Algorithmic Operator Algebras via Normal Forms for Tensors

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ABSTRACT

We propose a general algorithmic approach to noncommutative operator algebras generated by linear operators. Ore algebras are a well-established tool covering many cases arising in applications. However, integro-differential operators, for example, do not fit this structure. Instead of using (parametrized) Gröbner bases in noncommutative polynomial algebras as has been used so far in the literature, we use Bergman's basis-free analog in tensor algebras. This allows for a finite reduction system with unique normal forms. To have a smaller reduction system, we develop a generalization of Bergman's setting, which also makes the algorithmic verification of the confluence criterion more efficient. We provide an implementation in Mathematica and we illustrate both versions of the tensor setting using integro-differential operators as an example.

Keywords

operator algebra; tensor algebra; integro-differential operators; noncommutative Gröbner basis; reduction systems

1. INTRODUCTION

Skew polynomials are used in the literature for an algebraic and algorithmic treatment of many common operators like differential and difference operators; see e.g. [6] or the recent overview [8]. Normal forms for skew polynomials are given by the standard polynomial basis. However, normal forms for univariate integral operators are of the form $f\int g$. We show that tensor algebras and their quotients are useful for algebraic modeling of and algorithmic computations with linear operators. Tensor algebras naturally capture the multilinearity of composition of linear operators. In addition, they allow basis-free treatment of multiplication operators. Moreover, for integro-differential operators, they also cover general rings of constants which do not have to be fields.

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For computing in quotients of tensor algebras, we use Bergman's analog [3] of Gröbner bases, which we recall in Section 2. Obstructions for general algorithms, like deciding existence of finite Gröbner bases, are inherited from the noncommutative polynomial algebra case. In this paper, we focus on verification of confluence based on Bergman's criterion, for which we need to compute in the tensor algebra. However, for a confluent reduction system, determining normal forms reduces to a combinatorial problem on words.

In Section 3, we apply Bergman's original result to integro-differential operators in a direct way. Motivated by shortcomings of that, we develop a two-level tensor setting in Section 4. Its flexibility in modeling operator algebras allows for a smaller reduction system and makes the computation more efficient, see Section 5. In each section, we comment about the computational aspects and our implementation. The Mathematica package `TenReS` [10] together with the examples of this paper is available at:

http://gregensburger.com/software/TenReS.zip

Using differential operators, we illustrate advantages of the tensor setting over construction in noncommutative polynomial algebras. Assuming no knowledge about their normal forms, we cannot use skew polynomials here. Throughout the paper, K denotes a commutative ring with unity.

Example 1. Recall that differential operators with polynomial coefficients (Weyl algebra) over a field $F \supseteq \mathbb{Q}$ can be defined as the quotient algebra $F\langle X, D\rangle/(DX - XD - 1)$ of $F\langle X, D\rangle$. For any differential K-algebra (R, ∂) over K, we introduce for each $f \in R$ an indeterminate $\lceil f\rceil$ and consider the noncommutative polynomial algebra $K\langle(\lceil f\rceil)_{f\in R}, \partial\rangle$. If we have a K-basis of R, then we just take basis elements as generators and use basis expansion in each step. However, this would not affect the main points of the following discussion. We factor out the two-sided ideal generated by the parametrized identities

$$\lceil f\rceil\lceil g\rceil = \lceil fg\rceil \quad \text{and} \quad \partial\lceil f\rceil = \lceil f\rceil\partial + \lceil\partial f\rceil$$

for $f, g \in R$, corresponding to the composition of multiplication operators and the Leibniz rule. Computing the parametrized S-polynomial between these two identities, we obtain after some reduction steps

$$\begin{aligned}\mathrm{S}(\partial\lceil f\rceil, \lceil f\rceil\lceil g\rceil) &= (\lceil f\rceil\partial + \lceil\partial f\rceil)\lceil g\rceil - \partial\lceil fg\rceil \\ &\to \lceil fg\rceil\partial + \lceil f\partial g\rceil + \lceil(\partial f)g\rceil - \lceil fg\rceil\partial - \lceil\partial(fg)\rceil\end{aligned}$$

for $f, g \in R$. For reducing this parametrized S-polynomial to zero, we would need the identity $\lceil\partial(fg)\rceil = \lceil(\partial f)g\rceil + \lceil f\partial g\rceil$ for $f, g \in R$, corresponding to addition and the Leibniz rule in the algebra R. However, such identities cannot directly be used for Gröbner bases in the noncommutative polynomial algebra without expansion w.r.t. a basis of R.

As mentioned above, for a basis-free treatment of the K-linear structure of R, we can use the tensor algebra on the module $R \oplus K\partial$ for differential operators. To deal with parametrized rules, one uses reduction rules defined by K-module homomorphisms. Corresponding to the two parametrized identities above, we need two homomorphisms defined by $f \otimes g \mapsto fg$ and $\partial \otimes f \mapsto f \otimes \partial + \partial f$. The S-polynomials formed from these two homomorphisms reduce to zero for all $f, g \in R$ due to the Leibniz rule in R:

$$\begin{aligned} \mathrm{S}(\partial \otimes f, f \otimes g) &= (f \otimes \partial + \partial f) \otimes g - \partial \otimes (fg) \\ &\to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg) = 0. \end{aligned}$$

2. GRÖBNER BASES FOR TENSORS

In this section, we introduce the setting and notions for Gröbner bases in tensor algebras following Bergman and using standard notation for reduction systems from [1]. We first outline the construction of the K-tensor algebra $K\langle M\rangle$ on a K-module M, which is a generalization of the noncommutative polynomial algebra on a set.

We denote the n-fold tensor product of M with itself over K by $M^{\otimes n} = M \otimes \ldots \otimes M$ (n factors). In particular, $M^{\otimes 1} = M$ and we interpret $M^{\otimes 0}$ as the free K-module $K\epsilon$, where ϵ denotes the empty tensor. Elements of the form $m_1 \otimes \ldots \otimes m_n \in M^{\otimes n}$ with $m_1, \ldots, m_n \in M$, are called *pure tensors* and they generate $M^{\otimes n}$ as a K-module. A K-module homomorphism from $M^{\otimes n}$, $n \geq 1$, to a K-module is uniquely determined by specifying it on the generators $m_1 \otimes \ldots \otimes m_n$, in such a way that the value is a K-multilinear function of $m_1, \ldots, m_n$. As a K-module, $K\langle M\rangle$ is defined as the direct sum $K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n}$ with multiplication $M^{\otimes r} \times M^{\otimes s} \to M^{\otimes(r+s)}$ given by

$$(m_1 \otimes \ldots \otimes m_r, \tilde{m}_1 \otimes \ldots \otimes \tilde{m}_s) \mapsto m_1 \otimes \ldots \otimes m_r \otimes \tilde{m}_1 \otimes \ldots \otimes \tilde{m}_s,$$

which can be extended to $K\langle M\rangle$ by bilinearity. With this multiplication $K\langle M\rangle$ is a K-algebra with ϵ being its one element. Note that for a free K-module M with basis X, the K-tensor algebra $K\langle M\rangle$ is isomorphic to the noncommutative polynomial algebra $K\langle X\rangle$. It has the set of all products $x_1 \otimes \ldots \otimes x_n$ for $x_1, \ldots, x_n \in X$ as a K-module basis.

Now we are ready to introduce the setting for Gröbner bases in tensor algebras following Bergman. Let $(M_x)_{x \in X}$ be a family of K-modules indexed by a set X. The modules M_x play the role of the indeterminates in the noncommutative polynomial algebra, where one would take the free module $M_x = Kx$ generated by the indeterminate x.

We denote the free monoid on X by $\langle X\rangle$ and its one element by ϵ. The free monoid $\langle X\rangle$ can also be regarded as the word monoid over the alphabet X with ϵ as the empty word. For every word $W = x_1 \ldots x_n \in \langle X\rangle$, we denote the tensor product of the corresponding modules by

$$M_W = M_{x_1} \otimes \cdots \otimes M_{x_n}.$$

In particular, we have $M_\epsilon = K\epsilon$ for the empty word/tensor ϵ. The pure tensors $m_1 \otimes \ldots \otimes m_n \in M_W$ with $m_i \in M_{x_i}$ play the role of the monomials in the noncommutative polynomial algebra. We consider the direct sum

$$M = \bigoplus_{x \in X} M_x \tag{1}$$

and the K-tensor algebra on M:

$$K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n} = \bigoplus_{W \in \langle X\rangle} M_W. \tag{2}$$

Every tensor $t \in K\langle M\rangle$ can be written as a linear combination of pure tensors. However, in contrast to linear combinations of monomials in the noncommutative polynomial algebra, this representation is not unique. Still, using module homomorphisms, one can define reductions analogous to polynomial reduction for commutative Gröbner bases.

Definition 2.1. *Let M be given by* (1). *A* reduction rule *for $K\langle M\rangle$ is given by a pair (W, h) of a word $W \in \langle X\rangle$ and a K-module homomorphism $h \colon M_W \to K\langle M\rangle$. For a reduction rule $r = (W, h)$ and words $A, B \in \langle X\rangle$, we define a* reduction *as the K-module homomorphism*

$$h_{A,r,B} \colon K\langle M\rangle \to K\langle M\rangle$$

acting as $\mathrm{id}_A \otimes h \otimes \mathrm{id}_B$ on M_{AWB} and the identity on all other M_V with $V \in \langle X\rangle$ and $V \neq AWB$.

For a tensor $a \otimes w \otimes b \in M_{AWB}$ with $a \in M_A$, $w \in M_W$, and $b \in M_B$, the reduction $h_{A,r,B}$ is given by

$$a \otimes w \otimes b \mapsto a \otimes h(w) \otimes b.$$

So, as for polynomial reduction, we “replace” the “monomial” w by the “tail” $h(w)$ given by the homomorphism h.

Let $t \in K\langle M\rangle$. A reduction $h_{A,r,B}$ acts trivially on t, i.e. $h_{A,r,B}(t) = t$, if the summand of t in M_{AWB} is zero, see Eq. (2). A reduction rule $r = (W, h)$ *reduces* t to $s \in K\langle M\rangle$ if a reduction $h_{A,r,B}$ for some $A, B \in \langle X\rangle$ acts nontrivially on t and $h_{A,r,B}(t) = s$ and we write $t \to_r s$.

A *reduction system* over X for $K\langle M\rangle$ is a set Σ of reduction rules. Every reduction system Σ induces a *reduction relation* $\to_\Sigma$ on tensors by defining $t \to_\Sigma s$ for $t, s \in K\langle M\rangle$ if $t \to_r s$ for some reduction rule $r \in \Sigma$. Fixing a reduction system Σ, we say that $t \in K\langle M\rangle$ *can be reduced* to $s \in K\langle M\rangle$ by Σ if $t = s$ or there exists a finite sequence of reduction rules $r_1, \ldots, r_n$ in Σ such that

$$t \to_{r_1} t_1 \to \ldots \to_{r_{n-1}} t_{n-1} \to_{r_n} s$$

and we write $t \xrightarrow{*}_\Sigma s$. In other words, $\xrightarrow{*}_\Sigma$ denotes the reflexive transitive closure of the reduction relation $\to_\Sigma$.

The set of *irreducible words* $\langle X\rangle_{\mathrm{irr}} \subseteq \langle X\rangle$ consists of those words having no subwords from the set $\{W \mid (W, h) \in \Sigma\}$. We define the submodule of *irreducible tensors* as

$$K\langle M\rangle_{\mathrm{irr}} = \bigoplus_{W \in \langle X\rangle_{\mathrm{irr}}} M_W. \tag{3}$$

We also need to consider partial orders on $\langle X\rangle$. A *monoid partial ordering* on $\langle X\rangle$ is a partial order $\leq$ on $\langle X\rangle$ such that $\epsilon \leq A$ and $B < \tilde{B} \Rightarrow ABC < A\tilde{B}C$ for all $A, B, \tilde{B}, C \in \langle X\rangle$. It is called *Noetherian* if there are no infinite descending chains. It is *compatible* with a reduction system Σ if for all reduction rules $(W, h) \in \Sigma$, we have

$$h(M_W) \subseteq \bigoplus_{V < W} M_V.$$

If a compatible monoid partial ordering is Noetherian, then there do not exist infinite sequences of reductions in Σ. In other words, the reduction relation $\to_\Sigma$ is *terminating* or

Noetherian. So, in that case, every $t \in K\langle M\rangle$ can be reduced in finitely many steps to an irreducible tensor

$$t \xrightarrow{*}_\Sigma s \in K\langle M\rangle_{\mathrm{irr}}$$

and such an s is called a *normal form* of t. In general, a tensor can have different normal forms. If $t \in K\langle M\rangle$ has a *unique normal form*, we denote it by $t{\downarrow_\Sigma}$. Note that such a unique normal form is unique as an element in $K\langle M\rangle_{\mathrm{irr}}$, but as tensor, it has several representations.

For ensuring unique normal forms for reduction systems on tensor algebras, we state below Bergman's analog of Buchberger's criterion for Gröbner bases [5]. In the context of Gröbner-Shirshov bases for various algebraic structures this is also referred to as the Composition-Diamond Lemma; see e.g. the recent survey [4] and in connection with integro-differential algebras also [7].

We study the cases when two different rules of a reduction system Σ act nontrivially on tensors in M_W for $W \in \langle X\rangle$.

Definition 2.2. *An* overlap ambiguity *is given by reduction rules* $(W,h),(\tilde{W},\tilde{h}) \in \Sigma$ *and nonempty words* $A,B,C \in \langle X\rangle$ *such that*

$$W = AB \quad \textit{and} \quad \tilde{W} = BC.$$

It is called resolvable *if for all* $a \in M_A$*,* $b \in M_B$*, and* $c \in M_C$*, the* S-polynomial *can be reduced to zero:*

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_\Sigma 0.$$

An inclusion ambiguity *is given by distinct reduction rules* $(W,h),(\tilde{W},\tilde{h}) \in \Sigma$ *and words* $A,B,C \in \langle X\rangle$ *with* $W = B$ *and* $\tilde{W} = ABC$*. It is called* resolvable *if for all* $a \in M_A$*,* $b \in M_B$*, and* $c \in M_C$*, the* S-polynomial *can be reduced to zero:* $a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \xrightarrow{*}_\Sigma 0$.

With slight abuse of notation, we refer to S-polynomials of an overlap or inclusion ambiguity, respectively, by

$$\mathrm{S}(A\underline{B}, \underline{B}C) \quad \text{or} \quad \mathrm{S}(\underline{B}, A\underline{B}C).$$

A reduction system Σ induces the two-sided ideal

$$I_\Sigma := (t - h(t) \mid (W,h) \in \Sigma \text{ and } t \in M_W) \subseteq K\langle M\rangle. \quad (4)$$

For studying operator algebras, we want to compute in the factor algebra $K\langle M\rangle/I_\Sigma$. If all ambiguities are resolvable, we can do this constructively using reductions in $K\langle M\rangle$ and the normal forms with respect to $\to_\Sigma$. This is the *confluence criterion* (condition 1. below) that we will check algorithmically, for a brief discussion see the following subsection.

Theorem 2.3. *([3]) Let* $(M_x)_{x\in X}$ *be a family of* K*-modules indexed by a set* X*, and let* $M = \bigoplus_{x\in X} M_x$*. Let* Σ *be a reduction system on* $K\langle M\rangle$ *and* $\leq$ *be a Noetherian monoid partial ordering on* $\langle X\rangle$ *that is compatible with* Σ*. Then the following are equivalent:*

1. *All ambiguities of* Σ *are resolvable.*
2. *Every* $t \in K\langle M\rangle$ *has a unique normal form* $t{\downarrow_\Sigma}$*.*
3. $K\langle M\rangle/I_\Sigma$ *and* $K\langle M\rangle_{irr}$ *are isomorphic as* K*-modules.*

If these conditions hold, then we can define a multiplication on $K\langle M\rangle_{irr}$ *by* $s \cdot t := (s \otimes t){\downarrow_\Sigma}$ *so that* $K\langle M\rangle/I_\Sigma$ *and* $K\langle M\rangle_{irr}$ *are isomorphic as* K*-algebras.*

Example 2. We revisit Example 1 to study it formally in the tensor algebra setting. Let (R,∂) be a differential K-algebra. We consider the K-module $M_F = R$ of "functions" from the differential algebra R (indexed by the letter F), the free K-module $M_D = K\partial$ generated by ∂ (indexed by the letter D), and the direct sum $M = M_F \oplus M_D$. Then words over the alphabet $X = \{F, D\}$ index the direct summands of the K-tensor algebra $K\langle M\rangle$.

We interpret elements $f \in R$ as multiplication operators, ∂ as the derivation on R, and the tensor product $\otimes$ as the composition of linear operators. Hence we consider the reduction system $\Sigma = \{r_{FF}, r_{DF}\}$ with the two reduction rules

$$r_{FF} = (FF, f{\otimes}g \mapsto fg) \text{ and } r_{DF} = (DF, \partial{\otimes}f \mapsto f{\otimes}\partial{+}\partial f)$$

corresponding to the composition of multiplication operators and the Leibniz rule. Note that the K-module homomorphisms corresponding to r_{FF} and r_{DF} map $M_{FF} \to M_F$ and $M_{DF} \to M_{FD} \oplus M_F$, respectively.

So any monoid partial ordering $\leq$ on $\langle X\rangle$ with $DF > FD$ is compatible with Σ, e.g. the degree-lexicographic ordering with $D > F$. There are two overlap ambiguities. The S-polynomials of the first reduce to zero in two reduction steps:

$$\begin{aligned}\mathrm{S}(F\underline{F}, \underline{F}F) &= (fg) \otimes h - f \otimes (gh) \\ &\to_{r_{FF}} fgh - f \otimes (gh) \to_{r_{FF}} fgh - fgh = 0.\end{aligned}$$

We already have seen in Example 1 that the S-polynomials $\mathrm{S}(D\underline{F}, \underline{F}F)$ reduce to zero. Hence by Theorem 2.3 every $t \in K\langle M\rangle$ has a unique normal form $t{\downarrow_\Sigma}$ in $K\langle M\rangle_{\mathrm{irr}}$ given by

$$K\epsilon \oplus M_F \oplus M_D \oplus (M_F \otimes M_D) \oplus M_D^{\otimes 2} \oplus (M_F \otimes M_D^{\otimes 2}) \oplus \ldots$$

since $\langle X\rangle_{\mathrm{irr}} = \{\epsilon, F, D, FD, D^2, FD^2, \ldots\}$. In other words, $t{\downarrow_\Sigma}$ can be written as a K-linear combination of pure tensors of the form $\epsilon,\ f,\ \partial,\ f \otimes \partial,\ \partial \otimes \partial,\ f \otimes \partial \otimes \partial, \ldots$

2.1 Computational Aspects

Considering the algorithmic aspects of Theorem 2.3, we assume that we have a finite reduction system Σ over a finite alphabet X. Moreover, a compatible Noetherian monoid partial order has to be assumed on $\langle X\rangle$.

Generating the finite set of ambiguities is just a simple combinatorial task in the word monoid $\langle X\rangle$. Determining $\langle X\rangle_{\mathrm{irr}}$ is a purely combinatorial problem on words as well, albeit less simple. Since modules M_W are generated by pure tensors, it suffices to work with S-polynomials defined by pure tensors constructed from general elements of the basic modules M_x. The result of a reduction step, i.e. the application of a homomorphism from the reduction system, needs to be simplified in the tensor algebra. This involves application of properties of the tensor product and of identities in the modules, like the Leibniz rule in the example above. In practice, the reduction to zero often can be detected heuristically without having a canonical simplifier in the modules.

The package `TenReS` provides routines to generate all ambiguities and corresponding S-polynomials of a reduction system given by the user. It also includes routines for computing in the tensor algebra. Computations in the modules (1) and homomorphisms of reduction rules have to be implemented by the user in each concrete case.

3. INTEGRO-DIFFERENTIAL OPERATORS

Integro-differential operators were introduced in [14, 16] to study boundary problems using noncommutative polynomial algebras and a basis of the coefficient algebra; see also the survey [17]. For polynomial coefficients, also generalized Weyl algebras [2] and skew polynomials [13] have been used to study them. In this section, we apply Bergman's tensor setting presented above to the construction of normal forms for integro-differential operators (IDO) over an arbitrary integro-differential algebra. First, we recall the definition of an integro-differential algebra [17, 9].

Definition 3.1. *Let (R, ∂) be a differential algebra over K such that $1 \in R$ and $\partial R = R$. Moreover, let $\int\colon R \to R$ be an K-linear operation on R such that*

$$\partial \textstyle\int f = f \tag{5}$$

for all $f \in R$. We call $(R, \partial, \int)$ an integro-differential algebra *over K if the* evaluation $\mathrm{E}\colon R \to K$ *defined by* $\mathrm{E} := \mathrm{id} - \int\partial$ *is multiplicative, i.e. for all $f, g \in R$ we have*

$$\mathrm{E}fg = (\mathrm{E}f)\mathrm{E}g. \tag{6}$$

For the rest of this paper, we fix an arbitrary integro-differential algebra $(R, \partial, \int)$ over its ring of constants K with evaluation $\mathrm{E} = \mathrm{id} - \int\partial$. Recall from [9] that in any integro-differential algebra, we have the direct sum decomposition

$$R = K \oplus \textstyle\int R$$

into constant and non-constant "functions". We consider the corresponding K-modules

$$M_K = K \quad \text{and} \quad M_{\tilde{F}} = \textstyle\int R \tag{7}$$

(indexed by the symbols K and $\tilde{F}$). Note that the elements of M_K and $M_{\tilde{F}}$ are not interpreted as functions but as multiplication operators induced by those functions. We also consider the set

$$\Phi := \{\varphi\colon R \to K \mid \varphi \text{ is } K\text{-linear and multiplicative}\} \tag{8}$$

of all characters on R, which we refer to as multiplicative "functionals". Note that $\mathrm{E} \in \Phi$ by definition. Again, we have a direct sum decomposition $K\Phi = K\mathrm{E} \oplus K\tilde{\Phi}$ where $\tilde{\Phi} := \Phi \setminus \{\mathrm{E}\}$. For the K-linear operators ∂, $\int$, E, and $\varphi \in \tilde{\Phi}$ we consider the free modules

$$M_D = K\partial, \quad M_I = K\textstyle\int, \quad M_E = K\mathrm{E}, \quad M_{\tilde{C}} = K\tilde{\Phi} \tag{9}$$

generated by them (indexed by the symbols D, I, E, and $\tilde{C}$). Now, let

$$M = M_K \oplus M_{\tilde{F}} \oplus M_D \oplus M_I \oplus M_E \oplus M_{\tilde{C}} \tag{10}$$

and $X = \{K, \tilde{F}, D, I, E, \tilde{C}\}$. In order to compute with these operators we need to collect the identities they satisfy in form of a reduction system. To this end, we first list basic identities (like the Leibniz rule) and some of their consequences (like integration by parts) that hold in R. For all $f, g \in R$ and $\varphi, \psi \in \Phi$:

$\varphi f g = (\varphi f)\varphi g$	$\partial \int g = g$
$\psi\varphi g = (\psi 1)\varphi g$	$\int \partial g = g - \mathrm{E}g$
$\mathrm{E}\int g = 0$	$\int f \varphi g = (\int f)\varphi g$
$\partial f g = f\partial g + (\partial f) g$	$\int f \partial g = fg - \int(\partial f)g - (\mathrm{E}f)\mathrm{E}g$
$\partial\varphi g = 0$	$\int f \int g = (\int f)\int g - \int(\int f)g$

All these identities correspond to identities for operators acting on $g \in R$. Together with the properties of multiplication operators we list them in Table 1 in the form of words $W \in \langle X \rangle$ and reduction homomorphisms $h\colon M_W \to K\langle M \rangle$ defined in terms of all $f, g \in M_{\tilde{F}}$ and $\varphi, \psi \in \Phi$.

K	$1 \mapsto \epsilon$
$\tilde{F}\tilde{F}$	$f \otimes g \mapsto fg$
$E\tilde{F}, \tilde{C}\tilde{F}$	$\varphi \otimes f \mapsto (\varphi f)\varphi$
$EE, E\tilde{C}, \tilde{C}E, \tilde{C}\tilde{C}$	$\psi \otimes \varphi \mapsto (\psi 1)\varphi$
EI	$\mathrm{E} \otimes \int \mapsto 0$
$D\tilde{F}$	$\partial \otimes f \mapsto f \otimes \partial + \partial f$
$DE, D\tilde{C}$	$\partial \otimes \varphi \mapsto 0$
DI	$\partial \otimes \int \mapsto \epsilon$
$IE, I\tilde{C}$	$\int \otimes \varphi \mapsto \int 1 \otimes \varphi$
ID	$\int \otimes \partial \mapsto \epsilon - \mathrm{E}$
II	$\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1$
$I\tilde{F}E, I\tilde{F}\tilde{C}$	$\int \otimes f \otimes \varphi \mapsto \int f \otimes \varphi$
$I\tilde{F}D$	$\int \otimes f \otimes \partial \mapsto f - \int \otimes \partial f - (\mathrm{E}f)\mathrm{E}$
$I\tilde{F}I$	$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f$

Table 1: Reduction rules for IDO

Definition 3.2. *Let $(R, \partial, \int)$ be an integro-differential algebra over its ring of constants K. We call*

$$R\langle \partial, \textstyle\int, \Phi\rangle := K\langle M \rangle / J$$

the algebra of integro-differential operators*, where J is the ideal induced by the reduction system obtained from Table 1.*

In order to compute in $R\langle \partial, \int, \Phi\rangle$ we want to analyze the reduction system defined by Table 1 according to Bergman's theorem above and determine normal forms of tensors.

Theorem 3.3. *Let $(R, \partial, \int)$ be an integro-differential K-algebra and let Φ be the set of multiplicative functionals as in Eq. (8). Let M be as in Eqs. (7), (9), and (10) and let the reduction system Σ be defined by Table 1.*

Then every $t \in K\langle M \rangle$ has a unique normal form $t\downarrow_\Sigma$, which is given by a K-linear combination of pure tensors

$$f \otimes \varphi \otimes \partial^{\otimes j} \quad \text{or} \quad f \otimes \varphi \otimes \textstyle\int \otimes g$$

where $j \in \mathbb{N}_0$, each of $f, g \in M_{\tilde{F}}$ and $\varphi \in \Phi$ may be absent, and $\varphi \otimes \int$ does not specialize to $\mathrm{E} \otimes \int$. Moreover,

$$R\langle \partial, \textstyle\int, \Phi\rangle \cong K\langle M \rangle_{irr}$$

as K-algebras, where the multiplication on $K\langle M \rangle_{irr}$ is defined by $s \cdot t := (s \otimes t)\downarrow_\Sigma$.

Proof. We use the alphabet $X = \{K, \tilde{F}, D, I, E, \tilde{C}\}$. For defining a Noetherian monoid partial order $\leq$ on $\langle X \rangle$ that is compatible with Σ, it is sufficient to require the Noetherian monoid partial order to satisfy

$$\tilde{F}\tilde{F} > K,\ D\tilde{F} > \tilde{F}D > K,\ I\tilde{F}D > E,\ ID > E,\ I > \tilde{F}.$$

For instance, we could use a degree-lexicographic order with $I > D > \tilde{C} > E > \tilde{F} > K$ or other degree-lexicographic orders with $D > \tilde{F}$ and $I > \tilde{F}$. Then by our software package we verify that all ambiguities of Σ are resolvable, see Section 3.1. Hence by Theorem 2.3 every element of $K\langle M \rangle$ has a unique normal form and $R\langle \partial, \int, \Phi\rangle \cong K\langle M \rangle_{\mathrm{irr}}$ as K-algebras.

It remains to determine the explicit form of elements in $K\langle M \rangle_{\mathrm{irr}}$. In order to do so, we determine the set of irreducible words $\langle X \rangle_{\mathrm{irr}}$ in $\langle X \rangle$. Irreducible words containing only the letters K and $\tilde{F}$ have to avoid the subwords K and $\tilde{F}\tilde{F}$, hence only the words ϵ and $\tilde{F}$ are left. The irreducible words containing only E and $\tilde{C}$ are exactly ϵ, E, and $\tilde{C}$, since they have to avoid the subwords $EE, E\tilde{C}, \tilde{C}E, \tilde{C}\tilde{C}$. Altogether, we see that the irreducible words containing only the letters $K, \tilde{F}, E, \tilde{C}$ are given by the set

$$\{\epsilon, \tilde{F}, E, \tilde{C}, \tilde{F}E, \tilde{F}\tilde{C}\},$$

since they have to avoid the subwords $E\tilde{F}$ and $\tilde{C}\tilde{F}$. Allowing also the letter D, we have to avoid the subwords $D\tilde{F}$, DE, and $D\tilde{C}$. Therefore, we can only append D^j with $j \in \mathbb{N}_0$ to the words in the set above in order to obtain all irreducible words not containing I. Finally, we also allow the letter I. Since subwords EI and DI have to be avoided, the first I in an irreducible word can only be preceded by ϵ, $\tilde{F}$, $\tilde{C}$, or $\tilde{F}\tilde{C}$. We also have to avoid the subwords $IE, I\tilde{C}, ID, II$, so any letter immediately following I has to be $\tilde{F}$. In addition, we have to avoid the subwords $I\tilde{F}E, I\tilde{F}\tilde{C}, I\tilde{F}D, I\tilde{F}I$, so I cannot be followed by a subword of length greater than one. Altogether, the irreducible words from $\langle X\rangle_{\mathrm{irr}}$ are of the form

$$\tilde{F}VD^j \quad \text{or} \quad \tilde{F}\tilde{C}I\tilde{F}$$

where $j \in \mathbb{N}_0$ and each of $\tilde{F}$, $\tilde{C}$, and $V \in \{E, \tilde{C}\}$ may be absent. The normal forms follow from (3). □

3.1 Computational aspects

We briefly discuss how our implementation of the verification of the confluence criterion for tensor algebras behaves on the reduction system Σ given by Table 1.

Using the package `TenReS`, we determine that there are 79 ambiguities. The corresponding S-polynomials are generated automatically from general elements of basic modules. Applying the tensor reduction rules from Σ together with identities in R, all S-polynomials are reduced to zero automatically. Here we just comment about the reduction process and refer to the example file for details.

There are 3 ambiguities for which the corresponding S-polynomials are zero anyway, for instance

$$\mathrm{S}(D\underline{E}, \underline{E}I) = 0 \otimes \textstyle\int - \partial \otimes 0 = 0.$$

The S-polynomials of 69 remaining ambiguities are reduced to zero by just applying automatically the implementation of rules from Σ and identities in R, e.g. for general $f \in M_{\tilde{F}}$:

$$\begin{aligned}\mathrm{S}(I\underline{D}, \underline{D}\tilde{F}) &= (\epsilon - \mathrm{E}) \otimes f - \textstyle\int \otimes (f \otimes \partial + \partial f) \\ &\to_{r_{E\tilde{F}}} f - (\mathrm{E}f)\mathrm{E} - \textstyle\int \otimes f \otimes \partial - \int \otimes \partial f \to_{r_{I\tilde{F}D}} 0.\end{aligned}$$

For the remaining 7 ambiguities the program has to do more in order to reduce the S-polynomials completely to zero. This is because after some reduction steps expressions in R occur, which do not belong to any of the two submodules M_K and $M_{\tilde{F}}$ in general, and no reduction applies that acts on that term. To remedy this, these expressions then have to be split further according to $R = M_K \oplus M_{\tilde{F}}$ in order to proceed with the reduction process. We give one such instance, which first can be reduced as follows.

$$\begin{aligned}\mathrm{S}(I\tilde{F}\underline{D}, \underline{D}I) &= (f - \textstyle\int \otimes \partial f - (\mathrm{E}f)\mathrm{E}) \otimes \int - \int \otimes f \otimes \epsilon \\ &\to_{r_{EI}} f \otimes \textstyle\int - \int \otimes (\partial f) \otimes \int - \int \otimes f\end{aligned}$$

Then there is no reduction that directly applies to any of these terms, since $\partial f \in R$ is neither in M_K nor in $M_{\tilde{F}}$ for general $f \in M_{\tilde{F}} = \int R$. So we write $f = \int c + \int g$ for appropriate $c \in M_K$ and $g \in M_{\tilde{F}}$. This replacement is done automatically in our implementation. Then the reduction process continues as usual.

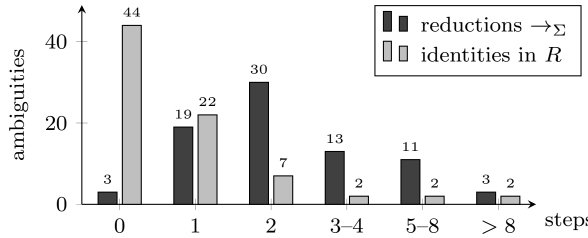

Figure 1: Histogram of computational steps used by our implementation for resolving each ambiguitiy

We collect more detailed statistics about the whole reduction process in Figure 1. For 3 ambiguities we do not need to apply reductions and for a total of 44 ambiguities no identities in R are applied in order to reduce the corresponding S-polynomial to 0. The maximal number of reductions used for one ambiguity is 12 and the maximal number of identities in R is 10, which both happens for $\mathrm{S}(I\tilde{F}\underline{D}, \underline{D}\tilde{F})$. Per ambiguity on average 2.84 reductions and 1.00 identities have been used in the computation.

Inspecting the computation more closely one sees also that many S-polynomial reductions actually are done more than once, due to the splitting (10). For instance the S-polynomials $\mathrm{S}(I\tilde{F}\underline{I}, \underline{I}E)$ and $\mathrm{S}(I\tilde{F}\underline{I}, \underline{I}\tilde{C})$ are reduced completely in the same way. Since that happens a lot for this particular reduction system we want to refine the setting in order to reduce the redundancy.

4. TWO-LEVEL TENSOR SETTING

In the example of the previous section the reduction system contained several reduction rules, where the homomorphism is defined by the same formula and the homomorphisms differ only by the choice of their domain. This leads to some redundancy in the investigation of ambiguities and S-polynomials. Sticking to the above definition of reduction systems for tensor algebras, this situation cannot be avoided.

To increase flexibility in formulating reduction systems for $K\langle M\rangle$, we generalize the setting by considering two decompositions of the K-module M at the same time, where one decomposition is a refinement of the other:

$$M = \bigoplus_{x \in X} M_x = \bigoplus_{y \in Y} M_y. \tag{11}$$

Definition 4.1. *Let $(M_x)_{x\in X}$ and $(M_y)_{y\in Y}$ be two families of K-modules indexed by sets X and Y, respectively. We call $(M_x)_{x\in X}$* a refinement *of $(M_y)_{y\in Y}$ if there exists a partition $(X_y)_{y\in Y}$ of X such that*

- *$X_y = \{y\}$ for all $y \in X \cap Y$ and*
- *$M_y = \bigoplus_{x\in X_y} M_x$ for all $y \in Y$.*

Following this definition, whenever a letter occurs in both alphabets X and Y the corresponding module is the same in both decompositions. If X_y is not a singleton, the module M_y is actually refined. We consider the *combined alphabet*

$$Z := X \cup Y.$$

For words $W = w_1 \dots w_n \in \langle Z\rangle$, we define the corresponding submodule of $K\langle M\rangle$ as before by $M_W := M_{w_1} \otimes \dots \otimes M_{w_n}$.

We define the *set of specializations* of W by replacing all its letters from $Y \setminus X$ by corresponding letters from X:

$$S(W) := \{V \in \langle X\rangle \mid |V| = n \wedge \forall i : (v_i = w_i \vee v_i \in X_{w_i})\}.$$

Remark 1. Note that for $V \in \langle X\rangle$ and $W \in \langle Z\rangle$ the modules M_V and M_W either intersect only in 0 or M_V is contained in M_W. Note further that $S(W)$ consists of all $V \in \langle X\rangle$ such that M_V is a submodule of M_W. Moreover,

$$M_W = \bigoplus_{V\in S(W)} M_V.$$

Remark 2. Reduction rules (over Z) are defined by replacing X with Z in Definition 2.1. A reduction system Σ over Z is simply a set of such reduction rules. We define the ideal I_Σ by (4), and we define the irreducible words $\langle X\rangle_{\mathrm{irr}}$ w.r.t. Σ as the set of words from $\langle X\rangle$ containing no subwords from the set $\bigcup\{S(W) \mid (W,h) \in \Sigma\}$. Based on $\langle X\rangle_{\mathrm{irr}}$, we define $K\langle M\rangle_{\mathrm{irr}}$ as in (3).

Modeling an operator algebra as $K\langle M\rangle/I_\Sigma$, this two-level setting provides much more flexibility in describing the ideal by a reduction system Σ. Previously, with just one decomposition (1), $K\langle M\rangle$ was a direct sum of the modules M_W, which are the possible domains for the homomorphisms of reduction rules (over X). Now, with two decompositions (11), the sum of all M_W is not direct anymore, allowing coexistence of reduction rules (over Z) that could not be used together in a reduction system before without splitting them into several rules.

In some of the following proofs for a reduction system Σ over Z, we need the *induced reduction system* Σ_X over X:

$$\Sigma_X := \bigcup_{(W,h)\in\Sigma} \{(V, h|_{M_V}) \mid V \in S(W)\}. \tag{12}$$

From (12), it is obvious that the following lemma holds.

Lemma 4.2. *Let Σ be a reduction system over Z inducing Σ_X over X. Then the reduction relations and the irreducible words are the same for Σ and for Σ_X.*

Definition 4.3. *We call a partial order $\leq$ on $\langle Z\rangle$* consistent with $(M_x)_{x\in X}$ being a refinement of $(M_y)_{y\in Y}$ *if for all words $V, W \in \langle Z\rangle$ with $V < W$ we also have $\tilde{V} < \tilde{W}$ for all specializations $\tilde{V} \in S(V)$ and $\tilde{W} \in S(W)$.*

Note that the above definition implies $\tilde{W} \not< W$ for all $\tilde{W} \in S(W)$, which can be seen by choosing $V = \tilde{W}$.

A monoid partial order $\leq$ on $\langle Z\rangle$ is *compatible* with a reduction system Σ over Z if for all $(W,h) \in \Sigma$ we have

$$h(M_W) \subseteq \sum_{\substack{V\in\langle Z\rangle\\ V<W}} M_V.$$

If in addition $\leq$ is also consistent with $(M_x)_{x\in X}$ being a refinement of $(M_y)_{y\in Y}$ then we have

$$\sum_{\substack{V\in\langle Z\rangle\\ V<W}} M_V = \bigoplus_{\substack{V\in\langle Z\rangle\\ V<W}} M_V = \bigoplus_{\substack{V\in\langle X\rangle, \tilde{W}\in S(W)\\ V<\tilde{W}}} M_V.$$

Lemma 4.4. *Let Σ be a reduction system over Z and let $\leq$ be a monoid partial order on $\langle Z\rangle$ consistent with $(M_x)_{x\in X}$ being a refinement of $(M_y)_{y\in Y}$ and compatible with Σ. Then the restricted order $\leq$ on $\langle X\rangle$ is compatible with Σ_X.*

We also generalize the notion of ambiguities to take specialization into account, see Definition 4.5. To this end, note first that for $W, \tilde{W} \in \langle Z\rangle$ having a common specialization, i.e. $S(W) \cap S(\tilde{W}) \neq \emptyset$, there exists $V \in \langle Z\rangle$ such that

$$S(V) = S(W) \cap S(\tilde{W}).$$

While the word V is not necessarily unique, the corresponding module M_V is unique and will be denoted by

$$M_{\min(W,\tilde{W})} := M_V.$$

Remark 3. The module $M_{\min(W,\tilde{W})}$ is the biggest module M_V, $V \in \langle Z\rangle$, contained in M_W and $M_{\tilde{W}}$. In fact, we even have $M_{\min(W,\tilde{W})} = M_W \cap M_{\tilde{W}}$.

Example 3. Consider disjoint sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ with $M_{y_1} = M_{x_1}$ and $M_{y_2} = M_{x_2} \oplus M_{x_3}$. Then $Z = \{x_1, x_2, x_3, y_1, y_2\}$ and $(M_x)_{x\in X}$ is a refinement of $(M_y)_{y\in Y}$ since we have the partition $\{\{x_1\}, \{x_2, x_3\}\}$ of X. The words $W = x_1y_2y_2$ and $\tilde{W} = y_1y_2x_2$ in $\langle Z\rangle$ satisfy $S(W) \cap S(\tilde{W}) = \{x_1x_2x_2, x_1x_3x_2\} \neq \emptyset$. Both choices $V = y_1y_2x_2$ and $V = x_1y_2x_2$ satisfy $S(V) = S(W) \cap S(\tilde{W})$ and $M_V = M_{x_1} \otimes M_{y_2} \otimes M_{x_2}$. So $M_{\min(W,\tilde{W})} = M_{x_1} \otimes M_{y_2} \otimes M_{x_2}$.

Definition 4.5. *Let $(W,h), (\tilde{W},\tilde{h}) \in \Sigma$ be two reduction rules and let $A, B_1, B_2, C \in \langle Z\rangle$ be nonempty words with*

$$W = AB_1, \quad \tilde{W} = B_2C, \quad B_1 \neq B_2, \\ \text{and} \quad S(B_1) \cap S(B_2) \neq \emptyset,$$

then we call this an overlap ambiguity with specialization. *Note that $B_1 \neq B_2$ ensures that this is not an overlap ambiguity (without specialization). An overlap ambiguity with specialization is called* resolvable *if for all $a \in M_A$, $b \in M_{\min(B_1,B_2)}$, and $c \in M_C$ the* S-polynomial *can be reduced to zero:*

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_\Sigma 0.$$

Similarly, an inclusion ambiguity with specialization *is given by two reduction rules $(W,h), (\tilde{W},\tilde{h}) \in \Sigma$ and words $A, B_1, B_2, C \in \langle Z\rangle$ with $W = B_1$, $\tilde{W} = AB_2C$, $B_1 \neq B_2$, and $S(B_1) \cap S(B_2) \neq \emptyset$. An inclusion ambiguity with specialization is called* resolvable *if for all $a \in M_A$, $b \in M_{\min(B_1,B_2)}$, and $c \in M_C$ the* S-polynomial *can be reduced to zero: $a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \xrightarrow{*}_\Sigma 0$.*

Again, we use $\mathrm{S}(A\underline{B_1}, \underline{B_2}C)$ or $\mathrm{S}(\underline{B_1}, A\underline{B_2}C)$, respectively, to refer to S-polynomials of an overlap or inclusion ambiguity with specialization.

Remark 4. Note that in total there now can be 4 types of ambiguities: in addition to the two types of ambiguities (without specialization) of Definition 2.2 there are also corresponding versions with specialization as defined above.

With these definitions we can prove the following generalization of Bergman's result. In order to prove properties of the reduction system Σ, we apply Bergman's result (Theorem 2.3) to the induced reduction system Σ_X over X.

Theorem 4.6. *Let $(M_x)_{x\in X}$ and $(M_y)_{y\in Y}$ be two families of K-modules indexed by sets X and Y, respectively, such that $(M_x)_{x\in X}$ is a refinement of $(M_y)_{y\in Y}$ and let $M = \bigoplus_{x\in X} M_x$. Let Σ be a reduction system over $Z := X \cup Y$ on $K\langle M\rangle$ and $\leq$ be a Noetherian monoid partial order on $\langle Z\rangle$ consistent with $(M_x)_{x\in X}$ being a refinement of $(M_y)_{y\in Y}$ and compatible with Σ. Then the following are equivalent:*

1. All ambiguities of Σ are resolvable.

2. Every $t \in K\langle M\rangle$ has a unique normal form $t{\downarrow_\Sigma}$.

3. $K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-modules.

If these conditions hold, then we can define a multiplication on $K\langle M\rangle_{irr}$ by $s \cdot t := (s \otimes t)\downarrow_\Sigma$ so that $K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-algebras.

Proof. Lemma 4.2 shows that we can replace the reduction system Σ by its refinement Σ_X without changing the reduction relation. Consequently, also the reduction ideals and normal forms agree, i.e. $I_\Sigma = I_{\Sigma_X}$ and $t{\downarrow_\Sigma} = s \Leftrightarrow t{\downarrow_{\Sigma_X}} = s$ for all $s, t \in K\langle M\rangle$. The lemma also implies that $\langle X\rangle_{\mathrm{irr}}$ and hence $K\langle M\rangle_{\mathrm{irr}}$ stay the same. Furthermore, we note that every S-polynomial of Σ_X is also an S-polynomial of Σ and, conversely, every S-polynomial of Σ is a linear combination of S-polynomials of Σ_X. Hence all ambiguities of Σ are resolvable if and only if all ambiguities of Σ_X are resolvable, since $\to_\Sigma$ and $\to_{\Sigma_X}$ are the same. Finally, Lemma 4.4 implies that Σ_X and the restriction of $\leq$ to $\langle X\rangle$ satisfy the assumptions of Theorem 2.3, which concludes the proof. □

4.1 Computational Aspects

Many properties that we discussed for Bergman's tensor setting also hold for the generalization we introduced above. For instance, determining ambiguities and irreducible words is done just on the level of words.

The main computational benefit of Theorem 4.6 compared to Theorem 2.3 lies in the fact that for the confluence criterion we only need to check ambiguities of Σ over the combined alphabet Z and no computations with the induced reduction system Σ_X are needed. Computing with the induced reduction system over X instead, generally would lead to a higher number of ambiguities, since one reduction rule in Σ can give rise to many reduction rules in Σ_X. Only for determination of irreducible words we restrict to $\langle X\rangle$.

The package `TenReS` also provides routines for generating all overlap and inclusion ambiguities with specialization together with their corresponding S-polynomials. It can also generate all irreducible words up to given length.

5. AN IMPROVED REDUCTION SYSTEM

In this section, we apply the two-level version of Bergman's tensor setting to our example, the tensor algebra of integro-differential operators. First, we define two alphabets

$$X = \{K, \tilde{F}, D, I, E, \tilde{C}\} \quad \text{and} \quad Y = \{F, D, I, C\},$$

with the K-modules defined in Eqs. (7) and (9) as well as

$$M_F = M_K \oplus M_{\tilde{F}} \quad \text{and} \quad M_C = M_E \oplus M_{\tilde{C}}. \tag{13}$$

Then the decomposition (10) is a refinement of

$$M = M_F \oplus M_D \oplus M_I \oplus M_C. \tag{14}$$

The reduction system Σ over the combined alphabet is given by Table 2, defined in terms of all $f, g \in R$ and $\varphi, \psi \in \Phi$.

Following (12), the induced reduction system Σ_X is obtained, according to (13), by splitting rules whose words contain F or C into "smaller" rules using $X_F = \{K, \tilde{F}\}$ and $X_C = \{E, \tilde{C}\}$. For example, the reduction rule $(CF, h) \in \Sigma$ is split into the rules $(W, h|_{M_W}) \in \Sigma_X$ where $W \in S(CF) = \{EK, E\tilde{F}, \tilde{C}K, \tilde{C}\tilde{F}\}$.

FF	$f \otimes g \mapsto fg$
CF	$\varphi \otimes f \mapsto (\varphi f)\varphi$
CC	$\psi \otimes \varphi \mapsto (\psi 1)\varphi$
DF	$\partial \otimes f \mapsto f \otimes \partial + \partial f$
DC	$\partial \otimes \varphi \mapsto 0$
DI	$\partial \otimes \int \mapsto \epsilon$
IC	$\int \otimes \varphi \mapsto \int 1 \otimes \varphi$
ID	$\int \otimes \partial \mapsto \epsilon - \mathrm{E}$
II	$\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1$
IFC	$\int \otimes f \otimes \varphi \mapsto \int f \otimes \varphi$
IFD	$\int \otimes f \otimes \partial \mapsto f - \int \otimes \partial f - (\mathrm{E}f)\mathrm{E}$
IFI	$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f$
K	$1 \mapsto \epsilon$
EI	$\mathrm{E} \otimes \int \mapsto 0$

Table 2: Reduction rules for IDO on two levels

Theorem 5.1. *Let $(R, \partial, \int)$ be an integro-differential K-algebra and let Φ be the set of multiplicative functionals as in Eq. (8). Let M be as in Eqs. (13) and (14) and let the reduction system Σ be defined by Table 2.*

Then every $t \in K\langle M\rangle$ has a unique normal form of the same type as in Theorem 3.3 and, as K-algebras,

$$K\langle M\rangle/I_\Sigma \cong K\langle M\rangle_{irr}.$$

Proof. We use the alphabet $Z = X \cup Y$ where $X = \{K, \tilde{F}, D, I, E, \tilde{C}\}$ and $Y = \{F, D, I, C\}$. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ, it is sufficient to require the order to satisfy

$$DF > FD,\ IFD > E,\ ID > E,\ I > \tilde{F}.$$

For instance, we could use a degree-lexicographic order with $I > D > C > F$ on $\langle Y\rangle$ or other degree-lexicographic orders with $D > F$ and $I > F$. We extend it to a partial monoid order on $\langle Z\rangle$ based on Definition 4.3 in order to make it consistent with $(M_x)_{x\in X}$ being a refinement of $(M_y)_{y\in Y}$. Then by our software package we verify that all ambiguities of Σ are resolvable, see Section 5.1. Hence by Theorem 4.6 every element of $K\langle M\rangle$ has a unique normal form and $K\langle M\rangle/I_\Sigma \cong K\langle M\rangle_{\mathrm{irr}}$ as K-algebras.

It remains to determine the explicit form of elements in $K\langle M\rangle_{\mathrm{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\mathrm{irr}}$ in $\langle X\rangle$. Note that $\tilde{\Sigma} \subset \Sigma_X$, where $\tilde{\Sigma}$ is given by Table 1. Hence the irreducible words w.r.t. Σ are among the irreducible words w.r.t. $\tilde{\Sigma}$. In the proof of Theorem 3.3 we already determined the latter to be of the form $\tilde{F}VD^j$ or $\tilde{F}\tilde{C}I\tilde{F}$, where $j \in \mathbb{N}_0$ and each of $\tilde{F}$, $\tilde{C}$, and $V \in \{E, \tilde{C}\}$ may be absent. All of them are also irreducible w.r.t. Σ since they do not contain subwords from $\{W \mid (W, h) \in \Sigma_X \setminus \tilde{\Sigma}\}$. The normal forms follow from (3). □

In order to show $R\langle \partial, \int, \Phi\rangle = K\langle M\rangle/I_\Sigma$, we prove the following lemma.

Lemma 5.2. *Let the reduction system $\tilde{\Sigma}$ over X be given by Table 1 and let the reduction system Σ over Z be given by Table 2. Then $I_{\tilde{\Sigma}}$ and I_Σ are the same.*

Proof. In view of Lemma 4.2, we show $I_{\tilde{\Sigma}} = I_{\Sigma_X}$, where the induced reduction system Σ_X is described immediately before Theorem 5.1. Comparing $\tilde{\Sigma}$ and Σ_X, we see that $\tilde{\Sigma} \subset \Sigma_X$, hence $I_{\tilde{\Sigma}} \subseteq I_{\Sigma_X}$. Conversely, we can verify, e.g. by the package `TenReS`, for all 10 elements $(W, h) \in \Sigma_X \setminus \tilde{\Sigma}$ that $t - h(t) \in I_{\tilde{\Sigma}}$ for all $t \in M_W$ implying $I_{\Sigma_X} \subseteq I_{\tilde{\Sigma}}$. □

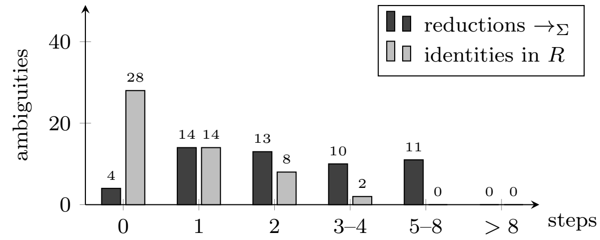

Figure 2: Histogram of computational steps used by our implementation for resolving each ambiguitiy

5.1 Computational aspects

In the following, we discuss computational details of the two-level tensor setting for integro-differential operators. Applying `TenReS` to the reduction system Σ given by Table 2, in total 52 ambiguities and corresponding S-polynomials are generated. In contrast to the computations for Table 1, we do not need to introduce any further splitting of expressions in R during reduction of the S-polynomials. For instance,

$$\mathrm{S}(IF\underline{D}, \underline{D}I) = (f - \textstyle\int \otimes \partial f - (\mathrm{E}f)\mathrm{E}) \otimes \textstyle\int - \textstyle\int \otimes f \otimes \epsilon$$
$$\to_{r_{EI}} f \otimes \textstyle\int - \textstyle\int \otimes \partial f \otimes \textstyle\int - \textstyle\int \otimes f \to_{r_{IFI}} \ldots \xrightarrow{*}_{r_K} 0.$$

There are 41 ambiguities without specialization, they only involve the first 12 reduction rules (over Y). The remaining 11 ambiguities consist of 4 overlap ambiguities with specialization and 7 inclusion ambiguities with specialization. They all involve the last two reduction rules (over X) in Table 2 and their S-polynomials are reduced to zero. For example,

$$\mathrm{S}(IF\underline{C}, \underline{E}I) = (\textstyle\int f \otimes \mathrm{E}) \otimes \textstyle\int - \textstyle\int \otimes f \otimes 0 \to_{r_{EI}} \textstyle\int f \otimes 0 = 0,$$
$$\mathrm{S}(\underline{K}, D\underline{F}) = \partial \otimes \epsilon \otimes \epsilon - 1 \otimes \partial \ \to_{r_K} \partial - \partial = 0.$$

In Figure 2 we present more detailed statistics on the whole reduction process. For 4 ambiguities, we do not need to apply any reduction, and 28 do not need identities in R. The maximal number of required reduction rules and identities from R are 8 and 4, respectively, which for instance is the case for the S-polynomial $\mathrm{S}(IF\underline{I}, \underline{I}FD)$. Per ambiguity on average 2.73 reductions and 0.73 identities have been used in the computation. So not only the number of ambiguities is less than before, but also the average effort per ambiguity has reduced.

We emphasize again that the confluence criterion of Theorem 4.6 directly works with the reduction system Σ, no computations with the induced reduction system Σ_X over X are needed. The 31 reduction rules of Σ_X would give rise to 149 overlap ambiguities and 11 inclusion ambiguities.

6. OUTLOOK

We showed that the new two-level tensor setting allows for a simpler study and a more general construction of integro-differential operators than via noncommutative polynomial algebras. So we are confident that this approach is also suitable for studying other operator algebras. Due to space limitations, we did not include our results for generalizations of integro-differential operators which contain more operators like linear substitutions, addressing the univariate case of [15], or arbitrary linear functionals. These operator algebras have applications to boundary problems and delay equations [12]. Beyond verification of confluence, we are studying an analog of Buchberger's algorithm [5] and Knuth-Bendix completion [11] for tensors, see [10] for an example.

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