only the letters K and $\tilde{\mathsf{R}}$ have to avoid the subwords K and $S(\mathsf{RR}) = \{\mathsf{KK}, \mathsf{K}\tilde{\mathsf{R}}, \tilde{\mathsf{R}}\mathsf{K}, \tilde{\mathsf{R}}\tilde{\mathsf{R}}\}$, hence only the words ϵ and $\tilde{\mathsf{R}}$ are left. The irreducible words containing only E and $\tilde{\mathsf{\Phi}}$ are exactly ϵ, E, and $\tilde{\mathsf{\Phi}}$, since they have to avoid the subwords $S(\mathsf{\Phi\Phi}) = \{\mathsf{EE}, \mathsf{E}\tilde{\mathsf{\Phi}}, \tilde{\mathsf{\Phi}}\mathsf{E}, \tilde{\mathsf{\Phi}}\tilde{\mathsf{\Phi}}\}$. Altogether, we see that the irreducible words containing only the letters K, $\tilde{\mathsf{R}}$, E, and $\tilde{\mathsf{\Phi}}$ are given by the set $\{\epsilon, \tilde{\mathsf{R}}, \mathsf{E}, \tilde{\mathsf{\Phi}}, \tilde{\mathsf{R}}\mathsf{E}, \tilde{\mathsf{R}}\tilde{\mathsf{\Phi}}\}$, since they also have to avoid the subwords $S(\mathsf{\Phi R}) = \{\mathsf{EK}, \mathsf{E}\tilde{\mathsf{R}}, \tilde{\mathsf{\Phi}}\mathsf{K}, \tilde{\mathsf{\Phi}}\tilde{\mathsf{R}}\}$. Allowing also the letter D, we have to avoid the subwords coming from $S(\mathsf{DR}) = \{\mathsf{DK}, \mathsf{D}\tilde{\mathsf{R}}\}$ and $S(\mathsf{D\Phi}) = \{\mathsf{DE}, \mathsf{D}\tilde{\mathsf{\Phi}}\}$. Therefore, we can only append words D^j with $j \in \mathbb{N}_0$ to the irreducible words determined so far, in order to obtain all elements of $\langle X\rangle_{\mathrm{irr}}$ not containing the letter I. Finally, we also consider the letter I. Since subwords EI and DI have to be avoided, the first occurrence of I in an irreducible word can only be preceded by ϵ, $\tilde{\mathsf{R}}$, $\tilde{\mathsf{\Phi}}$, or $\tilde{\mathsf{R}}\tilde{\mathsf{\Phi}}$. We also have to avoid the subwords $S(\mathsf{I\Phi}) = \{\mathsf{IE}, \mathsf{I}\tilde{\mathsf{\Phi}}\}$, ID, and II, so any letter immediately following I has to be $\tilde{\mathsf{R}}$. In addition, we have to avoid the subwords $S(\mathsf{IR\Phi}) = \{\mathsf{IKE}, \mathsf{IK}\tilde{\mathsf{\Phi}}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{E}, \mathsf{I}\tilde{\mathsf{R}}\tilde{\mathsf{\Phi}}\}$, $S(\mathsf{IRD}) = \{\mathsf{IKD}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{D}\}$, and $S(\mathsf{IRI}) = \{\mathsf{IKI}, \mathsf{I\tilde{R}I}\}$, so the letter I cannot be followed by a subword of length greater than one. Altogether, the elements of $\langle X \rangle_{\text{irr}}$ are of the form

$$\tilde{\mathsf{R}} V \mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}\tilde{\Phi}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\Phi}$, and $V \in S(\Phi) = \{\mathsf{E}, \tilde{\Phi}\}$ may be absent. The normal forms follow from (3.3). □

By means of the normal forms above, we are allowed to do coefficient comparison for operator identities in the ring of IDO.

Example 4.15. *For finding a family of right inverses for the operator $L = \partial - A$ we make the ansatz*

$$H = H_0 + H_1 \cdot \varphi_1 + H_2 \cdot \varphi_2 \cdot \partial + H_3 \cdot \textstyle\int \cdot H_4 + H_5 \cdot \varphi_3 \cdot \textstyle\int \cdot H_6.$$

In terms of normal forms given in Theorem 4.14 we can compute

$$\begin{aligned} L \cdot H = {} & (\partial H_5 - AH_5) \cdot \varphi_3 \cdot \textstyle\int \cdot H_6 + (\partial H_3 - AH_3) \cdot \textstyle\int \cdot H_4 \\ & + (\partial H_2 - AH_2) \cdot \varphi_2 \cdot \partial + (\partial H_1 - AH_1) \cdot \varphi_1 + H_0 \cdot \partial \\ & + \partial H_0 - AH_0 + H_3 H_4. \end{aligned}$$

Then, by coefficient comparison, for the blocks $H_0, H_1, H_2, H_3, H_4, H_5$ we obtain the following conditions.

$$\begin{aligned} \partial H_5 - AH_5 &= 0 \\ \partial H_3 - AH_3 &= 0 \\ \partial H_2 - AH_2 &= 0 \\ \partial H_1 - AH_1 &= 0 \\ H_0 &= 0 \\ \partial H_0 - AH_0 + H_3 H_4 &= 1 \end{aligned}$$

For solving these equations, we adjoin an invertible Φ such that $\partial\Phi - A\Phi = 0$ and let $H_1 = H_2 = H_3 = H_5 = \Phi$, $H_4 = \Phi^{-1}$. Then for an arbitrary element H_6 in $\mathcal{R}$ we obtain

$$H = \Phi \cdot \varphi_1 + \Phi \cdot \varphi_2 \cdot \partial + \Phi \cdot \textstyle\int \cdot \Phi^{-1} + \Phi \cdot \varphi_3 \cdot \textstyle\int \cdot H_6.$$

Hence, we find a family of right inverses for the operator L that contains the operator H in Example 4.9 and the Green's operator G in Theorem 5.5 as special cases.

4.3.1 Computational aspects

In the following, some computational details of the tensor setting with specialization for integro-differential operators are briefly discussed. For the reduction system Σ_{IDO}, by applying `TenReS` in total we obtain 52 ambiguities and corresponding S-polynomials. Among them, there are 4 ambiguities for which the corresponding S-polynomials are zero anyway, for instance

$$\mathrm{SP}(\mathsf{D}\underline{\Phi}, \underline{\mathsf{E}}\mathsf{I}) = 0 \otimes \textstyle\int - \partial \otimes 0 = 0.$$

Applying automatically the implementation of reduction rules from Σ_{IDO}, identities in $\mathcal{R}$ and identities in M_{D}, M_{I} and M_{Φ} we see that the S-polynomials of the 48 remaining ambiguities are reduced to zero. The complete computation is included in the example files of the package. Here we consider a few concrete instances of ambiguities. For example, we use the definition of E in $\mathcal{R}$ in the reduction of the following S-Polynomial

$$\begin{aligned}\mathrm{SP}(\mathsf{IR}\underline{\mathsf{D}}, \underline{\mathsf{D}}\Phi) &= (A - \textstyle\int \otimes \partial A - (\mathrm{E}A)\mathrm{E}) \otimes \varphi - \textstyle\int \otimes A \otimes 0 \\ &\to_{r_{\mathsf{IR}\Phi}} A \otimes \varphi - (\textstyle\int \partial A) \otimes \varphi - (\mathrm{E}A)\mathrm{E} \otimes \varphi \\ &= A \otimes \varphi - (A - \mathrm{E}A) \otimes \varphi - (\mathrm{E}A)\mathrm{E} \otimes \varphi \\ &= \mathrm{E}A \otimes \varphi - (\mathrm{E}A)\mathrm{E} \otimes \varphi \to_{r_{\Phi\Phi}} \mathrm{E}A \otimes \varphi - (\mathrm{E}A)\varphi \to_{r_{\mathsf{K}}} 0.\end{aligned}$$

As another example, we use the definition of the right multiplication in the $\mathcal{K}$-bimodule M_{I} in the following reduction

$$\begin{aligned}\mathrm{SP}(\mathsf{I}\underline{\Phi}, \underline{\Phi}\mathsf{R}) &= (\textstyle\int 1 \otimes \varphi) \otimes A - \textstyle\int \otimes (\varphi A)\varphi \to_{r_{\mathsf{I}\Phi}} \textstyle\int 1 \otimes \varphi \otimes A - \varphi A(\textstyle\int 1 \otimes \varphi) \\ &\to_{r_{\Phi\mathsf{R}}} \textstyle\int 1 \otimes (\varphi A)\varphi - \varphi A(\textstyle\int 1 \otimes \varphi) = (\textstyle\int 1 \varphi A) \otimes \varphi - \varphi A(\textstyle\int 1 \otimes \varphi) \\ &= (\varphi A)\textstyle\int 1 \otimes \varphi - \varphi A(\textstyle\int 1 \otimes \varphi) = 0.\end{aligned}$$

There are 41 ambiguities without specialization. The remaining 11 ambiguities consist of 4 overlap ambiguities with specialization and 7 inclusion ambiguities with specialization. For example,

$$\mathrm{SP}(\mathsf{IR}\underline{\Phi}, \underline{\mathsf{E}}\mathsf{I}) = (\textstyle\int A \otimes \mathrm{E}) \otimes \textstyle\int - \textstyle\int \otimes A \otimes 0 \to_{r_{\mathsf{EI}}} 0,$$

and

$$\mathrm{SP}(\underline{\mathsf{K}}, \mathsf{D}\underline{\mathsf{R}}) = \partial \otimes \epsilon \otimes \epsilon - 1 \otimes \partial \ \to_{r_{\mathsf{K}}} \partial - \partial = 0.$$

The reader should note the confluence criterion of Theorem 3.32 directly works with the reduction system Σ_{IDO}, no computations with the refined reduction system Σ_X on X given in (4.6) are required.

4.4 Vector-valued functions as an IDO-module

In Example 4.2, we showed that how to construct an integro-differential ring $(M_n(\mathcal{S}), \partial, \int)$ with ring of constants $\mathcal{K} = \{(a_{ij}) \mid a_{ij} \in \mathcal{S} \text{ and } \partial a_{ij} = 0\}$ from a commutative integro-differential ring $(\mathcal{S}, \partial, \int)$. Now, $\mathcal{S}^n$ becomes a left module over $(M_n(\mathcal{S}), \partial, \int)$. Recall that the abelian group

$$\mathrm{End}(\mathcal{S}^n) = \{\psi \colon \mathcal{S}^n \to \mathcal{S}^n \mid \psi \text{ is an additive map}\}$$

together with composition is a ring. We define a map $\mathcal{K} \mapsto \mathrm{End}(\mathcal{S}^n)$ by $K \mapsto \hat{K}$, where by $\hat{K}$ we denote the endomorphism sending any element $f \in \mathcal{S}^n$ to the matrix multiplication Kf. Since in particular $\mathcal{S}^n$ is a left $\mathcal{K}$-module, then for all $f \in \mathcal{S}^n$ and $K_1, K_2 \in \mathcal{K}$, we have

$$\hat{1}f = f, \quad (\hat{K_1} + \hat{K_2})f = \hat{K_1}f + \hat{K_2}f, \quad (\hat{K_1}\hat{K_2})f = \hat{K_1}(\hat{K_2}f)$$

and hence the map is a ring homomorphism. Hence, by Lemma 2.46, the ring $\mathrm{End}(\mathcal{S}^n)$ is a $\mathcal{K}$-ring. We also define a map $\mathcal{R} \to \mathrm{End}(\mathcal{S}^n)$ by $A \mapsto \hat{A}$, where $\hat{A}$ maps f to Af. In addition, we define a map $\{\partial, \int\} \cup \Phi \to \mathrm{End}(\mathcal{S}^n)$ by

$$\partial \mapsto \hat{\partial}, \quad \textstyle\int \mapsto \hat{\int}, \quad \varphi \mapsto \hat{\varphi},$$

where $\hat{\partial}, \hat{\int}, \hat{\varphi}$ denote endomorphisms on $\mathcal{S}^n$ such that

$$\hat{\partial}(f_i) := (\partial f_i), \quad \hat{\textstyle\int}(f_i) := (\textstyle\int f_i), \quad \hat{\varphi}(f_i) := (\varphi f_i),$$

for any $A \in M_n(\mathcal{S})$ and $(f_i) \in \mathcal{S}^n$. Let $\{\partial, \int\} \cup \Phi \to \mathcal{K}\partial \oplus \mathcal{K}\int \oplus \mathcal{K}\Phi$ be the inclusion map. By Theorem 2.5, there exists a unique $\mathcal{K}$-module homomorphism $\mathcal{K}\partial \oplus \mathcal{K}\int \oplus \mathcal{K}\Phi \to \mathrm{End}(\mathcal{S}^n)$. Analogous to Section 4.2, there exists a unique $\mathcal{K}$-bimodule homomorphism $M \to \mathrm{End}(\mathcal{S}^n)$ and a unique extension to a $\mathcal{K}$-ring homomorphism $\theta \colon \mathcal{K}\langle M \rangle \to \mathrm{End}(\mathcal{S}^n)$. We can also verify that $I_{\Sigma_0} \subseteq \mathrm{Ker}(\theta)$. For instance, for any $A \in \mathcal{R}$ and $\varphi \in \Phi$ we have

$$\theta(\varphi \otimes A - (\varphi A)\varphi) = \hat{\varphi}\hat{A} - (\varphi A)\hat{\varphi} = \hat{\varphi}\hat{A} - \widehat{(\varphi A)}\hat{\varphi} = 0.$$

To prove the last equality, we see that if $A = (a_{ij}) \in M_n(\mathcal{S})$ and $f = (f_i) \in \mathcal{S}^n$, then $Af = (g_i) \in \mathcal{S}^n$ where $g_i = \sum\limits_{k=1}^{n} a_{ik} f_k$. Since

$$\varphi g_i = \varphi(\sum_{k=1}^{n} a_{ik} f_k) = \sum_{k=1}^{n} \varphi a_{ik} f_k = \sum_{k=1}^{n} (\varphi a_{ik}) \varphi f_k,$$

then we can conclude that $\hat{\varphi}\hat{A}f = (\widehat{\varphi A})\hat{\varphi}f$ for any $f \in \mathcal{S}^n$. As another example, we have

$$\theta(\textstyle\int \otimes \partial - \epsilon + \mathrm{E}) = \hat{\textstyle\int}\hat{\partial} - \hat{\epsilon} + \hat{\mathrm{E}} = 0,$$

where correctness of the last equality is deduced by

$$\hat{\textstyle\int}\hat{\partial}(f_i) = \hat{\textstyle\int}(\partial f_i) = (\textstyle\int \partial f_i) = ((\epsilon - \mathrm{E})f_i) = (f_i) - (\mathrm{E}f_i) = \hat{\epsilon}(f_i) - \hat{\mathrm{E}}(f_i).$$

Similar to Section 4.2, by the factor theorem, existence of the unique $\mathcal{K}$-ring homomorphism $\overline{\theta}\colon M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle \to \mathrm{End}(\mathcal{S}^n)$ is proven and we can view $\mathcal{S}^n$ as an $M_n(\mathcal{R})\langle\partial, \textstyle\int, \Phi\rangle$-module:

Proposition 4.16. *The abelian group $\mathcal{S}^n$ with the scalar multiplication*

$$\begin{aligned} \bullet\colon M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle \times \mathcal{S}^n &\to \mathcal{S}^n \\ (\overline{L}, (f_i)) &\mapsto \overline{\theta}(\overline{L})(f_i) \end{aligned}$$

is a $M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle$-module which is called the module of vector-valued functions *over the ring $M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle$.*

Interpreting the method of variation of constants in terms of integro-differential operators amounts to constructing right inverses of differential operators. In fact, right inverses and particular solutions of inhomogeneous differential equations are closely related.

Example 4.17. *Let $(\mathcal{S}^n, +, \bullet)$ be the module of vector-valued functions over the ring $M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle$. Let H be a right inverse for $L = \partial - A$ in $M_n(\mathcal{S})\langle\partial, \textstyle\int, \Phi\rangle$. Then $H \bullet f$ is a particular solution for the system of inhomogeneous differential equation $L \bullet y = f$ where $y, f \in \mathcal{S}^n$, since*

$$L \bullet (H \bullet f) = (LH) \bullet f = 1 \bullet f = f.$$

We recall from Example 4.9 that if $\Phi \in M_n(\mathcal{S})$ is invertible with $\partial\Phi = A\Phi$, then the operator

$$H = \Phi \cdot \textstyle\int \cdot \Phi^{-1}$$

is a right inverse of L. Therefore, for this H, we observe that

$$H \bullet f = \Phi \cdot \textstyle\int \cdot \Phi^{-1} \bullet f$$

is a particular solution of $L \bullet y = f$.

4.5 Integro-differential operators with linear substitutions

In the following, we describe the construction of the ring of integro-differential operators with linear substitutions (IDOLS) obtained by extending the ring of IDO. A significant motivation for studying this ring comes from the work in [40] where such operators and their commutation rules are used for an algorithmic approach to Artstein's integral transformation of linear differential systems with delayed inputs to linear differential system without delays. IDOLS also addresses the univariate case in [49], where algebraic aspects of multivariate integration with linear substitutions are studied. Moreover, they provide an algebraic setting for dealing with delay differential equations and the corresponding initial and linear boundary problems in general.

A delay differential equation is an ordinary differential equation where the derivative at a certain time depends on the solution at prior times; for more details see [25, 54]. A general first-order constant delay equation has the form

$$y'(x) = A(x, y(x), y(x - b_1), y(x - b_2), \dots, y(x - b_n))$$

where the time delays b_j for $1 \leq j \leq n$ are positive constants. A homogeneous linear first-order time-delay equation with one constant delay has the form

$$y'(x) = A_0(x)y(x) + A_1(x)y(x - b).$$

The chain rule and integration by substitution from calculus describe the interaction of linear substitutions $A(ax - b)$ with differentiation and integration. More formally, let $\sigma_{a,b}$ denote the linear substitution operator mapping a smooth function $A(x)$ to $A(ax - b)$ for a nonzero constant a and an arbitrary constant b. Then we observe that

$$\partial_x \sigma_{a,b} A(x) = a\dot{A}(ax - b) = a\sigma_{a,b}\partial_x A(x).$$

and

$$\int_0^x \sigma_{a,b} A(t)\, dt = \frac{1}{a}\int_{-b}^{ax-b} A(t)\, dt = \frac{1}{a}\sigma_{a,b}\int_0^x A(t)\, dt - \frac{1}{a}\mathrm{E}\sigma_{a,b}\int_0^x A(t)\, dt.$$

Following these identities, we want to define an integro-differential ring with linear substitutions. In what follows, we denote by

$$\mathcal{C} = \mathcal{K} \cap \mathcal{Z}(\mathcal{R})$$

the ring of elements of $\mathcal{K}$ which commute with all elements of $\mathcal{R}$ and $\mathcal{C}^*$ denotes its group of units. For finding a proper algebraic setting, we add an axiomatization of linear substitution operations to an integro-differential ring.

Definition 4.18. *Let* $(\mathcal{R}, \partial, \int)$ *be an integro-differential ring with constants* $\mathcal{K}$ *and let*

$$\mathcal{S} := \{\sigma_{a,b} \mid a \in \mathcal{C}^*,\ b \in \mathcal{C}\}$$

where $\sigma_{a,b}\colon \mathcal{R} \to \mathcal{R}$ *are multiplicative* $\mathcal{K}$*-bimodule homomorphisms on* $\mathcal{R}$ *fixing the constants* $\mathcal{K}$ *such that*

$$\sigma_{1,0}A = A, \quad \sigma_{a,b}\sigma_{c,d}A = \sigma_{ac,bc+d}A \tag{4.9}$$

and

$$\partial\sigma_{a,b}A = a\sigma_{a,b}\partial A \tag{4.10}$$

for all $a, c \in \mathcal{C}^*$, $b, d \in \mathcal{C}$ *and* $A \in \mathcal{R}$. *Then we call* $(\mathcal{R}, \partial, \int, \mathcal{S})$ *an* integro-differential ring with linear substitutions.

Remark 4.19. *The set* $\mathcal{S}$ *along with composition can be considered as a group of* $\mathcal{K}$*-bimodule homomorphisms on* $\mathcal{R}$. *The neutral element is* $\sigma_{1,0}$ *and the inverse for* $\sigma_{a,b} \in \mathcal{S}$ *is given by*

$$\sigma_{a,b}^{-1} = \sigma_{a^{-1},-ba^{-1}}.$$

So the elements in $\mathcal{S}$ *are actually automorphisms.*

In the following, we fix an integro-differential ring with linear substitutions $(\mathcal{R}, \partial, \int, \mathcal{S})$ with constants $\mathcal{K}$ and evaluation $\mathrm{E} = \mathrm{id} - \int\partial$. We consider the modules M_{K}, $M_{\tilde{\mathsf{R}}}$, M_{D}, M_{I}, M_{E}, $M_{\tilde{\mathsf{\Phi}}}$, M_{R}, and $M_{\mathsf{\Phi}}$ which are introduced in (4.3), (4.5), and (4.7). In addition, we add the free left $\mathcal{K}$-module

$$M_{\mathsf{G}} := \mathcal{K}\mathcal{S}.$$

We also view it as a $\mathcal{K}$-bimodule with the right multiplication defined by $K_1\sigma_{a,b} \cdot K_2 = K_1K_2\sigma_{a,b}$ with $K_1, K_2 \in \mathcal{K}$. It has the direct sum decomposition

$$M_{\mathsf{G}} = M_{\mathsf{N}} \oplus M_{\tilde{\mathsf{G}}}$$

such that $M_{\mathsf{N}} := \mathcal{K}\sigma_{1,0}$ is the $\mathcal{K}$-bimodule generated by the trivial substitution $\sigma_{1,0} = \mathrm{id}$ and $M_{\tilde{\mathsf{G}}}$ is the $\mathcal{K}$-bimodule generated by all linear substitutions in $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{\sigma_{1,0}\}$. Therefore, we take the alphabets

$$X := \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{D}, \mathsf{I}, \mathsf{E}, \tilde{\mathsf{\Phi}}, \mathsf{N}, \tilde{\mathsf{G}}\} \quad \text{and} \quad Z := X \cup \{\mathsf{R}, \mathsf{\Phi}, \mathsf{G}\}. \tag{4.11}$$

With the $\mathcal{K}$-bimodules

$$M_{\mathsf{R}} = M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}}, \quad M_{\mathsf{\Phi}} = M_{\mathsf{E}} \oplus M_{\tilde{\mathsf{\Phi}}}, \quad M_{\mathsf{G}} = M_{\mathsf{N}} \oplus M_{\tilde{\mathsf{G}}}, \tag{4.12}$$

we define

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus M_{\mathsf{I}} \oplus M_{\mathsf{\Phi}} \oplus M_{\mathsf{G}}. \tag{4.13}$$

Then $(M_z)_{z\in Z}$ is a decomposition with specialization.

Definition 4.20. *Let* $(\mathcal{R}, \partial, \int, \mathcal{S})$ *be an integro-differential ring with linear substitutions. We call*

$$\mathcal{R}\langle\partial, \textstyle\int, \Phi, \mathcal{S}\rangle := \mathcal{K}\langle M\rangle / I_{\tilde{\Sigma}_0}$$

the ring of integro-differential operators with linear substitutions*, where* $I_{\tilde{\Sigma}_0}$ *is the two-sided reduction ideal induced by the reduction system* $\tilde{\Sigma}_0 = \Sigma_0 \cup \Sigma_{LS}$*, where* Σ_0 *is the reduction system in Definition 4.6 and*

$$\Sigma_{\mathrm{LS}} = \{(\mathsf{N}, \sigma_{1,0} \mapsto \epsilon), (\mathsf{GR}, \sigma_{a,b} \otimes A \mapsto \sigma_{a,b}A \otimes \sigma_{a,b}), (\mathsf{G\Phi}, \sigma_{a,b} \otimes \varphi \mapsto \varphi), \\ (\mathsf{GG}, \sigma_{a,b} \otimes \sigma_{c,d} \mapsto \sigma_{ac,bc+d}), (\mathsf{DG}, \partial \otimes \sigma_{a,b} \mapsto a\sigma_{a,b} \otimes \partial\}.$$

Example 4.21. *Consider the differential time-delay system*

$$\dot{x}(t) - A_0(t)x(t) - A_1(t)x(t-h) = f(t)$$

corresponding to the operator $R := L + S$ *with differential part* $L = \partial - A_0$ *as in Example 4.9 and time-delay part* $S := -A_1 \cdot \delta$*, where by* δ *we denote the time-delay operator* $\sigma_{1,h}$*. For solving this system, like Example 4.9, we first note that the equation* $R \bullet x = f$ *is equivalent to the equation*

$$(H \cdot R) \bullet x = H \bullet f$$

similar to above. We have

$$H \cdot R = 1 - \Phi \mathrm{E} \Phi^{-1} \cdot \mathrm{E} - \Phi \cdot \textstyle\int \cdot \Phi^{-1} A_1 \delta.$$

Taking $H \cdot R = \epsilon - G$*, where*

$$G := P - H \cdot S = \Phi \mathrm{E} \Phi^{-1} \cdot \mathrm{E} + \Phi \cdot \textstyle\int \cdot \Phi^{-1} \cdot A_1 \delta,$$

we can rewrite $(H \cdot R) \bullet x = H \bullet f$ *as the recurrence equation*

$$x = G \bullet x + H \bullet f.$$

This is the operator interpretation of the method of steps, see e.g. [25]:

$$x(t) = \Phi(t)\Big(\Phi^{-1}(t_0)x(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)\big(f(s) + A_1(s)x(s-h)\big)ds\Big).$$

4.6 Completion of tensor reduction system for the ring of IDOLS

In the following, we describe the completion process for the ring of IDOLS. Consider the reduction system obtained by adjoining the reduction system Σ_{LS} given in Definition 4.20 to the confluent reduction system Σ_{IDO} from Definition 4.20. Similar to the completion process for the ring of IDO, we collect the required conditions for compatible partial orders on the word monoid $\langle Z \rangle$ in (4.11). Hence, we consider all the conditions we obtained for monoid partial orders compatible with Σ_{IDO}, together with the new conditions $\mathsf{GR} > \mathsf{RG}$, $\mathsf{DG} > \mathsf{GD}$, and the corresponding conditions for their specializations. Note that in each step of completion, among the new S-polynomials, we only take one of them in order to derive a new rule.

For the reduction rules r_{ID} and r_{DG} we get a non-resolvable overlap ambiguity with S-polynomials

$$\mathrm{SP}(\mathsf{I}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{G}) = \sigma_{a,b} - \mathrm{E} \otimes \sigma_{a,b} - a\textstyle\int \otimes \sigma_{a,b} \otimes \partial$$

which leads to introduce the new rule

$$(\mathsf{IGD}, \textstyle\int \otimes \sigma_{a,b} \otimes \partial \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b}).$$

In this step we should also add $\mathsf{I} > \mathsf{E}$ to the conditions we obtained for partial orders. The reduction rules r_{IGD} and r_{DI} have two overlap ambiguities with each other, one is resolvable and one is not. The latter has S-polynomial

$$\mathrm{SP}(\mathsf{IG}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{I}) = a^{-1}\sigma_{a,b}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} - \textstyle\int \otimes \sigma_{a,b}$$

which implies a new reduction rule as

$$(\mathsf{IG}, \textstyle\int \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int).$$

We observe that the rule r_{IG} reduces the S-polynomials $\mathrm{SP}(\mathsf{I}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{G})$ to zero. Therefore, we drop the obtained reduction rule r_{IGD} from the reduction system we are completing. Consequently, we do not consider any more the condition $\mathsf{I} > \mathsf{E}$. Instead, we add the weaker condition $\mathsf{IG} > \mathsf{EGI}$ to the list of conditions required for compatible monoid partial orders. For the rest, among the non resolvable ambiguities we only look at the overlap ambiguity between the reduction rules r_{IG} and r_{GR} with S-polynomials

$$\begin{aligned}\mathrm{SP}(\mathsf{I}\underline{\mathsf{G}}, \underline{\mathsf{G}}\mathsf{R}) &= a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \textstyle\int \otimes A - \int \otimes \sigma_{a,b} \otimes A \\ &\to_{r_{\mathsf{GR}}} a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \textstyle\int \otimes A - \int \otimes \sigma_{a,b}A \otimes \sigma_{a,b}.\end{aligned}$$

We are motivated to define a new reduction homomorphism on M_{IRG} that reduces $\int \otimes \sigma_{a,b} A \otimes \sigma_{a,b}$ to $a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int \otimes A$. Then, we replace A by $\sigma_{a,b}^{-1} A$ and arrive at the rule

$$(\mathsf{IRG}, \textstyle\int \otimes A \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int \otimes \sigma_{a,b}^{-1} A).$$

As a consequence, we obtain the identity

$$\textstyle\int A \sigma_{a,b} B = a^{-1}(\mathrm{id} - \mathrm{E}) \sigma_{a,b} \int (\sigma_{a,b}^{-1} A) B,$$

for any $A, B \in \mathcal{R}$ which corresponds to integration by substitution. We also need to consider the new condition $\mathsf{IRG} > \mathsf{EGIR}$. Since the rule r_{IRG} reduces the S-polynomials $\mathrm{SP}(\mathsf{IR\underline{D}}, \mathsf{\underline{D}G})$ to zero, then we drop the reduction rule r_{IGD} from our reduction system. Altogether, we get a confluent reduction system by adjoining the following table to Table 4.1.

Reduction rules in Σ_{LS}	
N	$\sigma_{1,0} \mapsto \epsilon$
GR	$\sigma_{a,b} \otimes A \mapsto \sigma_{a,b} A \otimes \sigma_{a,b}$
GΦ	$\sigma_{a,b} \otimes \varphi \mapsto \varphi$
GG	$\sigma_{a,b} \otimes \sigma_{c,d} \mapsto \sigma_{ac,bc+d}$
DG	$\partial \otimes \sigma_{a,b} \mapsto a \sigma_{a,b} \otimes \partial$
Consequences of reduction rules from $\Sigma_{\mathrm{IDO}} \cup \Sigma_{\mathrm{LS}}$	
IG	$\int \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int$
IRG	$\int \otimes A \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int \otimes \sigma_{a,b}^{-1} A$

Table 4.3: New reduction rules in IDOLS

The identities for IDOLS contain those identities we collected in Section 4.1 for IDO, as well as the identities involving the substitution operators. In the following table, we collect identities involving substitution operations that hold in $\mathcal{R}$. For all $A, B \in \mathcal{R}$, $\varphi \in \Phi$ and $\sigma_{a,b}, \sigma_{c,d} \in \mathcal{S}$ we have:

$\sigma_{1,0} B = B$	$\sigma_{a,b} \sigma_{c,d} B = \sigma_{ac,bc+d} B$
$\sigma_{a,b} A B = (\sigma_{a,b} A)(\sigma_{a,b} B)$	$\partial \sigma_{a,b} B = a \sigma_{a,b} \partial B$
$\sigma_{a,b} \varphi B = \varphi B$	$\int A \sigma_{a,b} B = a^{-1}(\mathrm{id} - \mathrm{E}) \sigma_{a,b} \int (\sigma_{a,b}^{-1} A) B$

Table 4.4: Identities in $\mathcal{R}$ corresponding to new rules in IDOLS

Similar to the previous example, the refined reduction system Σ_X is obtained, according to (4.12), by splitting rules whose words contain R, $\mathsf{\Phi}$ or G into "smaller" rules using $\mathcal{S}(\mathsf{R}) = \{\mathsf{K}, \tilde{\mathsf{R}}\}$, $\mathcal{S}(\mathsf{\Phi}) = \{\mathsf{E}, \tilde{\mathsf{\Phi}}\}$ and $\mathcal{S}(\mathsf{G}) = \{\mathsf{N}, \tilde{\mathsf{G}}\}$. Following Theorem 3.32, we determine normal forms of tensors in $\mathcal{R}\langle\partial, \int, \Phi, \mathcal{S}\rangle$.

Theorem 4.22. *Let $(\mathcal{R}, \partial, \int, \mathcal{S})$ be an integro-differential ring with linear substitutions and let M be as in (4.12) and (4.13), and let the reduction system be Σ_{IDOLS} defined by tables 4.1 and 4.3. Then every $t \in \mathcal{K}\langle M\rangle$ has a unique normal form given by a sum of pure tensors*

$$A \otimes \varphi \otimes \sigma_{a,b} \otimes \partial^j \qquad \text{or} \qquad A \otimes \varphi \otimes \sigma_{a,b} \otimes \textstyle\int \otimes B,$$

where $j \in \mathbb{N}_0$, each of $A, B \in M_{\tilde{\mathsf{R}}}$, $\varphi \in \Phi$ and $\sigma_{a,b} \in \tilde{\mathcal{S}}$ may be absent, and $\varphi \otimes \sigma_{a,b} \otimes \int$ does not specialize to $\mathrm{E} \otimes \int$. Moreover, defining the multiplication $s \cdot t := (s \otimes t) \downarrow_{\Sigma_{\mathrm{IDOLS}}}$ on $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ we have

$$\mathcal{R}\langle\partial, \textstyle\int, \Phi, \mathcal{S}\rangle \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}.$$

Proof. We consider the alphabets X and Z as defined in (4.11). Then $(M_z)_{z\in Z}$ is a decomposition with specialization for the module M, see Definition 3.20. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ_{IDOLS}, it is sufficient to require the order to satisfy

$$\mathsf{DR} > \mathsf{RD},\ \mathsf{IRD} > \mathsf{E},\ \mathsf{ID} > \mathsf{E},\ \mathsf{I} > \tilde{\mathsf{R}},\ \mathsf{GR} > \mathsf{RG},$$
$$\mathsf{DG} > \mathsf{GD},\ \mathsf{IG} > \mathsf{EGI},\ \mathsf{IRG} > \mathsf{EGIR}.$$

For instance, on the word monoid $\langle Y\rangle$ with $Y = \{\mathsf{R}, \mathsf{D}, \mathsf{I}, \mathsf{\Phi}, \mathsf{G}\}$, we first define an order by

$$V \leq W :\Leftrightarrow \tilde{V} \prec \tilde{W} \text{ or } \tilde{V} = \tilde{W} \text{ and } V \preceq W,$$

where $\tilde{V}$ and $\tilde{W}$ are obtained by removing all occurrences of $\mathsf{\Phi}$, cf. Remark 3.5, and $\preceq$ is the degree-lexicographic order with $\mathsf{I} \succ \mathsf{D} \succ \mathsf{G} \succ \mathsf{\Phi} \succ \mathsf{R}$ on $\langle Y\rangle$. Then, we extend $\leq$ to a monoid partial order on $\langle Z\rangle$ based on Definition 3.27 in order to make it consistent with specialization.

Then by the package `TenReS` we verify that all ambiguities of Σ_{IDOLS} are resolvable, see Section 4.6.1. Hence, by Theorem 3.32 every element of $\mathcal{K}\langle M\rangle$ has a unique normal form and $\mathcal{K}\langle M\rangle / I_{\Sigma_{\mathrm{IDOLS}}} \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}$ as $\mathcal{K}$-rings.

It remains to determine the explicit form of elements in $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\mathrm{irr}}$ in $\langle X\rangle$. Note that $\Sigma_{\mathrm{IDO}} \subset \Sigma_{\mathrm{IDOLS}}$ and thus the irreducible words w.r.t. Σ_{IDOLS} are among the irreducible words w.r.t. Σ_{IDO}. In Theorem 4.14, we already determined the irreducible words that do not contain the letters N and $\tilde{\mathsf{G}}$ to be of the form

$$\tilde{\mathsf{R}} V \mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}\tilde{\mathsf{\Phi}}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\Phi}$, and $V \in S(\Phi)$ may be absent.

The irreducible words containing only N and $\tilde{\mathsf{G}}$ are exactly ϵ and $\tilde{\mathsf{G}}$, since they have to avoid the subwords N and $S(\mathsf{GG}) = \{\mathsf{NN}, \mathsf{N\tilde{G}}, \mathsf{\tilde{G}N}, \mathsf{\tilde{G}\tilde{G}}\}$. The irreducible words in $\langle X\rangle_{\text{irr}}$ also have to avoid subwords from $S(\mathsf{GR})$, $S(\mathsf{G\Phi})$, $S(\mathsf{DG})$, $S(\mathsf{IG})$, and $S(\mathsf{IRG})$. Hence they are of the form

$$\tilde{\mathsf{R}}V\tilde{\mathsf{G}}\mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}V\tilde{\mathsf{G}}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\mathsf{G}}$, and $V \in S(\Phi)$ may be absent and $V\tilde{\mathsf{G}}\mathsf{I}$ does not specialize to EI. The normal forms follow from (3.3). □

4.6.1 Computational aspects

In the following, we briefly mention some computational details of the tensor setting with specialization for integro-differential operators with linear substitutions. Applying `TenReS` to the reduction system Σ_{IDOLS}, in total 87 ambiguities and corresponding S-polynomials are generated. All ambiguities are resolvable and the automatic verification can be found in the example files of the package. There are 66 ambiguities without specialization. For instance,

$$\mathrm{SP}(\mathsf{IR\underline{\Phi}}, \mathsf{\underline{E}I}) = (\textstyle\int A \otimes \mathrm{E}) \otimes \int - \int \otimes A \otimes 0 \to_{r_{\mathsf{EI}}} \int A \otimes 0 = 0,$$

and

$$\begin{aligned}\mathrm{SP}(\mathsf{I\underline{G}}, \mathsf{\underline{G}R}) &= (a^{-1}\sigma_{a,b} \otimes \textstyle\int - a^{-1}\mathrm{E} \otimes \sigma_{a,b} \otimes \int) \otimes A - \int \otimes (\sigma_{a,b}A \otimes \sigma_{a,b})\\ &= a^{-1}(\sigma_{a,b} \otimes \textstyle\int \otimes A - \mathrm{E} \otimes \sigma_{a,b} \otimes \int \otimes A) - \int \otimes \sigma_{a,b}A \otimes \sigma_{a,b}\\ &\to_{r_{\mathsf{IRG}}} 0.\end{aligned}$$

Among the remaining 21 ambiguities we have 5 overlap ambiguities with specialization and 16 inclusion ambiguities with specialization. They all involve the following three reduction rules (on X)

$$(\mathsf{K}, 1 \mapsto \epsilon),\ (\mathsf{EI}, \mathrm{E} \otimes \textstyle\int \mapsto 0),\ (\mathsf{N}, \sigma_{1,0} \mapsto \epsilon)$$

and their S-polynomials can be reduced to zero. For example,

$$\mathrm{SP}(\mathsf{\underline{N}}, \mathsf{D\underline{G}}) = \partial \otimes \epsilon - \sigma_{1,0} \otimes \partial \to_{r_{\mathsf{N}}} \partial - \partial = 0,$$

and

$$\begin{aligned}\mathrm{SP}(\mathsf{\underline{N}}, \mathsf{IR\underline{G}}) &= \textstyle\int \otimes A - (\epsilon - \mathrm{E}) \otimes \sigma_{1,0} \otimes \int \otimes A\\ &\to_{r_{\mathsf{N}}} \mathrm{E} \otimes \sigma_{1,0} \otimes \textstyle\int \otimes A \to_{r_{\mathsf{N}}} \mathrm{E} \otimes \int \otimes A \to_{r_{\mathsf{EI}}} 0.\end{aligned}$$

Chapter 5

Applications of the rings of IDO and IDOLS

Boundary problems play an important rule in science and engineering. However, they are usually treated numerically and are rarely considered in symbolic computation, from both a theoretical and practical perspective using computer algebra systems. For solving regular two-point linear boundary problems with constant coefficients, a symbolic approach was developed in [47]. A generalization of this method for differential algebras was proposed in [51], where also a factorization method applicable to linear boundary problems for ordinary differential equations was developed, see also [34, 50, 52]. In analogy to the symbolic methods for finding Green's operators of ordinary boundary problems of order n, our tensor setting for IDO allows us to propose similar Green's operators for first-order systems.

Using the algebraic framework for IDOLS and its implementation based on `TenReS`, classes of linear DTD systems and their transformations can be studied. Moreover, by formal computations in the ring of IDOLS also operators having rectangular matrices as their coefficients can be considered. In Section 5.2, following section 4 of [13], we apply our implementation of IDOLS to largely automatize the computations of [40] for recovering Artstein's transformation. This transformation is used for proving the equivalence between linear differential systems with delayed inputs and linear differential systems without time-delays. Furthermore, we propose a generalization that transforms the solution spaces of differential time-delay systems and solution spaces of the corresponding differential systems into each other without distinguishing state and control variables.

5.1 Green's operators of first-order systems

In this section, we study Green's operators of first-order linear ordinary boundary problems in the ring of integro-differential operators. Throughout this section, we let $\mathcal{R}\langle\partial, \int, \Phi\rangle$ be a ring of integro-differential operators over an integro-differential ring $(\mathcal{R}, \partial, \int)$ with constants $\mathcal{K}$. Let $(\mathcal{F}, +, \bullet)$ be a left $\mathcal{R}\langle\partial, \int, \Phi\rangle$-module. Moreover, we assume $\mathcal{F}$ is a right $\mathcal{K}$-module such that

$$(L \bullet f)K = L \bullet (fK),$$

for all $L \in \mathcal{R}\langle\partial, \int, \Phi\rangle$, $f \in \mathcal{F}$, and $K \in \mathcal{K}$. In other words, $\mathcal{F}$ is an $\mathcal{R}\langle\partial, \int, \Phi\rangle$-$\mathcal{K}$-bimodule. In particular, $\mathcal{F} = \mathcal{R}$ satisfies these properties because all operations on $\mathcal{R}$ are right $\mathcal{K}$-linear. Given a first-order differential operator $L = \partial - A \in \mathcal{R}\langle\partial\rangle$ and a linear boundary condition β from the right ideal generated by Φ in $\mathcal{R}\langle\partial, \int, \Phi\rangle$, acting as right $\mathcal{K}$-linear functional $\mathcal{R} \to \mathcal{K}$, we say $G \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ is a Green's operator for the linear boundary problem (L, β), if it satisfies

$$L \cdot G = 1, \quad \text{and} \quad \beta \cdot G = 0.$$

Then, for any $f \in \mathcal{F}$, we see that $u = G \bullet f \in \mathcal{F}$ is a solution of the inhomogeneous differential equation $L \bullet y = f$ satisfying the boundary condition

$$\beta \bullet u = 0.$$

We have already seen in Example 4.15 how to compute a family of right inverses, denoted by H, for first-order differential operators in the ring of integro-differential operators. In the following, we will see how to modify H to compute an integro-differential operator G that also satisfies the additional equation given by the boundary condition.

We are required to know how to define and compute with boundary conditions and idempotents as particular integro-differential operators. Recall that for any $A \in \mathcal{R}$ and $\varphi \in \Phi$, we have the identity $\varphi \cdot A = (\varphi A)\varphi$. Based on that, we can construct idempotents, i.e., elements satisfying $P^2 = P$ in the ring of integro-differential operators $\mathcal{R}\langle\partial, \int, \Phi\rangle$. Let $\varphi \in \Phi$ and assume that there exists $A \in \mathcal{R}$ with $\varphi A = 1$. Then

$$P = A \cdot \varphi$$

is idempotent in $\mathcal{R}\langle\partial, \int, \Phi\rangle$. We have

$$P \cdot P = (A \cdot \varphi) \cdot (A \cdot \varphi) = A \cdot (\varphi A)\varphi \cdot \varphi = A \cdot 1\varphi = A \cdot \varphi = P.$$

Moreover, $1 - P$ is idempotent since $(1 - P)^2 = 1 - 2P + P^2 = 1 - P$ and

$$\begin{aligned}\varphi \cdot (1 - P) = \varphi \cdot (1 - A \cdot \varphi) &= \varphi - \varphi \cdot A \cdot \varphi \\ &= \varphi - (\varphi A)\varphi \cdot \varphi = \varphi - (\varphi A)\varphi = \varphi - 1\varphi = 0.\end{aligned}$$

We considered so far only basic linear functionals $\varphi \in \Phi$. However, all elements in the right ideal generated by Φ act as right $\mathcal{K}$-linear functionals $\mathcal{R} \to \mathcal{K}$. Their normal forms are $\mathcal{K}$-linear combinations of products

$$\varphi \cdot \partial^j \quad \text{and} \quad \varphi \cdot \textstyle\int \cdot B, \tag{5.1}$$

where $j \in \mathbb{N}_0$, $B \in \mathcal{R}$, and $\int$ may be absent, leading to the following definition.

Definition 5.1. *The elements of the $\mathcal{K}$-bimodule generated by all the normal forms presented in* (5.1) *are called* boundary conditions *in $\mathcal{R}\langle\partial, \int, \Phi\rangle$.*

With the next lemma, the above considerations about idempotents and projectors generalize immediately to general boundary conditions in $\mathcal{R}\langle\partial, \int, \Phi\rangle$ as stated in the theorem below.

Lemma 5.2. *Let $\beta, \gamma \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ be boundary conditions. Then for any $A \in \mathcal{R}$, we have*

$$\beta \cdot A \cdot \gamma = (\beta \bullet A)\gamma.$$

Proof. By Definition 5.1, every boundary condition can be written as a $\mathcal{K}$-linear combination of the normal forms as in (5.1). Hence, we have to check the validity of the identity in the lemma only for these normal forms. We can assume without loss of generality that the boundary condition is of the form $\gamma = \varphi$ for some $\varphi \in \Phi$, since the above statement can be multiplied from the right by any appropriate element of the ring $\mathcal{R}\langle\partial, \int, \Phi\rangle$. In the following, for the boundary condition β, we consider different cases: if $\beta = \psi$ for some $\psi \in \Phi$, then the lemma holds since

$$\psi \cdot A \cdot \varphi = (\psi A)\psi \cdot \varphi = (\psi A)\varphi = (\psi \bullet A)\varphi.$$

If $\beta = \psi \cdot B$, then we compute

$$\begin{aligned}\beta \cdot A \cdot \gamma &= \psi \cdot B \cdot A \cdot \varphi = \psi \cdot BA \cdot \varphi \\ &= (\psi BA)\varphi = ((\psi \cdot B) \bullet A)\varphi = (\beta \bullet A)\gamma.\end{aligned}$$

For the case $\beta = \psi \cdot \int \cdot B$, we compute

$$\begin{aligned}\beta \cdot A \cdot \gamma &= \psi \cdot \textstyle\int \cdot B \cdot A \cdot \varphi = \psi \cdot \int \cdot BA \cdot \varphi = \psi \cdot (\int \cdot BA) \cdot \varphi \\ &= (\psi \cdot \textstyle\int \cdot BA) \cdot \varphi = ((\psi \cdot \int \cdot B) \bullet A)\varphi = (\beta \bullet A)\gamma.\end{aligned}$$

For $\beta = \psi \cdot \partial$, we compute

$$\begin{aligned}\beta \cdot A \cdot \gamma &= \psi \cdot \partial \cdot A \cdot \varphi = \psi \cdot A \cdot \partial \cdot \varphi + \psi \cdot \partial A \cdot \varphi \\ &= 0 + \psi \cdot \partial A \cdot \varphi = ((\psi \cdot \partial) \bullet A)\varphi = (\beta \bullet A)\gamma\end{aligned}$$

and similarly for $\beta = \psi \cdot \partial^j$ with $j > 1$. □

Theorem 5.3. *Let $\beta \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ be a boundary condition. Assume there exists $A \in \mathcal{R}$ with $\beta \bullet A = 1$. Then, the operator*

$$P = A \cdot \beta$$

is idempotent. Moreover, the idempotent $1 - P$ satisfies

$$\beta \cdot (1 - P) = 0.$$

Proof. By Lemma 5.2, we see that

$$P \cdot P = (A \cdot \beta) \cdot (A \cdot \beta) = A \cdot ((\beta \bullet A) \cdot \beta) = A \cdot 1 \cdot \beta = A \cdot \beta = P.$$

Moreover, we have

$$\begin{aligned} \beta \cdot (1 - P) &= \beta \cdot (1 - A \cdot \beta) = \beta - \beta \cdot A \cdot \beta \\ &= \beta - (\beta \bullet A) \cdot \beta = \beta - 1 \cdot \beta = \beta - \beta = 0. \end{aligned}$$ □

Definition 5.4. *A first order linear boundary problem is given by a pair (L, β), where $L \in \mathcal{R}\langle\partial\rangle$ is a monic differential operator and $\beta \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ is a boundary condition. An integro-differential operator $G \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ is a Green's operator for (L, β) if*

$$L \cdot G = 1 \quad \textit{and} \quad \beta \cdot G = 0.$$

The following theorem presents a formula for computing a Green's operator of a first-order boundary problem from a given right inverse of the defining differential operator and solutions which "interpolate" the boundary condition. We have already seen in Example 4.15 how to compute a family of right inverses for a monic first-order differential operator in the ring of integro-differential operators assuming that we have a solution for the homogeneous equation which is invertible in $\mathcal{R}$. In addition, every boundary condition $\beta \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ is a $\mathcal{K}$-linear combination of operators of the form $\varphi \cdot N$, with $N \in \mathcal{R}\langle\partial, \int, \Phi\rangle$ according to (5.1). Hence, we see the reduction $\partial \cdot \varphi = 0$ for $\varphi \in \Phi$ implies

$$\partial \cdot \beta = 0. \tag{5.2}$$

Theorem 5.5. *Let (L, β) be a first-order linear boundary problem. Let H be a right inverse of $L = \partial - A \in \mathcal{R}\langle\partial, \int, \Phi\rangle$, let $W \in \mathcal{R}$ such that $L \bullet W = 0$ with $\beta \bullet W$ invertible. Let $P = W(\beta \bullet W)^{-1} \cdot \beta$. Then*

$$G := (1 - P) \cdot H \in \mathcal{R}\langle\partial, \textstyle\int, \Phi\rangle$$

is a Green's operator for (L, β).

Proof. First, we assume $\beta \bullet W = 1$. Then $P = W \cdot \beta$ is idempotent and by Theorem 5.3,

$$\beta \cdot G = \beta \cdot (1 - P) \cdot H = 0 \cdot H = 0.$$

So, it remains to show that G is a right inverse of L. Since

$$L \cdot W = (\partial - A) \cdot W = W \cdot \partial + \partial W - AW = W \cdot \partial,$$

then, by equation (5.2), we have

$$L \cdot W \cdot \beta = W \cdot \partial \cdot \beta = W \cdot 0 = 0.$$

Therefore, we have $L \cdot P = 0$ and

$$L \cdot G = L \cdot (1 - P) \cdot H = L \cdot H - L \cdot P \cdot H = 1 + 0 \cdot H = 1.$$

This means G is a Green's operator for (L, β). If $\beta \bullet W \neq 1$, we take $\tilde{W} = W(\beta \bullet W)^{-1}$. Then, we have

$$L \bullet \tilde{W} = L \bullet (W(\beta \bullet W)^{-1}) = (L \bullet W)(\beta \bullet W)^{-1} = 0,$$

and

$$\beta \bullet \tilde{W} = \beta \bullet (W(\beta \bullet W)^{-1}) = (\beta \bullet W)(\beta \bullet W)^{-1} = 1,$$

and we are reduced to the previous case. □

5.2 Symbolic computations with IDOLS

The goal for this section is to algebraize and automatize symbolic computations with linear differential time-delay (DTD) systems of the form

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - h), \tag{5.3}$$

where $A_0(t)$ and $A_1(t)$ are square matrices and $h > 0$. With the explicit matrices $A_0(t)$ and $A_1(t)$ in (5.3), a standard approach utilizes the ring of integro-differential operators with linear substitutions $\mathcal{D} = \mathcal{A}\langle \partial, \textstyle\int, \Phi, \delta \rangle$ where by δ we denote the shift operator $\sigma_{1,h}$, and the coefficient ring $\mathcal{A}$ consists of scalar functions. By Theorem 4.22, all sums of terms of the form $f\varphi\delta^i\partial^j$ or $f\varphi\delta^i \textstyle\int g$ live in this ring. We can also use the matrix of DTD operators

$$\partial I_n - A_0 - A_1\delta \in \mathcal{D}^{n\times n},$$

for representing the system of (5.3) where, for shorter notation, its coefficients are collected into matrices $A_0, A_1 \in \mathcal{A}^{n\times n}$. However, the formula above is a matrix of operators in $\mathcal{D}^{n\times n}$. Within the *algebraic analysis* approach, described in Section 2.5, we can study the system by means of the matrix of operators, of its associated module, and of the properties of some function space $\mathcal{F}$ that can be considered as a left $\mathcal{D}$-module.

Note that considering a fixed linear differential time-delay system for studying certain (control) problems is not always sufficient. Rather, we should look at whole classes of systems as, for instance, the set of all systems of the form of (5.3) where the matrices A_0 and A_1 are general matrices of generic size. However, such matrices can not be treated rigorously by means of matrices of operators. This requires us to perform formal computations with undetermined matrices as coefficients. To this end, we directly equip our operators with coefficients from some abstract ring $\mathcal{R}$, whose elements represent matrices. In other words, we consider the ring of operators $\mathcal{R}\langle\partial, \textstyle\int, \Phi, \delta\rangle$. Then, the above system is represented by the operator

$$R = \partial - A_0 - A_1 \cdot \delta \in \mathcal{R}\langle\partial, \textstyle\int, \Phi, \delta\rangle$$

with coefficients $A_0, A_1 \in \mathcal{R}$ representing matrices. Following this method, it is possible to consider general classes of systems at once and results on these classes can be obtained directly. To this end, computer algebra methods have been developed at the level of the ring $\mathcal{R}\langle\partial, \textstyle\int, \Phi, \delta\rangle$, see Section 4.6.

By formal computations in the ring of IDOLS, operators having rectangular matrices as their coefficients can also be treated. To illustrate these use-cases, we apply the implementation of IDOLS to largely automatize the computations of [40] for recovering Artstein's transformation and we make these computations available online. We also generalize Artstein's transformation to differential time-delay systems where state and control variables are no longer distinguished.

5.2.1 Rectangular coefficients

It is clear in analysis that we can not add or multiply operators with different domains and codomains (e.g., they have rectangular matrices as their coefficients) together. However, the ring $\mathcal{R}$ of coefficients is just an abstract ring without restriction on addition and multiplication. If some elements of $\mathcal{R}$ stand for matrices of different formats, then their sum in $\mathcal{R}$ can not be interpreted as a matrix. A similar situation arises when multiplying two elements in $\mathcal{R}$ that stand for matrices with incompatible formats. This problem carries over to the whole ring of operators $\mathcal{R}\langle\partial, \textstyle\int, \Phi, \mathcal{S}\rangle$ where the generators ∂, $\textstyle\int$, as well as elements from Φ and $\mathcal{S}$ can be interpreted as operators acting on objects of any size.

Still, by applying the reduction system given in Tables 4.1 and 4.3, valid expressions in $\mathcal{R}\langle\partial, \int, \Phi, \mathcal{S}\rangle$ are transformed into valid expressions, i.e. we can interpret them as actual operators with domain and codomain. We can see this by observing that in each reduction rule the right hand side is interpreted automatically as an operator with the same domain and codomain whenever the left hand side is. In order to do that, one should keep in mind that for every $A \in \mathcal{R}$ which can be interpreted as an operator $A : \mathcal{F}^m \longrightarrow \mathcal{F}^n$, the operations $\partial, \int$, and all $\varphi \in \Phi$ and $\sigma_{a,b} \in \mathcal{S}$ give operators ∂A, $\int A$, φA and $\sigma_{a,b}A$ with the same domain and codomain. The generators $\partial, \int$, and all $\varphi \in \Phi$ and $\sigma_{a,b} \in \mathcal{S}$ in $\mathcal{R}\langle\partial, \int, \Phi, \mathcal{S}\rangle$ are interpreted as operators from any $\mathcal{F}^n$ to itself, as indicated above. For example, we now explicitly check that the Leibniz rule transforms valid expressions into valid expressions. If some $A \in \mathcal{R}$ can be interpreted as a multiplication operator $A : \mathcal{F}^m \longrightarrow \mathcal{F}^n$, then in $\partial \cdot A$ the derivation is interpreted as an operator $\partial : \mathcal{F}^n \longrightarrow \mathcal{F}^n$ and the Leibniz rule

$$\partial \cdot A = A \cdot \partial + \partial A$$

yields $A \cdot \partial$, where the derivation is interpreted as an operator $\partial : \mathcal{F}^m \longrightarrow \mathcal{F}^m$, and ∂A which both map from $\mathcal{F}^m$ to $\mathcal{F}^n$. Analogous to the discussion above, we can check that in each of the identities in $\mathcal{R}$, whenever one term can be interpreted as operator from some $\mathcal{F}^m$ to some $\mathcal{F}^n$ also the other terms can be interpreted as having the same domain and codomain. A formalization of symbolic computations with operators having different domains and codomains will be presented in a future publication.

Example 5.6. *Consider the rectangular differential system*

$$A_1(t)\dot{x}(t) - A_0(t)x(t) = f(t)$$

corresponding to the operator $L := A_1 \cdot \partial - A_0$. *Like Example 4.15, we make the irreducible ansatz* $H := H_1 \cdot \int \cdot H_2$ *for a right inverse of* L, *with undetermined multiplication operators* H_1 *and* H_2. *Then, using the reduction system, we write the product* $L \cdot H$ *in irreducible form.*

$$\begin{aligned} L \cdot H &= (A_1 \cdot \partial - A_0) \cdot H_1 \cdot \textstyle\int \cdot H_2 \\ &= (A_1 H_1 \cdot \partial + A_1 \partial H_1) \cdot \textstyle\int \cdot H_2 - A_0 H_1 \cdot \textstyle\int \cdot H_2 \\ &= A_1 H_1 H_2 + (A_1 \partial H_1 - A_0 H_1) \cdot \textstyle\int \cdot H_2 \end{aligned}$$

Comparing coefficients in $L \cdot H = 1$ *yields*

$$A_1 H_1 H_2 = 1 \quad \textit{and} \quad A_1 \partial H_1 - A_0 H_1 = 0.$$

To solve these equations, we adjoin Θ *and* $\tilde{\Theta}$ *such that*

$$A_1 \Theta \tilde{\Theta} = 1, \quad A_1 \partial\Theta - A_0 \Theta = 0,$$

and we let $H_1 = \Theta$ *and* $H_2 = \tilde{\Theta}$.

5.2.2 Recovering Artstein's transformation

In this section, following the work of [40], we show how our framework can be used to recover and prove Artstein's transformation for DTD control systems of the form

$$\dot{x}(t) = A(t)x(t) + B_0(t)u(t) + B_1(t)u(t-h). \tag{5.4}$$

In order to apply the algebraic framework introduced in Section 4.5, we write this control system as a differential time-delay system where coefficient matrices have block structure:

$$\begin{pmatrix} I_n & 0 \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} A(t) & B_0(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} 0 & B_1(t) \end{pmatrix} \begin{pmatrix} x(t-h) \\ u(t-h) \end{pmatrix}. \tag{5.5}$$

We show how to use our setting to find by ansatz (and then prove) a transformation from the DTD system (5.4) to the differential system

$$\dot{z}(t) = E(t)z(t) + F(t)v(t), \tag{5.6}$$

considered as a differential system with block structure

$$\begin{pmatrix} I_n & 0 \end{pmatrix} \begin{pmatrix} \dot{z}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} E(t) & F(t) \end{pmatrix} \begin{pmatrix} z(t) \\ v(t) \end{pmatrix}. \tag{5.7}$$

The systems (5.5) and (5.7) correspond to the operators

$$R' = R'_0 \cdot \partial + R'_1 + R'_2 \cdot \delta, \tag{5.8}$$

$$R = R_0 \cdot \partial + R_1, \tag{5.9}$$

where

$$R'_0 = \begin{pmatrix} I_n & 0 \end{pmatrix}, \quad R'_1 = \begin{pmatrix} -A & -B_0 \end{pmatrix}, \quad R'_2 = \begin{pmatrix} 0 & -B_1 \end{pmatrix},$$
$$R_0 = \begin{pmatrix} I_n & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -E & -F \end{pmatrix}.$$

Based on statement (i) in Theorem 2.84, we are required to find operators P and Q such that

$$R \cdot P = Q \cdot R'. \tag{5.10}$$

We choose $Q = Q_0$ where Q_0 is a multiplication operator, and consider the following ansatz for the operator P

$$P = P_0 \cdot \delta \cdot \textstyle\int \cdot P_1 + P_2 \cdot \int \cdot P_3 + P_4 \cdot \delta + P_5, \tag{5.11}$$

where the multiplication operators $P_0, P_1, P_2, P_3, P_4, P_5$ have undetermined blocks $P_{11}, P_{22}, a_0, a_1, a_2, a_3, a_4, a_5$ as follows.

$$P_0 = \begin{pmatrix} a_0 \\ 0 \end{pmatrix},\ P_1 = \begin{pmatrix} 0 & a_1 \end{pmatrix},\ P_2 = \begin{pmatrix} a_2 \\ 0 \end{pmatrix},\ P_3 = \begin{pmatrix} 0 & a_3 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & a_4 \\ 0 & 0 \end{pmatrix},\ P_5 = \begin{pmatrix} P_{11} & a_5 \\ 0 & P_{22} \end{pmatrix} \tag{5.12}$$

With this ansatz and using our setting, we discover conditions for the coefficients in order to have (5.10). First, the irreducible forms for the left and the right hand sides of (5.10) are computed. Then, by coefficient comparison, the following conditions are obtained for the blocks $a_0, a_1, a_2, a_3, a_5, P_{11}, P_{22}, Q_0$:

$$\partial a_0 - E a_0 = 0, \tag{5.13}$$
$$\partial a_2 - E a_2 = 0, \tag{5.14}$$
$$P_{11} = Q_0, \tag{5.15}$$
$$a_4 = 0, \tag{5.16}$$
$$a_5 = 0, \tag{5.17}$$
$$a_0 \delta a_1 + \partial a_4 - E a_4 = -Q_0 B_1, \tag{5.18}$$
$$\partial P_{11} - E P_{11} = -Q_0 A, \tag{5.19}$$
$$\partial a_5 + a_2 a_3 - E a_5 - F P_{22} = -Q_0 B_0. \tag{5.20}$$

For solving these equations, following [40], we set $a_4 = a_5 = 0$ and we let P_{11} be such that (5.19) holds. Furthermore, we set $Q_0 = P_{11}$ and we let Φ be invertible such that

$$\partial \Phi = E \Phi. \tag{5.21}$$

Then, for arbitrary constants c_0 and c_2, we assume that

$$a_0 = \Phi c_0 \quad \text{and} \quad a_2 = \Phi c_2. \tag{5.22}$$

This solves six of the above equations. The remaining equations are (5.18) and (5.20) which can be written as

$$c_0 a_1 = -\delta^{-1} \Phi^{-1} P_{11} B_1, \tag{5.23}$$
$$c_2 a_3 = \Phi^{-1}(F P_{22} - P_{11} B_0), \tag{5.24}$$

and we assume that c_0, c_2, a_1, a_3 are such that they satisfy these equations. Considering these assumptions, it is easy to verify that all conditions (5.13) through (5.20) are satisfied, also (5.10) can be verified directly. With these assumptions, equation (5.11) can be rewritten as

$$P = -\begin{pmatrix}\Phi\\0\end{pmatrix}\cdot\delta\cdot\textstyle\int\cdot\begin{pmatrix}0 & \delta^{-1}\Phi^{-1}P_{11}B_1\end{pmatrix} + \begin{pmatrix}\Phi\\0\end{pmatrix}\cdot\textstyle\int\cdot\begin{pmatrix}0 & \Phi^{-1}(FP_{22}-P_{11}B_0)\end{pmatrix} + \begin{pmatrix}P_{11} & 0\\0 & P_{22}\end{pmatrix}. \quad (5.25)$$

In other words, we obtain the invertible transformation

$$\begin{pmatrix}z(t)\\v(t)\end{pmatrix} = \begin{pmatrix}P_{11}(t)x(t)\\P_{22}(t)u(t)\end{pmatrix} + \begin{pmatrix}\Phi(t)\\0\end{pmatrix}\int_{t_0}^{t}\Phi^{-1}(s)T_2(s)u(s)\,ds - \begin{pmatrix}\Phi(t)\\0\end{pmatrix}\int_{t_0}^{t-h}\Phi^{-1}(s+h)P_{11}(s+h)B_1(s+h)u(s)\,ds,$$

where $T_2(t) := F(t)P_{22}(t) - P_{11}(t)B_0(t)$, as in Theorem 5 of [40].

5.2.3 Generalization of Artstein's transformation

The algebraic framework we utilize for transforming equation (5.4) can also be used to transform the differential time-delay control system

$$A_1(t)\dot{x}(t) + A_0(t)x(t) + B_1(t)x(t-h) = 0, \quad (5.26)$$

where state and control variables are not distinguished, to the differential system

$$E_1(t)\dot{z}(t) + E_0(t)z(t) = 0, \quad (5.27)$$

with the corresponding operators

$$R' = A_1\cdot\partial + A_0 + B_1\cdot\delta, \quad (5.28)$$

$$R = E_1\cdot\partial + E_0. \quad (5.29)$$

Although equation (5.5) is an instance of equation (5.26) and equation (5.7) is an instance of (5.27), in contrast to Subsection 5.2.2, we no longer impose a block structure. This generalization (and variants of it) and possible applications will be investigated in future works. For finding a transformation, we again need to find operators P and Q such that

$$R\cdot P = Q\cdot R'. \quad (5.30)$$

We choose $Q = Q_0$ where Q_0 is a multiplication operator, and consider the following ansatz for the operator P:

$$P = P_0 \cdot \delta \cdot \textstyle\int \cdot P_1 + P_2 \cdot \int \cdot P_3 + P_5. \tag{5.31}$$

Using our framework, we compute the normal forms for the left and the right hand sides of (5.30).

$$\begin{aligned} R \cdot P =&(E_0P_0 + E_1\partial P_0) \cdot \delta \cdot \textstyle\int \cdot P_1 \\ &+ (E_0P_2 + E_1\partial P_2) \cdot \textstyle\int \cdot P_3 + E_1P_5 \cdot \partial \\ &+ E_1P_0\delta P_1 \cdot \delta + E_1\partial P_5 + E_1P_2P_3 + E_0P_5 \end{aligned} \tag{5.32}$$

$$Q \cdot R' = Q_0A_1 \cdot \partial + Q_0B_1 \cdot \delta + Q_0A_0 \tag{5.33}$$

Then, by coefficient comparison, we get the following sufficient conditions:

$$E_0P_0 + E_1\partial P_0 = 0, \tag{5.34}$$
$$E_0P_2 + E_1\partial P_2 = 0, \tag{5.35}$$
$$E_1P_5 = Q_0A_1, \tag{5.36}$$
$$E_1P_0\delta P_1 = Q_0B_1, \tag{5.37}$$
$$E_1\partial P_5 + E_1P_2P_3 + E_0P_5 = Q_0A_0. \tag{5.38}$$

For solving these equations, we need to introduce an analog of the fundamental system and a few right inverses. We start by solving (5.34) and (5.35). Let Θ be such that

$$E_1\partial\Theta + E_0\Theta = 0.$$

Then for constants C_0 and C_2, we assume that

$$P_0 = \Theta C_0, \quad P_2 = \Theta C_2. \tag{5.39}$$

For solving equation (5.36) to (5.38) we take $\tilde{\Theta}$ such that

$$E_1\Theta\tilde{\Theta} = 1.$$

With an arbitrary multiplication operator N_5, the operator

$$P_5 = \Theta\tilde{\Theta}Q_0A_1 + (1 - \Theta\tilde{\Theta}E_1)N_5, \tag{5.40}$$

solves equation (5.36). For solving equation (5.37) assume that P_1 can be chosen such that we have

$$C_0P_1 = \delta^{-1}\tilde{\Theta}Q_0B_1, \tag{5.41}$$

and for solving equation (5.38) assume that P_3 can be chosen such that

$$C_2 P_3 = \tilde{\Theta}\big(Q_0 A_0 - E_0 P_5 - E_1 \partial P_5\big). \tag{5.42}$$

Therefore, equation (5.31) can be rewritten as

$$P = \Theta \cdot \delta \cdot \textstyle\int \cdot \delta^{-1} \tilde{\Theta} Q_0 B_1 + \Theta \cdot \int \cdot \tilde{\Theta}\big(Q_0 A_0 - E_0 P_5 - E_1 \partial P_5\big) + P_5. \tag{5.43}$$

If, in addition, $\tilde{\Theta}$ has a right inverse and

$$Q_0 A_0 - E_0 P_5 - E_1 \partial P_5 + E_1 \Theta \delta^{-1} \tilde{\Theta} Q_0 B_1 = 0, \tag{5.44}$$

the operator simplifies to $P = \Theta \cdot (\delta - 1) \cdot \int \cdot \delta^{-1} \tilde{\Theta} Q_0 B_1 + P_5$.

Theorem 5.7. *Consider the time-delay system*

$$A_1(t)\dot{x}(t) + A_0(t)x(t) + B_1(t)x(t-h) = 0,$$

of size $n \times m$ *and the differential system*

$$E_1(t)\dot{z}(t) + E_0(t)z(t) = 0,$$

of size $k \times l$. *Let* $\Theta(t)$ *and* $\tilde{\Theta}(t)$ *such that*

$$E_1(t)\dot{\Theta}(t) + E_0(t)\Theta(t) = 0 \quad \text{and} \quad E_1(t)\Theta(t)\tilde{\Theta}(t) = I_k.$$

For arbitrary matrices $Q_0(t)$ *of size* $k \times n$ *and* $N_5(t)$ *of size* $l \times m$, *we define*

$$P_5(t) = \Theta(t)\tilde{\Theta}(t)Q_0(t)A_1(t) + \Big(I_l - \Theta(t)\tilde{\Theta}(t)E_1(t)\Big)\, N_5(t),$$
$$T(t) = Q_0(t)A_0(t) - E_0(t)P_5(t) - E_1(t)\dot{P}_5(t).$$

Then, the transformation

$$z(t) = P_5(t)x(t) + \Theta(t)\int_{t_0}^{t} \tilde{\Theta}(s)T(s)x(s)\,ds + \Theta(t)\int_{t_0}^{t-h} \tilde{\Theta}(s+h)Q_0(s+h)B_1(s+h)x(s)\,ds \tag{5.45}$$

is a homomorphism mapping solutions of the time-delay system to solutions of the differential system. In particular, if $\tilde{\Theta}(t)$ *has a right inverse and*

$$Q_0(t)A_0(t) - E_0(t)P_5(t) - E_1(t)\dot{P}_5(t) + E_1(t)\Theta(t)\tilde{\Theta}(t+h)Q_0(t+h)B_1(t+h) = 0,$$

then the transformation simplifies to

$$z(t) = P_5(t)x(t) - \Theta(t)\int_{t-h}^{t} \tilde{\Theta}(s+h)Q_0(s+h)B_1(s+h)x(s)\,ds.$$

Note that we can instantiate Theorem 5.7 to obtain the Artstein's transformation (5.25) as follows: we consider special cases of R', R, P, and Θ as follows: for the systems R' and R we take

$$A_1 = \begin{pmatrix} I_n & 0 \end{pmatrix}, \; A_0 = \begin{pmatrix} -A & -B_0 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & -B_1 \end{pmatrix},$$
$$E_1 = \begin{pmatrix} I_n & 0 \end{pmatrix}, \; E_0 = \begin{pmatrix} -E & -F \end{pmatrix}.$$

For the operator P in (5.43), we choose

$$\Theta = \begin{pmatrix} \Phi \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\Theta} = \Phi^{-1},$$

where Φ is a fundamental system of the equation $\partial\Phi = E\Phi$, and we let $Q_0 = P_{11}$ and

$$N_5 = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix}.$$

Therefore, in Artstein's case equation (5.40) corresponds to $Q = P_{11}$, $a_5 = 0$. Furthermore, equation (5.42) corresponds to the two equations

$$\partial P_{11} = EP_{11} - P_{11}A, \quad \text{and} \quad a_2a_3 + P_{11}B_0 - FP_{22} - Ea_5 + \partial a_5 = 0.$$

5.2.4 Inverse of Artstein's transformation

In this section we investigate sufficient conditions for existence of a right inverse of the operator P in (5.31), which allows to transform the solution space of the system R into the solution space of the system R'. The ansatz we consider for such a right inverse is constructed based on the following lemma.

Lemma 5.8. *Let A and B be two operators and assume that for operators A_0, A_1, and B_0 we have*

$$A = A_1 + A_0 \quad \text{and} \quad B = -B_0A_1B_0 + B_0$$

such that $A_1B_0A_1 = 0$. Then $A_0B_0 = 1$ implies $AB = 1$, and $B_0A_0 = 1$ implies $BA = 1$.

Proof. We see that

$$AB = -A_1B_0A_1B_0 - A_0B_0A_1B_0 + A_1B_0 + A_0B_0,$$

$$BA = -B_0A_1B_0A_1 - B_0A_1B_0A_0 + B_0A_1 + B_0A_0.$$

Now by assumption $A_1B_0A_1 = 0$. Hence clearly $A_0B_0 = 1$ implies $AB = 1$ and $B_0A_0 = 1$ implies $BA = 1$. ☐

In addition, we need to consider a few more inverses: let $\tilde{P}$ be a right inverse of P and let $\tilde{Q}$ be a left inverse of Q. Then from (5.30) we obtain the following condition.

$$\tilde{Q} \cdot R \cdot P \cdot \tilde{P} = R' \cdot \tilde{P} \tag{5.46}$$

Based on Lemma 5.8 one possible choice for $\tilde{P}$ can be as follows:

$$\tilde{P} = -\tilde{P}_5 P_0 \cdot \delta \cdot \textstyle\int \cdot P_1 \tilde{P}_5 - \tilde{P}_5 P_2 \cdot \int \cdot P_3 \tilde{P}_5 + \tilde{P}_5 \tag{5.47}$$

where $\tilde{P}_5$ is a right inverse of P_5 with

$$(P_0 \cdot \delta \cdot \textstyle\int \cdot P_1 + P_2 \cdot \int \cdot P_3) \cdot \tilde{P}_5 \cdot (P_0 \cdot \delta \cdot \int \cdot P_1 + P_2 \cdot \int \cdot P_3) = 0.$$

Then, by Lemma 5.8, after computing normal forms and coefficient comparison, we realize that under the following assumptions

$$\begin{cases} P_1 \tilde{P}_5 P_2 = 0 \\ P_3 \tilde{P}_5 P_2 = 0 \\ P_1 \tilde{P}_5 P_0 = 0 \\ P_3 \tilde{P}_5 P_0 = 0 \end{cases} \tag{5.48}$$

equation $P \cdot \tilde{P} = 1$ for P in (5.31) and $\tilde{P}$ in (5.47) holds. In particular, considering the operator P obtained in (5.43) with

$$\begin{aligned} &P_0 = \Theta, \quad P_1 = \delta^{-1} \tilde{\Theta} Q_0 B_1, \\ &P_2 = \Theta, \quad P_3 = \tilde{\Theta}\big(Q_0 A_0 - E_0 P_5 - E_1 \partial P_5\big) \end{aligned} \tag{5.49}$$

conditions (5.48) are satisfied if

$$\big(\delta^{-1} \tilde{\Theta} Q_0 B_1\big) \tilde{P}_5 \Theta = 0, \quad \tilde{\Theta}\big(Q_0 A_0 - E_0 P_5 - E_1 \partial P_5\big) \tilde{P}_5 \Theta = 0. \tag{5.50}$$

Theorem 5.9. *In addition to the assumptions of Theorem 5.7, assume that there exists $\tilde{P}_5(t)$ such that*

$$\begin{aligned} \tilde{\Theta}(t+h) Q_0(t+h) B_1(t+h) \tilde{P}_5(t) \Theta(t) = 0, \\ \tilde{\Theta}(t) T(t) \tilde{P}_5(t) \Theta(t) = 0. \end{aligned}$$

Then, if $\tilde{P}_5(t)$ is a right (resp. left) inverse of $P_5(t)$, the transformation

$$\begin{aligned} x(t) = \tilde{P}_5(t) \bigg(z(t) - \Theta(t) \int_{t_0}^{t} \tilde{\Theta}(s) T(s) \tilde{P}_5(s) z(s)\, ds \\ - \Theta(t) \int_{t_0}^{t-h} \tilde{\Theta}(s+h) Q_0(s+h) B_1(s+h) \tilde{P}_5(s) z(s)\, ds \bigg) \end{aligned} \tag{5.51}$$

is a right (resp. left) inverse of the transformation (5.45).

Chapter 6

Inversive sum-difference operators

Difference equations arise as mathematical models for describing real life situations in diverse subjects such as probability theory, statistical problems, combinatorial analysis, geometry, electrical networks, and so on. We use the theory of difference algebra for studying difference (or functional) equations from the algebraic point of view. The structure of difference algebra is analogous to differential algebra but concerned with difference equations rather than differential equations. As an independent subject, it was initiated by J. F. Ritt [44] and his student R. M. Cohn [17]. For a self-contained reference in the area of difference algebra and algebraic structures with operators, we suggest the reader to look at [37].

An inversive sum-difference ring is a generalization of an inversive difference ring by adding the operations summation and evaluation satisfying certain properties. One of our main contributions described in this chapter is the construction of the ring of inversive sum-difference operators (SDO) over an inversive sum-difference ring by applying tensor reduction systems. The ring of inversive SDO allows us to do symbolic computations with systems of linear difference equations effectively. In fact, we proceed as in the discussions for IDO in Chapter 4. We define this ring in Section 6.1 and illustrate some computations by giving an algebraic proof for the discrete version of the variation of constants method. Then, in Section 6.2, we complete the given reduction system to a confluent one and find normal forms. In addition, by making ansatz and normal form computations we discover a family of right inverses for first-order difference operators. Finally, some computational aspects are mentioned. Some related references for this chapter are also given in Section 1.2.

6.1 Inversive sum-difference rings

A bi-infinite sequence $(A_n)_{n\in\mathbb{Z}}$ in a ring $\mathcal{R}$ is simply a map from $\mathbb{Z}$ to $\mathcal{R}$. The set of all bi-infinite sequences in $\mathcal{R}$ together with componentwise addition and multiplication becomes a ring. On such a ring, we can define the operations shift forward σ, shift backward $\bar{\sigma}$, and evaluation E by

$$\sigma(A_n)_{n\in\mathbb{Z}} := (A_{n+1})_{n\in\mathbb{Z}},\ \ \bar{\sigma}(A_n)_{n\in\mathbb{Z}} := (A_{n-1})_{n\in\mathbb{Z}},\ \ \mathrm{E}(A_n)_{n\in\mathbb{Z}} := (A_0)_{n\in\mathbb{Z}}.$$

We can also define the summation operation Σ as

$$\Sigma(A_n)_{n\in\mathbb{Z}} := \big(\sum_{k=0}^{n-1} A_k\big)_{n\in\mathbb{Z}},$$

with the convention

$$\sum_{k=0}^{n-1} A_k = -\sum_{k=n-1}^{0} A_k, \quad \text{for all } n \leq 1.$$

One can easily check that the operations above satisfy the following identities.

$$\begin{aligned}
\sigma\bar{\sigma}(A_n)_{n\in\mathbb{Z}} &= \bar{\sigma}\sigma(A_n)_{n\in\mathbb{Z}} = (A_n)_{n\in\mathbb{Z}},\\
\sigma\Sigma(A_n)_{n\in\mathbb{Z}} &= (A_n)_{n\in\mathbb{Z}} + \Sigma(A_n)_{n\in\mathbb{Z}},\\
(A_0)_{n\in\mathbb{Z}} &= (A_n)_{n\in\mathbb{Z}} + \Sigma\big((A_n)_{n\in\mathbb{Z}} - \sigma(A_n)_{n\in\mathbb{Z}}\big)
\end{aligned}$$

Moreover, we can also consider the forward and backward difference operators Δ and ∇ defined by

$$\begin{aligned}
\Delta(A_n)_{n\in\mathbb{Z}} &:= (A_{n+1} - A_n)_{n\in\mathbb{Z}},\\
\nabla(A_n)_{n\in\mathbb{Z}} &:= (A_n - A_{n-1})_{n\in\mathbb{Z}}.
\end{aligned}$$

Since we have the identities

$$\Delta = \sigma - \mathrm{id}, \quad \nabla = \mathrm{id} - \bar{\sigma},$$

we do not consider them as basic operators in the setting. The relations among σ, $\bar{\sigma}$, and Σ motivate us to define an inversive sum-difference ring. But, first let us recall the definition of a difference ring and its ring of constants.

Definition 6.1. *Let $\mathcal{R}$ be a ring. We call $(\mathcal{R}, \sigma)$ a* difference *ring if σ is an injective endomorphism of $\mathcal{R}$. For a difference ring $(\mathcal{R}, \sigma)$, we also define its ring of constants by*

$$\mathcal{K} := \{A \in \mathcal{R} \mid \sigma A = A\}.$$

Definition 6.2. *Let $(\mathcal{R}, \sigma)$ be a difference ring. We call $(\mathcal{R}, \sigma, \bar{\sigma})$ an* inversive difference *ring if $\bar{\sigma}$ is an injective endomorphism of $\mathcal{R}$ with $\sigma\bar{\sigma} = \bar{\sigma}\sigma = \mathrm{id}$.*

Definition 6.3. *Let $(\mathcal{R}, \sigma, \bar{\sigma})$ be an inversive difference ring with ring of constants $\mathcal{K}$. Let $\Sigma\colon \mathcal{R} \to \mathcal{R}$ be a $\mathcal{K}$-bimodule homomorphism such that*

$$\sigma\Sigma A = A + \Sigma A \tag{6.1}$$

for all $A \in \mathcal{R}$. We call $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$ an inversive sum-difference *ring if the* evaluation

$$\mathrm{E}A := A + \Sigma(A - \sigma A) \tag{6.2}$$

is multiplicative, i.e. for all $A, B \in \mathcal{R}$ we have $\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B$.

The following lemma shows that in any inversive sum-difference ring, the evaluation E maps to the constants and acts as the identity on them, in particular, it is also a homomorphism of rings. Moreover, the ring $\mathcal{R}$ can be decomposed as direct sum of constant and non-constant "functions".

Lemma 6.4. *Let $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$ be an inversive sum-difference ring with ring of constants $\mathcal{K}$. Then, we have $\mathrm{E}1 = 1$, $\mathrm{E}A \in \mathcal{K}$ for all $A \in \mathcal{R}$, and*

$$\mathcal{R} = \mathcal{K} \oplus \Sigma\mathcal{R},$$

as direct sum of $\mathcal{K}$-bimodules.

Proof. We first compute $\mathrm{E}1 = 1 + \Sigma 1 - \Sigma\sigma 1 = 1$ and

$$\begin{aligned}\sigma\mathrm{E}A &= \sigma(A + \Sigma A - \Sigma\sigma A) = \sigma A + \sigma\Sigma A - \sigma\Sigma\sigma A \\ &= \sigma A + (A + \Sigma A) - (\sigma A + \Sigma\sigma A) = A + \Sigma A - \Sigma\sigma A = \mathrm{E}A.\end{aligned}$$

For any $A \in \mathcal{R}$, we have

$$A = \mathrm{E}A + A - \mathrm{E}A = \mathrm{E}A + \Sigma(\sigma A - A)$$

and hence $\mathcal{R} = \mathcal{K} + \Sigma\mathcal{R}$. Let $A \in \mathcal{K} \cap \Sigma\mathcal{R}$ and $B \in \mathcal{R}$ where $A = \Sigma B$. Then

$$0 = \sigma A - A = \sigma\Sigma B - \Sigma B = B + \Sigma B - \Sigma B = B,$$

which implies $A = 0$. □

Example 6.5. *Let $(\mathcal{S}, \sigma, \bar{\sigma}, \Sigma)$ be a commutative inversive sum-difference ring and let $\mathcal{R} = M_m(\mathcal{S})$. We leave to the reader to check that the ring $\mathcal{R}$ together with the maps $\sigma\colon \mathcal{R} \to \mathcal{R}$ and $\bar{\sigma}\colon \mathcal{R} \to \mathcal{R}$ defined by $\sigma A = (\sigma a_{ij})$ and $\bar{\sigma} A = (\bar{\sigma} a_{ij})$, for any $A = (a_{ij}) \in \mathcal{R}$ where $i, j = 1, \ldots, m$, respectively* *is an inversive difference ring with ring of constants given by matrices with constant entries, i.e.,*

$$\mathcal{K} = \{(c_{ij}) \in \mathcal{R} \mid \sigma c_{ij} = c_{ij}\}.$$

We define also $\Sigma\colon \mathcal{R} \to \mathcal{R}$ *componentwise by*

$$\Sigma A = (\Sigma a_{ij}).$$

Then the map Σ *satisfies* (6.1)*: let* $A = (a_{ij})$ *be an arbitrary element of* $\mathcal{R}$*. Since for any* $a_{i,j} \in \mathcal{S}$*, we have* $\sigma\Sigma a_{ij} = a_{ij} + \Sigma a_{ij}$ *then* $\sigma\Sigma A = A + \Sigma A$*. Moreover, the map* $\mathrm{E}\colon \mathcal{R} \to \mathcal{K}$ *defined by*

$$\mathrm{E}A = (\mathrm{E}a_{ij})$$

is multiplicative: for $A = (a_{ij})$ *and* $B = (b_{ij})$*, if* $AB = (c_{ij})$ *then* $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$ *and hence*

$$\mathrm{E}c_{ij} = \mathrm{E}(\sum_{k=1}^{m} a_{ik}b_{kj}) = \sum_{k=1}^{m} \mathrm{E}a_{ik}b_{kj} = \sum_{k=1}^{m}(\mathrm{E}a_{ik})\mathrm{E}b_{kj}.$$

This implies that $\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B$ *and thus* $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$ *is an inversive sum-difference ring over its ring of constants.*

For the rest of this section, we fix an inversive sum-difference ring $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$ and we denote its ring of constants by $\mathcal{K}$. By an operator, we understand in the following a $\mathcal{K}$-bimodule homomorphism from $\mathcal{R}$ to $\mathcal{R}$. For example, the operations σ, $\bar{\sigma}$, Σ, and E can be viewed as operators.

Following Lemma 6.4, we consider the direct sum decomposition $\mathcal{R} = \mathcal{K} \oplus \Sigma\mathcal{R}$ and the corresponding $\mathcal{K}$-bimodules

$$M_{\mathsf{K}} = \mathcal{K} \quad \text{and} \quad M_{\tilde{\mathsf{R}}} = \Sigma\mathcal{R} \tag{6.3}$$

(indexed by the letters K and $\tilde{\mathsf{R}}$). We do not interpret the elements of M_{K} and $M_{\tilde{\mathsf{R}}}$ as functions but as left multiplication operators $B \mapsto AB$ induced by those functions. For the operators σ, $\bar{\sigma}$, Σ, and E we consider the free left $\mathcal{K}$-modules

$$M_{\mathsf{F}} = \mathcal{K}\sigma, \quad M_{\mathsf{B}} = \mathcal{K}\bar{\sigma}, \quad M_{\mathsf{S}} = \mathcal{K}\Sigma, \quad M_{\mathsf{E}} = \mathcal{K}\mathrm{E} \tag{6.4}$$

generated by them (indexed by the letters F, B, S, and E). We view these modules as $\mathcal{K}$-bimodules with right multiplication defined by

$$c\alpha \cdot d = cd\alpha$$

where $\alpha \in \{\sigma, \bar{\sigma}, \Sigma, \mathrm{E}\}$ and $c, d \in \mathcal{K}$, since the generators of these modules correspond to left $\mathcal{K}$-linear operators. We define two alphabets

$$X = \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{F}, \mathsf{B}, \mathsf{S}, \mathsf{E}\} \quad \text{and} \quad Z = X \cup \{\mathsf{R}\}, \tag{6.5}$$

with the $\mathcal{K}$-bimodules $(M_x)_{x \in X}$ defined in (6.3) and (6.4) as well as

$$M_{\mathsf{R}} = M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}}. \tag{6.6}$$

Now, we define the module M by

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{F}} \oplus M_{\mathsf{B}} \oplus M_{\mathsf{S}} \oplus M_{\mathsf{E}}, \tag{6.7}$$

which turns $(M_z)_{z \in Z}$ into a decomposition with specialization.

Definition 6.6. *Let $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$ be a inversive sum-difference ring with constants $\mathcal{K}$. Then*

$$\mathcal{R}\langle \sigma, \bar{\sigma}, \Sigma, \mathrm{E}\rangle := \mathcal{K}\langle M \rangle / I_{\Sigma_0}$$

is called the ring of inversive sum-difference *operators, where I_{Σ_0} is the two-sided reduction ideal induced by the reduction system*

$$\begin{aligned}\Sigma_0 = \{&(\mathsf{K}, 1 \mapsto \epsilon), (\mathsf{RR}, A \otimes B \mapsto AB), (\mathsf{ER}, \mathrm{E} \otimes A \mapsto (\mathrm{E}A)\mathrm{E}),\\ &(\mathsf{BR}, \bar{\sigma} \otimes A \mapsto \bar{\sigma}A \otimes \bar{\sigma}), (\mathsf{BF}, \bar{\sigma} \otimes \sigma \mapsto \epsilon), (\mathsf{FR}, \sigma \otimes A \mapsto \sigma A \otimes \sigma),\\ &(\mathsf{FB}, \sigma \otimes \bar{\sigma} \mapsto \epsilon), (\mathsf{FS}, \sigma \otimes \Sigma \mapsto \Sigma + \epsilon), (\mathsf{SF}, \Sigma \otimes \sigma \mapsto \Sigma + \epsilon - \mathrm{E})\}.\end{aligned}$$

As in Remark 4.8, we define $\mathcal{K}$-bimodule homomorphisms based on reduction rules in Σ_0. In the following, we verify the variation of constants formula for difference equations algebraically, see for example [1].

Example 6.7. *Consider the difference system*

$$x_{n+1} - A_n x_n = f_n, \quad n \in \mathbb{Z}$$

where $A = (A_n)_{n \in \mathbb{Z}} = ((a_{ijn})_{i,j})_{n \in \mathbb{Z}}$ is a sequence in $\mathcal{R}$, the ring of sequences of matrices of size m having bi-infinite sequences as entries. The system above corresponds to the operator $L = \sigma - A \in \mathcal{R}\langle \sigma, \bar{\sigma}, \Sigma, \mathrm{E}\rangle$. Let $\Phi \in \mathcal{R}$ be an invertible solution of $Lx = 0$. Then $H := \Phi \otimes \Sigma \otimes \sigma\Phi^{-1}$ is a right inverse of L, since independent of the size m we have

$$\begin{aligned}L \otimes H &= (\sigma - A) \otimes \Phi \otimes \Sigma \otimes \sigma\Phi^{-1}\\ &\to_{r_{\mathsf{RR}}} \sigma \otimes \Phi \otimes \Sigma \otimes \sigma\Phi^{-1} - A\Phi \otimes \Sigma \otimes \sigma\Phi^{-1}\\ &\to_{r_{\mathsf{FR}}} \sigma\Phi \otimes \sigma \otimes \Sigma \otimes \sigma\Phi^{-1} - A\Phi \otimes \Sigma \otimes \sigma\Phi^{-1}\\ &\to_{r_{\mathsf{FS}}} \sigma\Phi \otimes \sigma\Phi^{-1} \to_{r_{\mathsf{RR}}} \Phi\Phi^{-1} \to_{r_{\mathsf{K}}} \varepsilon.\end{aligned}$$

This is exactly the formula $x = Hf$ for a particular solution of $Lx = f$ that is obtained from a fundamental matrix by variation of constants:

$$x_n = \Phi_n \sum_{m=0}^{n-1} \Phi_{m+1}^{-1} f_m.$$

6.2 Completion and normal forms

In the following, we describe a completion process for the ring of inversive SDO. Consider the reduction system Σ_0 given in Definition 6.6. We collect the conditions that any compatible partial order on $\langle Z \rangle$ in (6.5) has to satisfy. Analogous to the completion of IDO, we restrict ourselves to monoid partial orders where we have the condition $\epsilon < A$, for any $A \in \langle Z \rangle$. We also require to consider the additional properties $\mathsf{FR} > \mathsf{RF}$, $\mathsf{BR} > \mathsf{RB}$, and $\mathsf{SF} > \mathsf{E}$. Then, in order to obtain the minimal partial order which is consistent with specialization, we also have to consider

$$\mathsf{FK} > \mathsf{KF},\ \mathsf{BK} > \mathsf{KB}, \mathsf{FK} > \tilde{\mathsf{R}}\mathsf{F}, \mathsf{BK} > \tilde{\mathsf{R}}\mathsf{B}, \mathsf{F}\tilde{\mathsf{R}} > \mathsf{KF}, \\ \mathsf{B}\tilde{\mathsf{R}} > \mathsf{KB}, \mathsf{F}\tilde{\mathsf{R}} > \tilde{\mathsf{R}}\mathsf{F}, \mathsf{B}\tilde{\mathsf{R}} > \tilde{\mathsf{R}}\mathsf{B}.$$

The rules r_{FS}, and r_{SF} have two overlap ambiguities with each other. One has S-polynomial

$$\begin{aligned} \mathrm{SP}(\mathsf{F}\underline{\mathsf{S}}, \underline{\mathsf{S}}\mathsf{F}) &= (\Sigma + \epsilon) \otimes \sigma - \sigma \otimes (\Sigma + \epsilon - \mathrm{E}) \\ &\to_{r_{\mathsf{SF}}} \Sigma + \epsilon - \mathrm{E} - \sigma \otimes \Sigma + \sigma \otimes \mathrm{E} \\ &\to_{r_{\mathsf{FS}}} -\mathrm{E} + \sigma \otimes \mathrm{E}, \end{aligned}$$

which gives rise to the new rule

$$(\mathsf{FE}, \sigma \otimes \mathrm{E} \mapsto \mathrm{E}).$$

Analogous to Section 4.2, we can prove that for any inversive sum-difference ring $(\mathcal{R}, \sigma, \bar{\sigma}, \Sigma)$, the ring $\mathcal{R}$ is a left module over the corresponding ring of inversive SDO. Consequently, by applying the operator identity induced by each reduction rule on any $B \in \mathcal{R}$, correctness of the corresponding identity in $\mathcal{R}$ follows, see Remark 4.11. The other has S-polynomial

$$\mathrm{SP}(\mathsf{S}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{S}) = (\Sigma + \epsilon - \mathrm{E}) \otimes \Sigma - \Sigma \otimes (\Sigma + \epsilon) = -\mathrm{E} \otimes \Sigma,$$

and this gives rise to the new rule

$$(\mathsf{ES}, \mathrm{E} \otimes \Sigma \mapsto 0).$$

The rules r_{BF}, and r_{FS} have a non-resolvable overlap ambiguity with S-polynomials

$$\mathrm{SP}(\mathsf{B}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{S}) = \epsilon \otimes \Sigma - \bar{\sigma} \otimes (\Sigma + \epsilon) = \Sigma - \bar{\sigma} \otimes \Sigma - \bar{\sigma},$$

which yields the new rule

$$(\mathsf{BS}, \bar{\sigma} \otimes \Sigma \mapsto \Sigma - \bar{\sigma}).$$

The rules r_{SF}, and r_{FB} have a non-resolvable overlap ambiguity with S-polynomials

$$\begin{aligned}\mathrm{SP}(\mathsf{S\underline{F}},\mathsf{\underline{F}B}) &= (\Sigma+\epsilon-\mathrm{E})\otimes\bar{\sigma}-\Sigma\otimes\epsilon\\ &= \Sigma\otimes\bar{\sigma}+\bar{\sigma}-\mathrm{E}\otimes\bar{\sigma}-\Sigma,\end{aligned}$$

and hence we obtain the new rule

$$(\mathsf{SB},\Sigma\otimes\bar{\sigma}\mapsto\Sigma-\bar{\sigma}+\mathrm{E}\otimes\bar{\sigma}).$$

Moreover, we have to add the condition $\mathsf{SF}>\mathsf{EF}$ and drop the weaker condition $\mathsf{SF}>\mathsf{E}$ from the conditions required for monoid partial orders. The rules r_{SB}, and r_{BR} have a non-resolvable overlap ambiguity with S-polynomials

$$\begin{aligned}\mathrm{SP}(\mathsf{S\underline{B}},\mathsf{\underline{B}R}) &= -\bar{\sigma}\otimes A+\Sigma\otimes A+\mathrm{E}\otimes\bar{\sigma}\otimes A-\Sigma\otimes\bar{\sigma}A\otimes\bar{\sigma}\\ &\to_{r_{\mathsf{BR}}} -\bar{\sigma}A\otimes\bar{\sigma}+\Sigma\otimes A+\mathrm{E}\otimes\bar{\sigma}A\otimes\bar{\sigma}-\Sigma\otimes\bar{\sigma}A\otimes\bar{\sigma}\\ &\to_{r_{\mathsf{ER}}} -\bar{\sigma}A\otimes\bar{\sigma}+\Sigma\otimes A+(\mathrm{E}\bar{\sigma}A)\mathrm{E}\otimes\bar{\sigma}-\Sigma\otimes\bar{\sigma}A\otimes\bar{\sigma}.\end{aligned}$$

While we could reduce further, by using r_{K} for example, we will not be able to reduce to zero for all $A\in\mathcal{R}$. We would like to have a new reduction homomorphism on M_{SRB} that reduces $\Sigma\otimes\bar{\sigma}A\otimes\bar{\sigma}$ to

$$-\bar{\sigma}A\otimes\bar{\sigma}+\Sigma\otimes A+(\mathrm{E}\bar{\sigma}A)\mathrm{E}\otimes\bar{\sigma}.$$

Replacing A by σA, we arrive at the definition

$$(\mathsf{SRB},\Sigma\otimes A\otimes\bar{\sigma}\mapsto -A\otimes\bar{\sigma}+\Sigma\otimes\sigma A+(\mathrm{E}A)\mathrm{E}\otimes\bar{\sigma}).$$

In addition, we add the condition $\mathsf{SR}>\mathsf{E}$ to the list of conditions for the monoid orders we had so far. The rules r_{SF}, and r_{FR} have a non-resolvable overlap ambiguity with S-polynomials

$$\begin{aligned}\mathrm{SP}(\mathsf{S\underline{F}},\mathsf{\underline{F}R}) &= A+\Sigma\otimes A-\mathrm{E}\otimes A-\Sigma\otimes\sigma A\otimes\sigma\\ &\to_{r_{\mathsf{ER}}} A+\Sigma\otimes A-(\mathrm{E}A)\mathrm{E}-\Sigma\otimes\sigma A\otimes\sigma.\end{aligned}$$

We want to have a new reduction homomorphism on M_{SRF} that reduces $\Sigma\otimes\sigma A\otimes\sigma$ to $A+\Sigma\otimes A-(\mathrm{E}A)\mathrm{E}$. Then replacing A by $\bar{\sigma}A$, we arrive at the definition

$$(\mathsf{SRF},\Sigma\otimes A\otimes\sigma\mapsto\bar{\sigma}A+\Sigma\otimes\bar{\sigma}A-(\mathrm{E}\bar{\sigma}A)\mathrm{E}).$$

That allows to reduce all the S-polynomials of the overlap ambiguity of r_{SF}, and r_{FR} to zero. This rule gives rise to a non-resolvable overlap ambiguity with r_{FS} among others. The corresponding S-polynomials can be reduced to

$$\begin{aligned}\mathrm{SP}(\mathsf{SR}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{S}) &= \Sigma \otimes \bar{\sigma}A \otimes \Sigma + \bar{\sigma}A \otimes \Sigma \\ &\quad - (\mathrm{E}\bar{\sigma}A)\mathrm{E} \otimes \Sigma - \Sigma \otimes A - \Sigma \otimes A \otimes \Sigma \\ &\to_{r_{\mathsf{ES}}} \Sigma \otimes (\bar{\sigma}A - A) \otimes \Sigma + \bar{\sigma}A \otimes \Sigma - \Sigma \otimes A.\end{aligned}$$

We would like to have a new reduction homomorphism on M_{SRS} that reduces $\Sigma \otimes (\bar{\sigma}A - A) \otimes \Sigma$ to $\Sigma \otimes A - \bar{\sigma}A \otimes \Sigma$. Replacing A by $\Sigma\sigma A$, we arrive at the definition

$$(\mathsf{SRS}, \Sigma \otimes A \otimes \Sigma \mapsto \Sigma A \otimes \Sigma - \Sigma \otimes \Sigma A - \Sigma \otimes A),$$

where we consider the condition $\mathsf{S} > \tilde{\mathsf{R}}$ for the partial orders. The rules r_{BF}, and r_{FE} have a non-resolvable overlap ambiguity with S-polynomials

$$\mathrm{SP}(\mathsf{B}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{E}) = \epsilon \otimes \mathrm{E} - \bar{\sigma} \otimes \mathrm{E} = \mathrm{E} - \bar{\sigma} \otimes \mathrm{E},$$

and hence we obtain from it the new rule

$$(\mathsf{BE}, \bar{\sigma} \otimes \mathrm{E} \mapsto \mathrm{E}).$$

The rules r_{SF}, and r_{FE} have a non-resolvable overlap ambiguity with S-polynomials

$$\mathrm{SP}(\mathsf{S}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{E}) = (\Sigma + \epsilon - \mathrm{E}) \otimes \mathrm{E} - \Sigma \otimes \mathrm{E} = \mathrm{E} - \mathrm{E} \otimes \mathrm{E},$$

and hence we obtain from it the new rule

$$(\mathsf{EE}, \mathrm{E} \otimes \mathrm{E} \mapsto \mathrm{E}).$$

Finally, we consider the inclusion ambiguity (with specialization) of this new rule with r_{K}, which has irreducible S-polynomials

$$\begin{aligned}\mathrm{SP}(\mathsf{K}, \mathsf{S}\underline{\mathsf{K}}\mathsf{S}) &= \Sigma \otimes \epsilon \otimes \Sigma - (\Sigma 1 \otimes \Sigma - \Sigma \otimes \Sigma 1 - \Sigma \otimes 1) \\ &= \Sigma \otimes \Sigma - \Sigma 1 \otimes \Sigma + \Sigma \otimes \Sigma 1 + \Sigma.\end{aligned}$$

At this point, the leading term is not determined by our partial order above. We decide to have the new rule

$$(\mathsf{SS}, \Sigma \otimes \Sigma \mapsto \Sigma 1 \otimes \Sigma - \Sigma \otimes \Sigma 1 - \Sigma)$$

and extend $\leq$ accordingly to have it compatible with the new rule. Similarly, the overlap ambiguity of r_{SRF} and r_{FE} gives rise to the rule r_{SRE}, which in turn has an inclusion ambiguity with r_{K} giving rise to r_{SE}. Thereby, we obtain the reduction system given in Table 6.1.

Reduction rules in Σ_0	
K	$1 \mapsto \epsilon$
RR	$A \otimes B \mapsto AB$
ER	$\mathrm{E} \otimes A \mapsto (\mathrm{E}A)\mathrm{E}$
BR	$\bar{\sigma} \otimes A \mapsto \bar{\sigma}A \otimes \bar{\sigma}$
BF	$\bar{\sigma} \otimes \sigma \mapsto \epsilon$
FR	$\sigma \otimes A \mapsto \sigma A \otimes \sigma$
FB	$\sigma \otimes \bar{\sigma} \mapsto \epsilon$
FS	$\sigma \otimes \Sigma \mapsto \Sigma + \epsilon$
SF	$\Sigma \otimes \sigma \mapsto \Sigma + \epsilon - \mathrm{E}$
Consequences of reduction rules in Σ_0	
EE	$\mathrm{E} \otimes \mathrm{E} \mapsto \mathrm{E}$
ES	$\mathrm{E} \otimes \Sigma \mapsto 0$
BE	$\bar{\sigma} \otimes \mathrm{E} \mapsto \mathrm{E}$
BS	$\bar{\sigma} \otimes \Sigma \mapsto \Sigma - \bar{\sigma}$
FE	$\sigma \otimes \mathrm{E} \mapsto \mathrm{E}$
SE	$\Sigma \otimes \mathrm{E} \mapsto \Sigma 1 \otimes \mathrm{E}$
SB	$\Sigma \otimes \bar{\sigma} \mapsto \Sigma - \bar{\sigma} + \mathrm{E} \otimes \bar{\sigma}$
SS	$\Sigma \otimes \Sigma \mapsto \Sigma 1 \otimes \Sigma - \Sigma \otimes \Sigma 1 - \Sigma$
SRE	$\Sigma \otimes A \otimes \mathrm{E} \mapsto \Sigma A \otimes \mathrm{E}$
SRB	$\Sigma \otimes A \otimes \bar{\sigma} \mapsto -A \otimes \bar{\sigma} + \Sigma \otimes \sigma A + (\mathrm{E}A)\mathrm{E} \otimes \bar{\sigma}$
SRF	$\Sigma \otimes A \otimes \sigma \mapsto \bar{\sigma}A + \Sigma \otimes \bar{\sigma}A - (\mathrm{E}\bar{\sigma}A)\mathrm{E}$
SRS	$\Sigma \otimes A \otimes \Sigma \mapsto \Sigma A \otimes \Sigma - \Sigma \otimes \Sigma A - \Sigma \otimes A$

Table 6.1: Reduction rules for inversive SDO

The whole completion process for Table 6.1 can be found in the example file of the `TenReS` package. The following table represents identities in the coefficient ring $\mathcal{R}$ corresponding to the reduction rules in Σ_0 and their consequences discovered in the completion process.

Identities in $\mathcal{R}$ corresponding to reduction rules in Σ_0	
$\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B$	$\sigma\bar{\sigma}B = B$
$\bar{\sigma}AB = (\bar{\sigma}A)\bar{\sigma}B$	$\sigma\Sigma B = B + \Sigma B$
$\bar{\sigma}\sigma B = B$	$\Sigma\sigma B = \Sigma B + B - \mathrm{E}B$
$\sigma AB = (\sigma A)\sigma B$	
Identities in $\mathcal{R}$ corresponding to consequences of reduction rules in Σ_0	
$\mathrm{E}\mathrm{E}B = \mathrm{E}B$	$\Sigma A\mathrm{E}B = (\Sigma A)\mathrm{E}B$
$\mathrm{E}\Sigma B = 0$	$\Sigma A\bar{\sigma}B = -A\bar{\sigma}B + \Sigma(\sigma A)B + (\mathrm{E}A)\mathrm{E}\bar{\sigma}B$
$\bar{\sigma}\mathrm{E}B = \mathrm{E}B$	$\Sigma A\sigma B = (\bar{\sigma}A)B + \Sigma(\bar{\sigma}A)B - (\mathrm{E}\bar{\sigma}A)\mathrm{E}B$
$\bar{\sigma}\Sigma B = \Sigma B - \bar{\sigma}B$	$\Sigma A\Sigma B = (\Sigma A)\Sigma B - \Sigma(\Sigma A)B - (\Sigma A)B$
$\sigma\mathrm{E}B = \mathrm{E}B$	

Table 6.2: Identities in $\mathcal{R}$ corresponding to reduction rules for inversive SDO

The identities that do not follow immediately from the definitions are $\mathrm{E}\Sigma B = 0$, summation by parts

$$\Sigma A\bar{\sigma}B = -A\bar{\sigma}B + \Sigma(\sigma A)B + (\mathrm{E}A)\mathrm{E}\bar{\sigma}B,$$

$$\Sigma A\sigma B = (\bar{\sigma}A)B + \Sigma(\bar{\sigma}A)B - (\mathrm{E}\bar{\sigma}A)\mathrm{E}B,$$

and the Rota-Baxter identity with weight 1, i.e.

$$\Sigma A\Sigma B = (\Sigma A)\Sigma B - \Sigma(\Sigma A)B - (\Sigma A)B.$$

These identities can be verified as explained in Remark 4.11.

Example 6.8. *Analogous to Example 4.13, we continue Example 6.7 by doing computations in the confluent reduction system Σ_{SDO}. The equation $Lx = f$ is equivalent to the equation $(H \otimes L)x = Hf$. We find the irreducible form*

$$\begin{aligned} H \otimes L &= \Phi \otimes \Sigma \otimes \sigma\Phi^{-1} \otimes (\partial - A) \\ &\to_{r_{\mathsf{RR}}} \Phi \otimes \Sigma \otimes \sigma\Phi^{-1} \otimes \partial - \Phi \otimes \Sigma \otimes \sigma\Phi^{-1}A \\ &\to_{r_{\mathsf{SRF}}} \Phi\Phi^{-1} + \Phi \otimes \Sigma \otimes \Phi^{-1} - \Phi(\mathrm{E}\Phi) \otimes \mathrm{E} - \Phi \otimes \Sigma \otimes (\sigma\Phi^{-1})A \\ &\to_{r_{\mathsf{K}}} \epsilon - \Phi\mathrm{E}\Phi^{-1} \otimes \mathrm{E}, \end{aligned}$$

where we used the identity $\Phi^{-1} - \sigma\Phi^{-1}A = 0$ obtained in Example . Defining the projector $P = \Phi(\mathrm{E}\Phi^{-1}) \otimes \mathrm{E}$ allows us to write $(H \otimes L)x = Hf$ as $x = Px + Hf$, which yields the general solution obtained by variation of constants:

$$x_n = \Phi_n\Phi_{n_0}^{-1}x_{n_0} + \Phi_n\sum_{m=0}^{n-1}\Phi_{m+1}^{-1}f_m.$$

In order to compute in $\mathcal{R}\langle\sigma,\bar{\sigma},\Sigma,\mathrm{E}\rangle$ we want to analyze the reduction system defined by Table 6.1 according to Theorem 3.32 for tensor rings and determine normal forms of tensors.

Theorem 6.9. *Let $(\mathcal{R},\sigma,\bar{\sigma},\Sigma)$ be an inversive sum-difference ring with constants $\mathcal{K}$. Let M be defined by* (6.6) *and* (6.7) *and let the reduction system Σ_{SDO} be defined by Table 6.1. Then every $t \in \mathcal{K}\langle M\rangle$ has a unique normal form $t\downarrow_{\Sigma_{\mathrm{SDO}}}$, which is given by a sum of pure tensors of the form*

$$A \otimes \mathrm{E} \otimes \sigma^i \quad or \quad A \otimes \mathrm{E} \otimes \bar{\sigma}^j \quad or \quad A \otimes \Sigma \otimes B$$

where $i, j \in \mathbb{N}_0$, each of $A, B \in M_{\tilde{\mathsf{R}}}$ and E may be absent. Moreover,

$$\mathcal{R}\langle\sigma,\bar{\sigma},\Sigma,\mathrm{E}\rangle \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}$$

as $\mathcal{K}$-rings, where the multiplication on $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$ is defined by

$$s \cdot t := (s \otimes t)\downarrow_{\Sigma_{\mathrm{SDO}}} .$$

Proof. We consider the alphabets X and Z given by (6.5). This turns $(M_z)_{z\in Z}$ into a decomposition with specialization for the module M, see definition of decomposition with specialization over a $\mathcal{K}$-bimodule. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ_{SDO}, it is sufficient to require the order to satisfy

$$\mathsf{BR} > \mathsf{RB},\ \mathsf{FR} > \mathsf{RF},\ \mathsf{SB} > \mathsf{BE},\ \mathsf{SR} > \mathsf{E},\ \mathsf{S} > \tilde{\mathsf{R}}.$$

For instance, we could use a degree-lexicographic order with $\mathsf{S} > \mathsf{F} > \mathsf{B} > \mathsf{E} > \mathsf{R}$ on $\langle\{\mathsf{R},\mathsf{F},\mathsf{B},\mathsf{S},\mathsf{E}\}\rangle \subseteq \langle Z\rangle$ or other degree-lexicographic orders with $\mathsf{F} > \mathsf{R}$, $\mathsf{B} > \mathsf{R}$, $\mathsf{S} > \mathsf{E}$, and $\mathsf{S} > \mathsf{R}$. We extend it to a monoid partial order on $\langle Z\rangle$ based on definition of refined order in order to make it consistent with specialization. Then by the package `TenReS` we verify that all ambiguities of Σ are resolvable. Hence, by Theorem 3.32 for tensor rings every element of $\mathcal{K}\langle M\rangle$ has a unique normal form and $\mathcal{K}\langle M\rangle/I_{\Sigma_{\mathrm{SDO}}} \cong \mathcal{K}\langle M\rangle_{\mathrm{irr}}$ as $\mathcal{K}$-rings.

It remains to determine the explicit form of elements in $\mathcal{K}\langle M\rangle_{\mathrm{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\mathrm{irr}}$ in $\langle X\rangle$. Irreducible words containing only the letters K and $\tilde{\mathsf{R}}$ have to avoid the subwords K and $S(\mathsf{RR}) = \{\mathsf{KK},\mathsf{K}\tilde{\mathsf{R}},\tilde{\mathsf{R}}\mathsf{K},\tilde{\mathsf{R}}\tilde{\mathsf{R}}\}$, hence only the words ϵ and $\tilde{\mathsf{R}}$ are left. In addition, the irreducible words containing only E are exactly ϵ, E. Altogether, we see that the irreducible words containing only the letters K, $\tilde{\mathsf{R}}$, and E are given by the set $\{\epsilon,\tilde{\mathsf{R}},\mathsf{E},\tilde{\mathsf{R}}\mathsf{E}\}$, since they also have to avoid the subwords $S(\mathsf{ER}) = \{\mathsf{EK},\mathsf{E}\tilde{\mathsf{R}}\}$. Allowing also the letters F and B, we have to avoid the subwords coming from $S(\mathsf{FR}) = \{\mathsf{FK},\mathsf{F}\tilde{\mathsf{R}}\}$, and $S(\mathsf{BR}) = \{\mathsf{BK},\mathsf{B}\tilde{\mathsf{R}}\}$. Therefore, we can only append words F^j and B^t with $j, t \in \mathbb{N}_0$ to the irreducible words determined so far, in order to obtain all elements of $\langle X \rangle_{\mathrm{irr}}$ not containing the letter S. Finally, we also consider the letter S. Since subwords ES, FS, and BS have to be avoided, the first occurrence of S in an irreducible word can only be preceded by ϵ or $\tilde{\mathsf{R}}$. We also have to avoid the subwords SE, SF, SB, and SS, so any letter immediately following S has to be $\tilde{\mathsf{R}}$. In addition, we have to avoid the subwords $S(\mathsf{SRE}) = \{\mathsf{SKE}, \mathsf{S\tilde{R}E}\}$, $S(\mathsf{SRB}) = \{\mathsf{SKB}, \mathsf{S\tilde{R}B}\}$, $S(\mathsf{SRF}) = \{\mathsf{SKF}, \mathsf{S\tilde{R}F}\}$, and $S(\mathsf{SRS}) = \{\mathsf{SKS}, \mathsf{S\tilde{R}S}\}$, so the letter S cannot be followed by a subword of length greater than one. Altogether, the elements of $\langle X \rangle_{\mathrm{irr}}$ are of the form

$$\tilde{\mathsf{R}}\mathsf{EF}^i \quad \text{or} \quad \tilde{\mathsf{R}}\mathsf{EB}^j \quad \text{or} \quad \tilde{\mathsf{R}}\mathsf{S}\tilde{\mathsf{R}},$$

where $i, j \in \mathbb{N}_0$ and $\tilde{\mathsf{R}}$ may be absent. The normal forms follow from definition of the $\mathcal{K}$-subbimodule of irreducible tensors. □

By means of the normal forms above we are allowed to do coefficient comparison for operator identities in the ring of SDO.

Example 6.10. *For finding a family of right inverses for the operator $L = \sigma - A$ we make the ansatz*

$$H = H_0 + H_1 \cdot \mathrm{E} + H_2 \cdot \mathrm{E} \cdot \sigma + H_3 \cdot \mathrm{E} \cdot \bar{\sigma} + H_4 \cdot \Sigma \cdot H_5.$$

In terms of normal forms given in Theorem 6.9 we can compute

$$\begin{aligned} L \cdot H = {} & (\sigma H_4 - AH_4) \cdot \mathrm{E} \cdot \Sigma \cdot H_5 + (\sigma H_3 - AH_3) \cdot \mathrm{E} \cdot \bar{\sigma} \\ & + (\sigma H_2 - AH_2) \cdot \mathrm{E} \cdot \sigma + (\sigma H_1 - AH_1) \cdot \mathrm{E} + \sigma H_0 \cdot \sigma \\ & + (\sigma H_4) H_5 - AH_0. \end{aligned}$$

Then, by coefficient comparison, for the blocks $H_0, H_1, H_2, H_3, H_4, H_5$ we obtain the following conditions.

$$\begin{aligned} \sigma H_4 - AH_4 &= 0 \\ \sigma H_3 - AH_3 &= 0 \\ \sigma H_2 - AH_2 &= 0 \\ \sigma H_1 - AH_1 &= 0 \\ \sigma H_0 &= 0 \\ (\sigma H_4) H_5 - AH_0 &= 1 \end{aligned}$$

For solving these equations, we adjoin an invertible Φ such that $\sigma\Phi - A\Phi = 0$ and let $H_1 = H_2 = H_3 = H_4 = \Phi$, $H_5 = \sigma\Phi^{-1}$. Then, we obtain

$$H = \Phi \cdot \mathrm{E} + \Phi \cdot \mathrm{E} \cdot \sigma + \Phi \cdot \mathrm{E} \cdot \bar{\sigma} + \Phi \cdot \Sigma \cdot \sigma\Phi^{-1},$$

as a family of right inverses for L involving the operator H in Example 6.7.

6.2.1 Computational aspects

In the following, we express some computational details of the tensor setting with specialization for the ring of inversive sum-difference operators. For the reduction system Σ_{SDO}, by applying `TenReS` in total we obtain 92 ambiguities and corresponding S-polynomials. Among them, there are 6 ambiguities for which the corresponding S-polynomials are zero anyway, for instance

$$\mathrm{SP}(\mathsf{B}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{B}) = \epsilon \otimes \bar{\sigma} - \bar{\sigma} \otimes \epsilon = 0.$$

Applying automatically the implementation of reduction rules from Σ_{SDO}, identities in $\mathcal{R}$ and identities in M_{B}, M_{F}, M_{S} and M_{E} we see that the S-polynomials of the 86 remaining ambiguities are reduced to zero. The complete computation is included in the example files of the package. Here we consider a few concrete instances of ambiguities. For example, we use the definition of E in $\mathcal{R}$ in the reduction of the following S-Polynomial

$$\begin{aligned}\mathrm{SP}(\mathsf{SR}\underline{\mathsf{F}}, \underline{\mathsf{F}}\mathsf{E}) &= (\bar{\sigma}A + \Sigma \otimes \bar{\sigma}A - (\mathrm{E}\bar{\sigma}A)\mathrm{E}) \otimes \mathrm{E} - \Sigma \otimes A \otimes \mathrm{E} \\ &\to_{r_{\mathsf{SRE}}} \bar{\sigma}A \otimes \mathrm{E} + \Sigma\bar{\sigma}A \otimes \mathrm{E} - (\mathrm{E}\bar{\sigma}A)\mathrm{E} \otimes \mathrm{E} - \Sigma A \otimes \mathrm{E} \\ &\to_{r_{\mathsf{EE}}} \bar{\sigma}A \otimes \mathrm{E} + \Sigma\bar{\sigma}A \otimes \mathrm{E} - (\mathrm{E}\bar{\sigma}A)\mathrm{E} - \Sigma A \otimes \mathrm{E} \\ &= \bar{\sigma}A \otimes \mathrm{E} + (\Sigma - \bar{\sigma} + \mathrm{E}\bar{\sigma})A \otimes \mathrm{E} - (\mathrm{E}\bar{\sigma}A)\mathrm{E} - \Sigma A \otimes \mathrm{E} = 0.\end{aligned}$$

As another example, we use the definition of the right multiplication in the $\mathcal{K}$-bimodule M_{S} in the following reduction

$$\begin{aligned}\mathrm{SP}(\mathsf{S}\underline{\mathsf{E}}, \underline{\mathsf{E}}\mathsf{R}) &= (\Sigma 1 \otimes \mathrm{E}) \otimes A - \Sigma \otimes (\mathrm{E}A)\mathrm{E} \to_{r_{\mathsf{SE}}} \Sigma 1 \otimes \mathrm{E} \otimes A - \mathrm{E}A(\Sigma 1 \otimes \mathrm{E}) \\ &\to_{r_{\mathsf{ER}}} \Sigma 1 \otimes (\mathrm{E}A)\mathrm{E} - \mathrm{E}A(\Sigma 1 \otimes \mathrm{E}) = (\Sigma 1 \mathrm{E}A) \otimes \mathrm{E} - \mathrm{E}A(\Sigma 1 \otimes \mathrm{E}) \\ &= (\mathrm{E}A)\Sigma 1 \otimes \Sigma - \mathrm{E}A(\Sigma 1 \otimes \mathrm{E}) = 0.\end{aligned}$$

There are 83 ambiguities without specialization. The remaining 13 are inclusion ambiguities with specialization. For example,

$$\mathrm{SP}(\mathsf{SR}\underline{\mathsf{E}}, \underline{\mathsf{E}}\mathsf{S}) = (\Sigma A \otimes \mathrm{E}) \otimes \Sigma - \Sigma \otimes A \otimes 0 \to_{r_{\mathsf{ES}}} 0,$$

and

$$\mathrm{SP}(\underline{\mathsf{K}}, \mathsf{F}\underline{\mathsf{R}}) = \sigma \otimes \epsilon \otimes \epsilon - 1 \otimes \sigma \ \to_{r_{\mathsf{K}}} \sigma - \sigma = 0.$$

Note that the confluence criterion of Theorem 3.32 directly works with the reduction system Σ_{SDO}. Therefore, no computations with the refined reduction system Σ_X on X, given in (6.5), are required.

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