Algorithmic operator algebras via normal forms in tensor rings[1]

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Abstract

We propose a general algorithmic approach to noncommutative operator algebras generated by additive operators using quotients of tensor rings that are defined by tensor reduction systems. Skew polynomials are a well-established tool covering many cases arising in applications. However, integro-differential operators over an arbitrary integro-differential algebra do not fit this structure, for example. Instead of using parametrized Gröbner bases in free algebras, as has been used so far in the literature, we use Bergman's basis-free analog in tensor rings. Since reduction rules are given by module homomorphisms, the tensor setting often allows for a finite reduction system. A confluent tensor reduction system enables effective computations based on normal forms. Using tensor rings, we can also model integro-differential operators with matrix coefficients, where constants are not commutative.

To have smaller reduction systems, we develop a generalization of Bergman's setting. It allows overlapping domains of reduction homomorphisms, which also make the algorithmic verification of the confluence criterion more efficient. Moreover, we discuss a heuristic approach to complete a given reduction system to a confluent one in analogy to Buchberger's algorithm and Knuth-Bendix completion. Integro-differential operators are used to illustrate the tensor setting, verification of confluence, and completion of tensor reduction systems. We also introduce a confluent reduction system and normal forms for integro-differential operators with linear substitutions, which have applications in delay differential equations. Verification of the confluence criterion and completion based on S-polynomial computations is supported by the Mathematica package `TenReS`.

Keywords: operator algebra; tensor ring; integro-differential operators; linear substitutions; noncommutative Gröbner basis; reduction systems; completion; confluence

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1. Introduction

Skew polynomial rings are used in the literature for an algebraic and algorithmic treatment of many common operators like differential and difference operators; see e.g. the works by Chyzak and Salvy (1998); Li (2002); Bueso et al. (2003); Chyzak et al. (2005); Levandovskyy (2005) or the recent overview by Gómez-Torrecillas (2014). Normal forms for skew polynomials are given by the standard polynomial basis. However, normal forms for univariate integral operators are sums of terms of the form $f\int g$. We show that quotients of tensor rings are useful for algebraic modeling of and algorithmic computations with additive operators. The framework provided uses a quotient of a tensor ring by a two-sided ideal for constructing a ring of operators, constructing quotients of such rings of operators by one-sided ideals would be a separate problem. Tensor rings naturally capture the multiadditivity of composition of additive operators. In addition, they allow basis-free treatment of multiplication operators resp. coefficients. In particular, the coefficient ring is not required to be finitely presented. Moreover, for integro-differential operators, they also cover arbitrary rings of constants which neither have to be fields nor commutative rings but need to contain a unit element.

We are not aware that tensor reduction systems in tensor rings have been used so far in the literature for an algorithmic treatment of operator algebras. For applications of noncommutative Gröbner bases in the free polynomial algebra to operator algebras, we refer to (Helton et al., 1998; Helton and Stankus, 1999; Rosenkranz et al., 2003) and the references on integro-differential operators in Section 4. An overview on Gröbner-Shirshov bases for various algebraic structures is given in (Bokut and Chen, 2014); see, in particular, (Guo et al., 2013; Gao et al., 2014, 2015; Gao and Guo, 2017) in connection with differential type, integro-differential, and Rota-Baxter type operators.

For computing in quotients of tensor rings by two-sided ideals, we use Bergman's analog (Bergman, 1978) of Gröbner bases in tensor rings, which we explain in Section 2 along with the underlying algebraic structures. Bergman's confluence criterion for tensor reduction systems involves computations in the tensor ring, but determining the structure of normal forms reduces to a combinatorial problem on words. We generalize Bergman's tensor setting in Section 3 by introducing the concept of specialization. As a first example for our setting with specialization, we present integro-differential operators (IDOs) over an arbitrary integro-differential ring in Section 4. There we give a confluent tensor reduction system together with the corresponding normal forms. In Section 5, we introduce IDOs with linear substitutions. For completing a tensor reduction system to a confluent one, we give a heuristic method along the lines of Buchberger's algorithm in Section 6 and we discuss various problems arising in this context. In each section, we comment about the computational aspects. The Mathematica package `TenReS` can be obtained at `http://gregensburger.com/softw/tenres/` along with example files; see also (Hossein Poor et al., 2016b) for further details on the package.

Throughout this paper rings are not necessarily commutative unless stated otherwise, but they are always assumed to have a unit element (of multiplication). Furthermore, we use operator notation, e.g. we write $\varphi 1$ instead of $\varphi(1)$ or $\partial fg = (\partial f)g + f\partial g$ for the Leibniz rule $\partial(fg) = \partial(f)g + f\partial(g)$. All our operators act from the left, in particular, a product AB acts on f as $(A \circ B)(f)$.

1.1. Comparison with conference paper

A two-level version of Bergman's setting in tensor algebras has been introduced already in (Hossein Poor et al., 2016a). In contrast, in the present paper we deal with the more general structure of tensor rings instead of tensor algebras. We introduce a generalization and simplification of the two-level tensor setting in Section 3. New aspects treated are deletion criteria for excluding ambiguities from consideration (see Section 2.3.1) and the heuristic completion process discussed in Section 6. The example presented in Section 4 is more general as it allows also noncommutative differential rings and Section 5 contains an entirely new example.

We also need to correct some minor mistakes in (Hossein Poor et al., 2016a). The definition of Φ in Eq. (8) should include the requirement that $\varphi 1 = 1$. Lemma 4.2 should be replaced by the weaker statement of Lemma 15 of the present paper, the proof of Theorem 4.6 needs to be adapted accordingly, cf. the proof of Theorem 20. Also, the equation immediately before Lemma 4.4 has to be replaced by the equation immediately before Lemma 17 in the present paper.

1.2. Introductory example

We use the well-known example of differential operators to briefly discuss several approaches for modelling rings of operators. Recall that differential operators with polynomial coefficients (Weyl algebra) over a field $K \supseteq \mathbb{Q}$ can be defined as the quotient algebra

$$K\langle X, D\rangle/(DX - XD - 1)$$

of the free polynomial algebra $K\langle X, D\rangle$ by a two-sided ideal; see for example (Coutinho, 1995). Let now (R, ∂) be a commutative differential ring and let K denote its ring of constants. If R is a finitely presented K-algebra, then also the differential operators $R\langle\partial\rangle$ are a finitely presented K-algebra analogous to the Weyl algebra.

Skew polynomials are a well-established approach that only introduces finitely many rules for differential operators over arbitrary differential rings R (e.g. rational functions): they are represented by defining a multiplication on normal forms $\sum f_i\partial^i$ based on the commutation rule

$$\partial \cdot f = f\partial + \partial f.$$

Viewed as construction by generators and relations, this amounts to (potentially) infinitely many relations, one for each generator of R.

In the following, we motivate and illustrate informally tensor reduction systems. For a commutative differential ring, the construction leads to a quotient of the tensor algebra as in (Hossein Poor et al., 2016a). The commutation rule for skew polynomials above corresponds to a reduction homomorphism for tensors below. The ring R is regarded as the coefficient ring of skew polynomials, whereas in the tensor construction below R is just considered as a K-module and we tensor over the ring K only. Hence for the multiplication in R, we need to introduce an additional reduction homomorphism for tensors.

Example 1. *Consider a commutative differential ring (R, ∂) and let K denote its ring of constants. By the Leibniz rule, the derivation $\partial\colon R \to R$ is a K-module homomorphism. Since R is commutative, also the multiplication operators induced by $f \in R$ mapping $g \mapsto fg$ are K-module homomorphisms. Let $M_{\mathsf{D}} = K\partial$ denote the free left K-module generated by the symbol ∂. The identities in the K-tensor algebra $K\langle M\rangle$ on the K-module $M = R \oplus K\partial$ reflect the identities coming from the K-linearity of the operators and their compositions, where the tensor product is interpreted as composition of operators.*

To incorporate the additional identities, we use reduction rules defined by K-module homomorphisms on certain submodules of the tensor algebra. Corresponding to the composition of multiplication operators and the Leibniz rule, we consider two homomorphisms defined by

$$f \otimes g \mapsto fg \quad \textit{and} \quad \partial \otimes f \mapsto f \otimes \partial + \partial f.$$

These two reduction rules induce the two-sided ideal $J = (f \otimes g - fg, \partial \otimes f - f \otimes \partial - \partial f \mid f, g \in R)$ which we use to define the K-algebra of differential operators as the quotient algebra

$$R\langle\partial\rangle = K\langle M\rangle/J.$$

We want to obtain unique normal forms in the quotient by applying the reduction rules above. A tensor of the form

$$\partial \otimes f \otimes g$$

corresponds to an overlap ambiguity of these two rules, since it can be reduced by the homomorphisms in different ways to obtain either

$$(f \otimes \partial + \partial f) \otimes g \quad \textit{or} \quad \partial \otimes (fg).$$

For checking resolvability of the ambiguity the S-polynomial formed by the difference of these alternatives should be reducible to zero. In the present case, it reduces to zero because of the Leibniz rule in R. More explicitly, for all $f, g \in R$ we have

$$\begin{aligned}
\mathrm{SP}(\partial \otimes f, f \otimes g) &= (f \otimes \partial + \partial f) \otimes g - \partial \otimes (fg) \\
&\to f \otimes g \otimes \partial + f \otimes \partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\
&\to fg \otimes \partial + f\partial g + (\partial f)g - fg \otimes \partial - \partial(fg) \\
&= f\partial g + (\partial f)g - \partial(fg) = 0.
\end{aligned}$$

Another ambiguity is expressed by tensors of the form $f \otimes g \otimes h$ and is resolvable as well. Since all ambiguities are resolvable, we obtain normal forms in terms of irreducible tensors

$$\partial^{\otimes j} \quad \textit{and} \quad f \otimes \partial^{\otimes j}.$$

For differential operators with matrix coefficients, we let R be a ring of matrices over some (commutative) differential ring. Then not only R is a noncommutative differential ring, but also its ring of constants K is no longer commutative and elements of K do not commute with elements of R. Consequently, R is not a K-algebra anymore. More generally, we consider an arbitrary differential ring R. It is a bimodule over its ring of constants K and tensoring over K leads to a construction of the differential operators as a quotient of the tensor ring instead of the tensor algebra.

Example 2. *For an arbitrary (not necessarily commutative) differential ring (R, ∂), ∂ is a K-bimodule homomorphism of R whereas multiplication operators $g \mapsto fg$ in general are only right K-module homomorphisms. We consider the K-tensor ring $K\langle M\rangle$ on the K-bimodule $M = R \oplus M_{\mathsf{D}}$, where M_{D} is a K-bimodule non-freely generated by ∂. The identities in the tensor ring $K\langle M\rangle$ reflect the identities coming from the additivity of the operators and their compositions. Reduction rules are K-bimodule homomorphisms defined by the same formulae as above. For details see Example 8 later.*

2. Tensor reduction systems

In this section, we describe analogs of Gröbner bases in tensor rings following Bergman (1978) using standard notation for rewriting systems from (Baader and Nipkow, 1998). First we outline the construction and some properties of the K-tensor ring $K\langle M\rangle$ on a K-bimodule M over a arbitrary ring K with unit element. If K is commutative and the left and right scalar multiplication on M agree, then $K\langle M\rangle$ is the tensor algebra on M, which is a generalization of the noncommutative polynomial algebra on a set of indeterminates. In contrast to the noncommutative polynomials, in the tensor ring the "coefficients" in K do not commute with the "indeterminates". For further details on tensor rings and proofs see, for example, (Cohn, 2003; Rowen, 1991). A Gröbner basis theory for free bimodules has been presented in (Kobayashi, 2005) and for bimodules over Poincaré-Birkhoff-Witt (PBW) algebras in (Román García and Román García, 2005; Levandovskyy, 2005).

2.1. Basics of tensor rings

From now, K denotes a ring (not necessarily commutative) with unit element. A K-bimodule is a left K-module M which is also a right K-module satisfying the associativity condition $(km)l = k(ml)$ for all $m \in M$ and $k, l \in K$. By a K-ring we understand a ring R that is a K-bimodule such that $(xy)z = x(yz)$ for any x, y, z in R or K. Even when K is commutative, the notion of K-ring is more general than the notion of K-algebra, because the action of K need not centralize the ring, that is, we do not require $kr = rk$ for $k \in K$ and $r \in R$. In other words, the difference can be described by saying that whereas a K-algebra (K commutative) is a ring R with a homomorphism from K to the center of R, a K-ring is a ring R with a ring homomorphism from K to R. In particular, if K is a subring of some ring R, then R is a K-ring.

We first recall basic properties of the tensor product on K-bimodules. Let $M_1, \ldots, M_n$ be K-bimodules. Given an abelian group $(A, +)$, we say that $\beta\colon M_1 \times \cdots \times M_n \to A$ is a balanced map if it is multiadditive and it satisfies

$$\beta(m_1, \ldots, m_i k, m_{i+1}, \ldots, m_n) = \beta(m_1, \ldots, m_i, k m_{i+1}, \ldots, m_n)$$

for all $k \in K$, $m_j \in M_j$, where $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. By the definition of the tensor product, there exists an abelian group $M_1 \otimes \cdots \otimes M_n$ together with a balanced map

$$\otimes\colon M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n.$$

We write $m_1 \otimes \cdots \otimes m_n$ for the image of $(m_1, \ldots, m_n)$ under $\otimes$. The universal property of the tensor product states that if $\beta\colon M_1 \times \cdots \times M_n \to A$ is any balanced map, then there exists a unique homomorphism $\overline{\beta}\colon M_1 \otimes \cdots \otimes M_n \to A$ such that

$$\overline{\beta}(m_1 \otimes \cdots \otimes m_n) = \beta(m_1, \ldots, m_n).$$

Note that, if $M_1, \ldots, M_n$ are K-bimodules, then $M_1 \otimes \cdots \otimes M_n$ is again a K-bimodule with scalar multiplications

$$k(m_1 \otimes \cdots \otimes m_n) = km_1 \otimes \cdots \otimes m_n \quad \text{and} \quad (m_1 \otimes \cdots \otimes m_n)k = m_1 \otimes \cdots \otimes m_n k.$$

We denote the tensor product of M with itself over K by $M^{\otimes n} = M \otimes \cdots \otimes M$ (n factors) and its elements are called *tensors*. In particular, $M^{\otimes 1} = M$ and we interpret $M^{\otimes 0}$ as the K-bimodule $K\epsilon$, where ϵ denotes the empty tensor. Elements of the form $m_1 \otimes \cdots \otimes m_n \in M^{\otimes n}$ with $m_1, \ldots, m_n \in M$, are called *pure tensors* and they generate $M^{\otimes n}$ as a K-bimodule. As a K-bimodule, the tensor ring $K\langle M\rangle$ is defined as the direct sum $K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n}$ with multiplication $M^{\otimes r} \times M^{\otimes s} \to M^{\otimes(r+s)}$ given by the balanced map

$$(m_1 \otimes \cdots \otimes m_r, \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_s,$$

which can be extended to $K\langle M\rangle$ by biadditivity. In general, the K-bimodule $K\langle M\rangle$ with this multiplication is a ring with ϵ being its unit element. Note that by the homomorphism $K \to K\langle M\rangle$ mapping $k \mapsto k\epsilon$ the tensor ring $K\langle M\rangle$ is a K-ring.

The K-tensor algebra on a K-module M with K commutative is a special case of the K-tensor ring by viewing M as a K-bimodule with identical scalar multiplication from left and right. Note that for a free K-module M with basis X, the K-tensor algebra $K\langle M\rangle$ is isomorphic to the noncommutative polynomial algebra $K\langle X\rangle$. It has the set of all products $x_1 \otimes \cdots \otimes x_n$ for $x_1, \ldots, x_n \in X$ as a K-module basis, i.e. elements in $K\langle X\rangle$ have a unique representation as K-linear combinations of such products.

The analogous situation for tensor rings is more involved. The free K-bimodule on a set X is given by $K \otimes_{\mathbb{Z}} \mathbb{Z}X \otimes_{\mathbb{Z}} K$, where $\mathbb{Z}X$ denotes the free left $\mathbb{Z}$-module on X. The K-tensor ring over the free K-bimodule on X is isomorphic to the free K-ring on X, which is generated as a K-bimodule by the set of all products $x_1 \otimes k_2 x_2 \otimes \cdots \otimes k_n x_n$ such that $x_1, \ldots, x_n \in X$ and $k_2, \ldots, k_n \in K$. Note that the representation of elements of the free K-ring on X in terms of such products is not unique, in contrast to the noncommutative polynomial algebra. Since bimodules have coefficients on both sides and coefficients do not commute with indeterminates, even the free K-bimodule generated by $\{x_1\}$ gives rise to non-uniqueness: $k_1 x_1 k_3 + k_2 x_1 k_1 = k_3 x_1 k_1 + k_1 x_1 k_2$ for $k_3 = k_1 + k_2 \in K$.

2.2. *Diamond Lemma in tensor rings*

Now we are ready to explain the setting for reduction systems in tensor rings following (Bergman, 1978, Sec. 6). Let $(M_x)_{x \in X}$ be a family of K-bimodules indexed by a set X. The modules M_x play the role of the indeterminates in the noncommutative polynomial algebra.

We denote the free monoid on X by $\langle X\rangle$ and its unit element by ϵ. The free monoid $\langle X\rangle$ can also be regarded as the word monoid over the alphabet X with ϵ as the empty word. For every word $W = x_1 \ldots x_n \in \langle X\rangle$, we denote the tensor product of the corresponding bimodules by

$$M_W := M_{x_1} \otimes \cdots \otimes M_{x_n}.$$

In particular, we have $M_\epsilon = K\epsilon$ for the empty word/tensor ϵ. The pure tensors $m_1 \otimes \cdots \otimes m_n \in M_W$ with $m_i \in M_{x_i}$ play the role of the monomials in the tensor ring. We consider the direct sum

$$M := \bigoplus_{x \in X} M_x \tag{1}$$

and the K-tensor ring on M:

$$K\langle M\rangle = \bigoplus_{n=0}^{\infty} M^{\otimes n} = \bigoplus_{W \in \langle X\rangle} M_W. \tag{2}$$

Every tensor $t \in K\langle M\rangle$ can be written as a sum of pure tensors. However, in contrast to linear combinations of monomials in the noncommutative polynomial algebra, this representation is not unique. This happens because already $M^{\otimes n}$ is not freely generated as a K-bimodule by the pure tensors, e.g. $m_1 \otimes m_3 + m_2 \otimes m_1 = m_3 \otimes m_1 + m_1 \otimes m_2$ in $M^{\otimes 2}$ for $m_3 = m_1 + m_2 \in M$. Still, using bimodule homomorphisms, one can define reductions analogous to polynomial reduction for (non-)commutative Gröbner bases.

Definition 3. *Let M be given by Eq. (1). A* reduction rule *for $K\langle M\rangle$ is given by a pair (W, h) of a word $W \in \langle X\rangle$ and a K-bimodule homomorphism $h\colon M_W \to K\langle M\rangle$. For a reduction rule $r = (W, h)$ and words $A, B \in \langle X\rangle$, we define a* reduction *as the K-bimodule homomorphism*

$$h_{A,r,B}\colon K\langle M\rangle \to K\langle M\rangle$$

acting as $\mathrm{id}_A \otimes h \otimes \mathrm{id}_B$ on M_{AWB} and the identity on all other M_V with $V \in \langle X\rangle$ and $V \neq AWB$.

For a pure tensor $a \otimes w \otimes b \in M_{AWB}$ with $a \in M_A$, $w \in M_W$, and $b \in M_B$, the reduction $h_{A,r,B}$ is given by

$$a \otimes w \otimes b \mapsto a \otimes h(w) \otimes b.$$

So, as for polynomial reduction, we "replace" the "leading monomial" w by the "tail" $h(w)$ given by the homomorphism h.

Let $t \in K\langle M\rangle$. A reduction $h_{A,r,B}$ acts trivially on t, i.e. $h_{A,r,B}(t) = t$, if the summand of t in M_{AWB} is zero, see Eq. (2). A reduction rule $r = (W, h)$ *reduces* t to $s \in K\langle M\rangle$ if a reduction $h_{A,r,B}$ for some $A, B \in \langle X\rangle$ acts non-trivially on t and $h_{A,r,B}(t) = s$ and we write $t \to_r s$.

A *reduction system* for $K\langle M\rangle$ is a set Σ of reduction rules. Every reduction system Σ induces a *reduction relation* $\to_\Sigma$ on tensors by defining $t \to_\Sigma s$ for $t, s \in K\langle M\rangle$ if $t \to_r s$ for some reduction rule $r \in \Sigma$. Fixing a reduction system Σ, we say that $t \in K\langle M\rangle$ *can be reduced* to $s \in K\langle M\rangle$ by Σ if $t = s$ or there exists a finite sequence of reduction rules $r_1, \ldots, r_n$ in Σ such that

$$t \to_{r_1} t_1 \to_{r_2} \cdots \to_{r_{n-1}} t_{n-1} \to_{r_n} s$$

and we write $t \overset{*}{\to}_\Sigma s$. In other words, $\overset{*}{\to}_\Sigma$ denotes the reflexive transitive closure of the reduction relation $\to_\Sigma$.

The set of *irreducible words* $\langle X\rangle_{\mathrm{irr}} \subseteq \langle X\rangle$ consists of those words having no subwords from the set $\{W \mid (W, h) \in \Sigma\}$. We define the K-subbimodule of *irreducible tensors* as

$$K\langle M\rangle_{\mathrm{irr}} = \bigoplus_{W \in \langle X\rangle_{\mathrm{irr}}} M_W. \tag{3}$$

We also need to consider partial orders on $\langle X\rangle$. A *semigroup partial order* on $\langle X\rangle$ is a partial order $\leq$ on $\langle X\rangle$ such that $B < \tilde{B} \Rightarrow ABC < A\tilde{B}C$ for all $A, B, \tilde{B}, C \in \langle X\rangle$. If in addition $\epsilon \leq A$, for all $A \in \langle X\rangle$, then it is called a *monoid partial order*. It is called *Noetherian* if there are no infinite descending chains.

Remark 4. *Note that a lexicographic order on $\langle X\rangle$ is not a semigroup order. However, a (weighted) degree-lexicographic order of the words is a semigroup (total) order on $\langle X\rangle$ and it is Noetherian if X is finite. Given a semigroup S with a semigroup partial order $\preceq$ on it and a semigroup homomorphism $\varphi\colon \langle X\rangle \to S$, we can define the induced semigroup partial order on $\langle X\rangle$ by*

$$V \leq W :\Leftrightarrow V = W \text{ or } \varphi(V) \prec \varphi(W).$$

For example, for $S = \mathbb{N}$ with the usual order and the homomorphism given by $\varphi(x_0) = 1$ for $x_0 \in X$ and $\varphi(x) = 0$ for $x \in X \setminus \{x_0\}$, the induced partial order just compares the degree *in x_0. Given two semigroups S_1 and S_2 with corresponding semigroup partial orders $\leq_1$ and $\leq_2$ respectively, we can combine them lexicographically to obtain a semigroup partial order on $S = S_1 \times S_2$ by*

$$(a_1, a_2) \leq (b_1, b_2) :\Leftrightarrow a_1 <_1 b_1 \text{ or } a_1 = b_1 \text{ and } a_2 \leq_2 b_2.$$

A semigroup partial order $\leq$ is *compatible* with a reduction system Σ if for all reduction rules $(W, h) \in \Sigma$,

$$h(M_W) \subseteq \bigoplus_{V<W} M_V.$$

If a compatible semigroup partial order is Noetherian, then there do not exist infinite sequences of reductions in Σ. In other words, the reduction relation $\rightarrow_\Sigma$ is *terminating* or *Noetherian*. So, in that case, every $t \in K\langle M\rangle$ can be reduced in finitely many steps to an irreducible tensor

$$t \xrightarrow{*}_\Sigma s \in K\langle M\rangle_{\mathrm{irr}}$$

and such an s is called a *normal form* of t. In general, a tensor can have different normal forms. If $t \in K\langle M\rangle$ has a *unique normal form*, we denote it by $t\downarrow_\Sigma$.

For ensuring unique normal forms for reduction systems on tensor rings, we state below Bergman's analog of Buchberger's criterion for Gröbner bases (Buchberger, 1965). In the context of Gröbner-Shirshov bases for various algebraic structures this is also referred to as the Composition-Diamond Lemma; see e.g. the survey by Bokut and Chen (2014).

Let Σ be a reduction system. We study the cases when two different reductions act non-trivially on tensors in M_W for $W \in \langle X\rangle$.

Definition 5. *An* overlap ambiguity *is given by two (not necessarily distinct) reduction rules $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and nonempty words $A, B, C \in \langle X\rangle$ such that*

$$W = AB \quad \text{and} \quad \tilde{W} = BC.$$

It is called resolvable *if for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ the* S-polynomial *can be reduced to zero:*

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \xrightarrow{*}_\Sigma 0.$$

An inclusion ambiguity *is given by distinct reduction rules $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and words $A, B, C \in \langle X\rangle$ with $W = B$ and $\tilde{W} = ABC$. It is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_B$, and $c \in M_C$ the* S-polynomial *can be reduced to zero: $a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \xrightarrow{*}_\Sigma 0$.*

With slight abuse of notation, we refer to S-polynomials of an overlap or inclusion ambiguity, respectively, by

$$\mathrm{SP}(A\underline{B}, \underline{B}C) \quad \text{or} \quad \mathrm{SP}(\underline{B}, A\underline{B}C).$$

A reduction system Σ induces the two-sided *reduction ideal*

$$I_\Sigma := (t - h(t) \mid (W, h) \in \Sigma \text{ and } t \in M_W) \subseteq K\langle M\rangle. \tag{4}$$

For studying operator algebras, we want to compute in the factor ring $K\langle M\rangle/I_\Sigma$. If all ambiguities are resolvable, then we can do this effectively using reductions in $K\langle M\rangle$ and the corresponding normal forms with respect to $\rightarrow_\Sigma$. This is the *confluence criterion* (condition 1. below) that we will check algorithmically, for a brief discussion see the following subsection.

Theorem 6. *([Bergman (1978)]) Let K be a ring, let $(M_x)_{x\in X}$ be a family of K-bimodules indexed by a set X, and let $M = \bigoplus_{x\in X} M_x$. Let Σ be a reduction system on $K\langle M\rangle$ and let $\leq$ be a Noetherian semigroup partial order on $\langle X\rangle$ that is compatible with Σ. Then the following are equivalent:*

1. *All ambiguities of Σ are resolvable.*
2. *Every $t \in K\langle M\rangle$ has a unique normal form $t\downarrow_\Sigma$.*
3. *$K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-bimodules.*

If these conditions hold, then we can define a multiplication on $K\langle M\rangle_{irr}$ by $s \cdot t := (s \otimes t)\downarrow_\Sigma$ so that $K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-rings.

Note that our definition of resolvability above differs from the definition used by Bergman. Actually, he uses two different notions for resolvability of ambiguities, wich we briefly describe below. Both of them are weaker than our Definition 5 in general. However, if every tensor has a unique normal form, then all three definitions of resolvability are equivalent. Hence Theorem 6 holds regardless which of these three notions of resolvability is used. One slightly weaker notion only requires the existence of a tensor $t \in K\langle M\rangle$ such that

$$h(a \otimes b) \otimes c \xrightarrow{*}_\Sigma t \xleftarrow{*}_\Sigma a \otimes \tilde{h}(b \otimes c) \quad \text{or} \quad a \otimes h(b) \otimes c \xrightarrow{*}_\Sigma t \xleftarrow{*}_\Sigma \tilde{h}(a \otimes b \otimes c),$$

respectively, in other words, the two different results of the reductions of $a \otimes b \otimes c$ are joinable. Another even weaker notion is the following, which depends on semigroup partial order $\leq$.

Definition 7. *We call an overlap or inclusion ambiguity with words $A, B, C \in \langle X\rangle$ $\leq$-resolvable if and only if all its S-polynomials are contained in the bimodule I_{ABC} generated by*

$$\bigcup_{\substack{V\in\langle X\rangle\\ V<ABC}} \{t - s \mid t \in M_V \text{ and } t \to_\Sigma s \in K\langle M\rangle\}.$$

If the semigroup partial order $\leq$ is compatible with Σ, then this bimodule is contained in a "truncation" $I_\Sigma \cap \bigoplus_{\substack{V\in\langle X\rangle\\ V<ABC}} M_V$ of the reduction ideal I_Σ.

Example 8. *We revisit Example 2 to study it formally in the tensor ring setting. Let (R, ∂) be a differential ring and let K denote its ring of constants. We consider the K-bimodule $M_\mathsf{R} = R$ (indexed by the letter R). In addition, we consider the free left K-module $M_\mathsf{D} = K\partial$ generated by ∂ (indexed by the letter D), which we view as a K-bimodule with right multiplication defined by*

$$c\partial \cdot d = cd\partial,$$

for all $c, d \in K$. This definition is based on left K-linearity of the operation ∂ on R:

$$(c\partial d)f = c\partial(df) = (cd\partial)f.$$

Let $M = M_\mathsf{R} \oplus M_\mathsf{D}$ be the module of basic operators. Then words over the alphabet $X = \{\mathsf{R}, \mathsf{D}\}$ index the direct summands of the K-tensor ring $K\langle M\rangle$.

We interpret elements $f \in R$ as multiplication operators, ∂ as the derivation on R, and the tensor product $\otimes$ as the composition of operators. So we consider the reduction system $\Sigma = \{r_\mathsf{RR}, r_\mathsf{DR}\}$ with the reduction rules

$$r_\mathsf{RR} = (\mathsf{RR}, f \otimes g \mapsto fg) \quad \text{and} \quad r_\mathsf{DR} = (\mathsf{DR}, \partial \otimes f \mapsto f \otimes \partial + \partial f)$$

corresponding to the composition of multiplication operators and the Leibniz rule. Then the ring of differential operators can be defined as the quotient

$$R\langle\partial\rangle = K\langle M\rangle/I_\Sigma$$

of the tensor ring by the two-sided reduction ideal. The informal definition of the reduction homomorphisms above can be made formal in the following way. First, since

$$\begin{aligned} M_{\mathsf{R}} \times M_{\mathsf{R}} &\to M_{\mathsf{R}} \\ (f, g) &\mapsto fg \end{aligned}$$

is a balanced map, it induces a well-defined homomorphism $M_{\mathsf{RR}} \to M_{\mathsf{R}}$ of abelian groups. This homomorphism can be verified to be even a K-bimodule homomorphism, which we use to define r_{RR}. Extending the definition

$$\beta(\partial, f) := f \otimes \partial + \partial f$$

by

$$\beta(c\partial, f) := \beta(\partial, cf),$$

we obtain a balanced map $\beta\colon M_{\mathsf{D}} \times M_{\mathsf{R}} \to M_{\mathsf{RD}} \oplus M_{\mathsf{R}}$, since

$$\beta(c\partial \cdot d, f) = \beta(cd\partial, f) = \beta(\partial, cdf) = \beta(c\partial, df).$$

Like above, β induces a K-bimodule homomorphism $M_{\mathsf{DR}} \to M_{\mathsf{RD}} \oplus M_{\mathsf{R}}$ constituting r_{DR}.

So any semigroup partial order $\leq$ on $\langle X\rangle$ with $\mathsf{RR} > \mathsf{R}$, as well as $\mathsf{DR} > \mathsf{RD}$ and $\mathsf{DR} > \mathsf{R}$ is compatible with Σ, e.g. the degree-lexicographic order with $\mathsf{D} > \mathsf{R}$. There are two overlap ambiguities. The S-polynomials of the first ambiguity reduce to zero in two steps:

$$\mathrm{SP}(\mathsf{R}\underline{\mathsf{R}}, \underline{\mathsf{R}}\mathsf{R}) = (fg) \otimes h - f \otimes (gh) \to_{r_{\mathsf{RR}}} (fg)h - f(gh) = 0.$$

We already have seen in Example 2 that the S-polynomials $\mathrm{SP}(\mathsf{D}\underline{\mathsf{R}}, \underline{\mathsf{R}}\mathsf{R})$ reduce to the Leibniz rule in R. Hence by Theorem 6 every $t \in K\langle M\rangle$ has a unique normal form $t\downarrow_\Sigma$ in $K\langle M\rangle_{\mathrm{irr}}$, where

$$K\langle M\rangle_{\mathrm{irr}} = K\epsilon \oplus M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus (M_{\mathsf{R}} \otimes M_{\mathsf{D}}) \oplus M_{\mathsf{D}}^{\otimes 2} \oplus (M_{\mathsf{R}} \otimes M_{\mathsf{D}}^{\otimes 2}) \oplus \ldots$$

since $\langle X\rangle_{\mathrm{irr}} = \{\epsilon, \mathsf{R}, \mathsf{D}, \mathsf{RD}, \mathsf{D}^2, \mathsf{RD}^2, \ldots\}$. In other words, $t\downarrow_\Sigma$ can be written as a sum of pure tensors of the form ϵ, f, ∂, $f \otimes \partial$, $\partial \otimes \partial$, $f \otimes \partial \otimes \partial, \ldots$ and we recover the well-known normal forms of differential operators.

Remark 9. *If some $\alpha \in M_x$ corresponds to a left K-linear operator, like $\partial \in M_{\mathsf{D}}$ above, then for the right scalar multiplication of left multiples of α, we always have*

$$c\alpha \cdot d = cd\alpha$$

with $c, d \in K$; see also Eq. (12). As soon as such an operator is present, the ring over which the tensors are formed has to contain K in order to incorporate the corresponding relations directly into the tensor ring.

2.3. *Computational Aspects*

Considering the algorithmic aspects of Theorem 6, we assume that we have a finite reduction system Σ over a finite alphabet X. Moreover, a compatible Noetherian semigroup partial order has to be assumed.

For generating the set of ambiguities, we only need to work in the word monoid $\langle X \rangle$. Likewise, determining the set of irreducible words $\langle X \rangle_{\mathrm{irr}}$ is a purely combinatorial problem on words as well, cf. the proofs of Theorems 27 and 32. For checking resolvability of ambiguities, it suffices to work with S-polynomials constructed from general elements of the basic bimodules M_x. The result of a reduction step, i.e. the application of a homomorphism from the reduction system, needs to be simplified in the tensor ring. This involves application of properties of the tensor product and of identities in the bimodules, like the Leibniz rule in the example above. In practice, the reduction to zero often can be detected heuristically without having a canonical simplifier in the bimodules.

The package `TenReS` provides routines to generate all ambiguities and corresponding S-polynomials of a reduction system given by the user. It also includes routines for computing in the tensor ring. Identities needed for computing in the bimodules of Eq. (1) have to be implemented by the user in each concrete case.

In contrast to specifying new identities in the polynomial resp. term algebra, already the constructive specification of reduction homomorphisms in the tensor setting is not clear in general.

2.3.1. *Deletion criteria*

For polynomial rings there are two classical deletion criteria for excluding critical pairs from consideration: the *product criterion* and the *chain criterion*. We want to consider their analogs for excluding ambiguities from the confluence check for tensor reduction systems.

There is no need for an analog of the product criterion as it is already built into the definition of ambiguities of tensor reduction rules. If rules $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ are such that no word of length less than $|W| + |\tilde{W}|$ contains both W and $\tilde{W}$ as subwords, then the rules do not have any ambiguities among them anyway. Hence we focus only on the chain criterion. The following lemma is an analog of Lemma 5.11 in (Mora, 1994).

Lemma 10. *Let $\leq$ be a semigroup partial order on $\langle X \rangle$ compatible with the reduction system Σ. Let $r_1, r_2 \in \Sigma$ have an overlap ambiguity with $A, B, C \in \langle X \rangle$, i.e. $r_1 = (AB, g)$ and $r_2 = (BC, h)$. Let $r_3 = (V, f) \in \Sigma$ where V is a subword of $W = ABC$ such that one of the following cases holds.*

1. *V is a subword of $A = LVR$ and the inclusion ambiguity of r_1 and r_3 with L, V, RB is $\leq$-resolvable.*
2. *V is a subword of $B = LVR$ and the two inclusion ambiguities of r_1 and r_3 with AL, V, R and of r_2 and r_3 with L, V, RC are $\leq$-resolvable.*
3. *V is a subword of $C = LVR$ and the inclusion ambiguity of r_2 and r_3 with BL, V, R is $\leq$-resolvable.*
4. *V is a subword of $AB = LVR$ (with nonempty V_1, V_2 such that $V = V_1V_2$ and $B = V_2R$) and the inclusion ambiguity of r_1 and r_3 with L, V, R as well as the overlap ambiguity of r_2 and r_3 with V_1, V_2, RC are $\leq$-resolvable.*
5. *V is a subword of $BC = LVR$ (with nonempty V_1, V_2 such that $V = V_1V_2$ and $B = LV_1$) and the overlap ambiguity of r_1 and r_3 with AL, V_1, V_2 as well as the inclusion ambiguity of r_2 and r_3 with L, V, R are $\leq$-resolvable.*

6. *There are nonempty L, R such that $V = LBR$ (with $A = A_1L$ and $C = RC_2$) and the overlap/inclusion ambiguity of r_1 and r_3 with A_1, LB, R as well as the overlap/inclusion ambiguity of r_2 and r_3 with L, BR, C_2 are $\leq$-resolvable.*

Then the overlap ambiguity of r_1 and r_2 with A, B, C is $\leq$-resolvable.

Proof. For all cases there are canonical choices for W_1, W_2 such that $W = W_1LVRW_2$ (resp. $W = W_1LBRW_2$ in the last case). For a pure tensor $t \in M_W$ we have that the corresponding S-polynomial is equal to $h_{\epsilon,r_1,C}(t) - h_{A,r_2,\epsilon}(t) = t_1 + t_2$ with $t_1 := h_{\epsilon,r_1,C}(t) - t_3$, $t_2 := t_3 - h_{A,r_2,\epsilon}(t)$, and $t_3 := h_{W_1L,r_3,RW_2}(t)$ (resp. $t_3 := h_{W_1,r_3,W_2}(t)$ in the last case). According to Definition 7, we show that $t_1, t_2 \in I_W$.

In Case 3, we directly verify $t_1 = g(a\otimes b)\otimes(c-h_{L,r_3,R}(c))-(a\otimes b-g(a\otimes b))\otimes h_{L,r_3,R}(c) \in I_W$ with $a \in M_A$, $b \in M_B$, and $c \in M_C$ such that $t = a\otimes b\otimes c$. Otherwise, by assumption, all S-polynomials of r_1 and r_3 are contained in I_{S_1}, where $S_1 = ABV_2$ in Case 5, $S_1 = ABR$ in Case 6, and $S_1 = AB$ in the remaining cases. Then, there is $m_1 \in M_{T_1}$, where $T_1 \in \langle X\rangle$ is such that $W = S_1T_1$, and an S-polynomial s_1 of r_1 and r_3 such that $t_1 = s_1 \otimes m_1$. Hence $t_1 \in I_{S_1} \otimes M_{T_1} \subseteq I_{S_1T_1} = I_W$.

Analogously, we directly verify $t_2 \in I_W$ in Case 1. In the remaining cases we let $S_2 := BC$ (resp. $S_2 := V_1BC$ in Case 4 and $S_2 := LBC$ in Case 6). Then, we have $t_2 = m_2 \otimes s_2$ for some S-polynomial s_2 of r_2 and r_3 and some $m_2 \in M_{T_2}$, where $T_2 \in \langle X\rangle$ is such that $W = T_2S_2$. We conclude $t_2 \in M_{T_2} \otimes I_{S_2} \subseteq I_W$, since $s_2 \in I_{S_2}$ by assumption.

Consequently, t_1 and t_2 are in I_W in all cases. Hence the same applies to the S-polynomial $h_{\epsilon,r_1,C}(t) - h_{A,r_2,\epsilon}(t)$ and the overlap ambiguity of r_1 and r_2 with A, B, C is $\leq$-resolvable. □

Note that V might be a subword of W in multiple ways, so we need to specify which ambiguities of r_1, r_3 resp. r_2, r_3 are $\leq$-resolvable in order to be able to conclude that the given ambiguity of r_1, r_2 is $\leq$-resolvable. A similar statement can be obtained for inclusion ambiguities of r_1 and r_2.

3. Tensor setting with specialization

Direct application of Bergman's tensor setting requires the sum in Eq. (1) to be direct. As a consequence, domains of reduction rules in a reduction system cannot overlap, even their tensor factors cannot overlap. In order to emulate overlapping domains (or factors), reduction rules have to be split into several smaller parts so that domains of those smaller rules do not overlap. Thus computations with such reduction systems can be inconvenient and inefficient in practice as the smaller rules technically are just individual rules that need to be applied separately. Moreover, this leads to some redundancy in the investigation of ambiguities and S-polynomials. Sticking to the above definition of reduction systems for tensor rings, this situation cannot be avoided.

Example 11. *Note that in Example 8 irreducible tensors still have some relations among them when acting as operators. For instance, $k\epsilon \in M^{\otimes 0}$ and $k \in M$ both act by multiplying with $k \in K$. So we need an additional reduction rule reducing $k \in M$ to $k\epsilon \in M^{\otimes 0}$ for $k \in K$. Fixing a direct complement $R = K \oplus \tilde{R}$ in R for defining the reduction rule*

$$r_{\mathsf{K}} = (\mathsf{K}, 1 \mapsto \epsilon),$$

would cause the splitting of the rule r_{RR} into four rules $r_{\mathsf{KK}}, r_{\mathsf{K\tilde{R}}}, r_{\mathsf{\tilde{R}K}}, r_{\mathsf{\tilde{R}\tilde{R}}}$ and similarly r_{DR} would split into two rules. The aim of this section is to introduce a framework that allows the rule r_{K} to coexist with r_{RR} and r_{DR}.

In order to remedy this situation, the aim of this section is to introduce a more flexible tensor setting where the definable reduction systems are much more general. While the induced reduction relations are also more general, the corresponding reduction ideals are not, however.

Definition 12. *Let M be a K-bimodule. We call a family $(M_z)_{z\in Z}$ of K-subbimodules of M a* decomposition with specialization*, if $M = \sum_{z\in Z} M_z$ and there exists a subset $X \subseteq Z$ such that*

1. *we have the direct sum decomposition $M = \bigoplus_{x\in X} M_x$ and*
2. *for every $z \in Z$ the corresponding module M_z satisfies*

$$M_z = \bigoplus_{x\in S(z)} M_x \tag{5}$$

where $S(z) := \{x \in X \mid M_x \subseteq M_z\}$ is the set of specializations *of z.*

Note that this definition implies $S(x) = \{x\}$ for $x \in X$. In the following, we define a framework for tensor reduction systems that are based on such a decomposition with specialization. To this end, we fix a K-bimodule M, alphabets $X \subseteq Z$, and a decomposition $(M_z)_{z\in Z}$ of M with specialization.

For words $W = w_1 \dots w_n \in \langle Z\rangle$ we define the corresponding subbimodule of $K\langle M\rangle$ as before by $M_W := M_{w_1} \otimes \dots \otimes M_{w_n}$. Because of Eq. (5), any M_W is then a direct sum of certain M_V, $V \in \langle X\rangle$. For a precise statement we can extend the notion of specialization from the alphabet Z to the whole word monoid $\langle Z\rangle$ by the definition below such that we have the following generalization of Eq. (5):

$$M_W = \bigoplus_{V\in S(W)} M_V.$$

Definition 13. *For $W = w_1 \dots w_n \in \langle Z\rangle$ we define the set of* specializations *of W by*

$$S(W) := \{v_1 \dots v_n \in \langle X\rangle \mid \forall i : v_i \in S(w_i)\}$$

Remark 14. *Note that for $V \in \langle X\rangle$ and $W \in \langle Z\rangle$ the bimodules M_V and M_W either intersect only in 0 or M_V is contained in M_W. Note further that the specializations of $W \in \langle Z\rangle$ are also given by*

$$S(W) = \{V \in \langle X\rangle \mid M_V \subseteq M_W\}.$$

Definition 3 carries over by replacing X with Z. For such a reduction system Σ over Z we define the reduction ideal I_Σ by Eq. (4) and we define $\langle X\rangle_{\mathrm{irr}}$ as the set of words from $\langle X\rangle$ containing no subwords from the set

$$\bigcup_{(W,h)\in\Sigma} S(W).$$

Based on $\langle X\rangle_{\mathrm{irr}}$ we define $K\langle M\rangle_{\mathrm{irr}}$ as in Eq. (3). Furthermore, for every reduction system Σ over Z we call its reformulation as a reduction system over X the *refined reduction system* Σ_X, which is given by

$$\Sigma_X := \bigcup_{(W,h)\in\Sigma} \{(V, h|_{M_V}) \mid V \in S(W)\}. \tag{6}$$

Lemma 15. *Let Σ be a reduction system over Z and let Σ_X be its refinement on X. Then the reduction ideals and the irreducible words are the same for Σ and for Σ_X. Moreover, also $K\langle M\rangle_{irr}$ stays the same.*

Proof. Follows immediately from the definitions. □

Note that, however, the refined reduction system does not define the same reduction relation. In general, we neither have $\to_{\Sigma_X} \subseteq \to_\Sigma$ nor $\to_\Sigma \subseteq \to_{\Sigma_X}$. We only have $\to_\Sigma \subseteq \overset{*}{\to}_{\Sigma_X}$ in general.

Definition 16. *We call a partial order $\leq$ on $\langle Z\rangle$* consistent with specialization *if for all words $V, W \in \langle Z\rangle$ with $V < W$ we also have $\tilde{V} < \tilde{W}$ for all specializations $\tilde{V} \in S(V)$ and $\tilde{W} \in S(W)$.*

Note that the above definition implies that W is incomparable to all elements in $S(W)$, except possibly W itself, which can be seen by considering the two cases $V \in S(W)$ and $W \in S(V)$ in the definition.

A semigroup partial order $\leq$ on $\langle Z\rangle$ is *compatible* with a reduction system Σ over Z if for all $(W, h) \in \Sigma$ we have

$$h(M_W) \subseteq \sum_{\substack{V\in\langle Z\rangle\\ V<W}} M_V.$$

If $\leq$ is consistent with specialization, then for any $\tilde{W} \in S(W)$ we have

$$\sum_{\substack{V\in\langle Z\rangle\\ V<W}} M_V \subseteq \bigoplus_{\substack{V\in\langle X\rangle\\ V<\tilde{W}}} M_V.$$

Lemma 17. *Let Σ be a reduction system over Z and let $\leq$ be a semigroup partial order on $\langle Z\rangle$ consistent with specialization and compatible with Σ. Then the restricted order $\leq$ on $\langle X\rangle$ is compatible with Σ_X.*

Proof. By definition of Σ_X, For any reduction rule $(\tilde{W}, \tilde{h}) \in \Sigma_X$ there is $(W, h) \in \Sigma$ such that $\tilde{W} \in S(W)$ and $\tilde{h} = h|_{M_{\tilde{W}}}$. So, by our assumptions, we have

$$\tilde{h}(M_{\tilde{W}}) = h(M_{\tilde{W}}) \subseteq h(M_W) \subseteq \sum_{\substack{V\in\langle Z\rangle\\ V<W}} M_V \subseteq \bigoplus_{\substack{V\in\langle X\rangle\\ V<\tilde{W}}} M_V. \qquad \square$$

We need to generalize the notion of ambiguities to account for the fact that the sum $K\langle M\rangle = \sum_{W\in\langle Z\rangle} M_W$ is not necessarily direct anymore.

Definition 18. *Let $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ be two (not necessarily distinct) reduction rules and let $A, B_1, B_2, C \in \langle Z\rangle$ be nonempty words with*

$$W = AB_1, \quad \tilde{W} = B_2C, \quad \textit{and} \quad S(B_1) \cap S(B_2) \neq \emptyset,$$

then we call this an overlap ambiguity. *An overlap ambiguity is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ the S-polynomial can be reduced to zero:*

$$h(a \otimes b) \otimes c - a \otimes \tilde{h}(b \otimes c) \overset{*}{\to}_\Sigma 0.$$

Similarly, an inclusion ambiguity *is given by two distinct reduction rules $(W, h), (\tilde{W}, \tilde{h}) \in \Sigma$ and words $A, B_1, B_2, C \in \langle Z\rangle$ with $W = B_1$, $\tilde{W} = AB_2C$, and $S(B_1) \cap S(B_2) \neq \emptyset$. An inclusion ambiguity is called* resolvable *if for all pure tensors $a \in M_A$, $b \in M_{B_1} \cap M_{B_2}$, and $c \in M_C$ the S-polynomial can be reduced to zero: $a \otimes h(b) \otimes c - \tilde{h}(a \otimes b \otimes c) \overset{*}{\to}_\Sigma 0$.*

If $B_1 \neq B_2$ for an overlap or inclusion ambiguity, then we say that the ambiguity is with specialization.

Again, we use $\mathrm{SP}(A\underline{B_1}, \underline{B_2}C)$ or $\mathrm{SP}(\underline{B_1}, A\underline{B_2}C)$, respectively, to refer to S-polynomials of an overlap or inclusion ambiguity.

Remark 19. *Note that in total there now can be four types of ambiguities: in addition to the two types of ambiguities (without specialization) of Definition 5 there are also corresponding versions with specialization as defined above.*

With these definitions we can prove the following generalization of Bergman's result. In order to prove properties of the reduction system Σ over Z we apply Bergman's result (Theorem 6) to the refined reduction system Σ_X over X.

Theorem 20. *Let M be a K-bimodule and let $(M_z)_{z\in Z}$ be a decomposition with specialization. Let Σ be a reduction system over Z on $K\langle M\rangle$ and let $\leq$ be a Noetherian semigroup partial order on $\langle Z\rangle$ consistent with specialization and compatible with Σ. Then the following are equivalent:*

1. *All ambiguities of Σ are resolvable.*
2. *Every $t \in K\langle M\rangle$ has a unique normal form $t\downarrow_\Sigma$.*
3. *$K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-bimodules.*

Moreover, if these conditions are satisfied, then we can define a multiplication on $K\langle M\rangle_{irr}$ by $s \cdot t := (s \otimes t)\downarrow_\Sigma$ so that $K\langle M\rangle/I_\Sigma$ and $K\langle M\rangle_{irr}$ are isomorphic as K-rings.

Proof. First, we prove the implication 2. $\Rightarrow$ 1. Any S-polynomial of an ambiguity of Σ is of the form $h(t) - \tilde{h}(t)$ for some pure tensor $t \in K\langle M\rangle$ and reductions h and $\tilde{h}$ of Σ. Let H_1 be a composition of reductions of Σ such that $H_1(h(t)) \in K\langle M\rangle_{\mathrm{irr}}$ and let H_2 be a composition of reductions of Σ such that $H_2(H_1(\tilde{h}(t))) \in K\langle M\rangle_{\mathrm{irr}}$. Then $H_2 \circ H_1$ reduces the S-polynomial to zero since t has a unique normal form w.r.t. Σ.

The rest of the proof is reduced to Theorem 6 via properties of the refined reduction system Σ_X. Lemma 15 shows that we can replace the reduction system Σ by its refinement Σ_X without changing the reduction ideal or $K\langle M\rangle_{\mathrm{irr}}$, hence statement 3. holds for Σ if and only if it holds for Σ_X. Furthermore, we note that every S-polynomial of Σ_X is also an S-polynomial of Σ and that $\overset{*}{\to}_\Sigma \subseteq \overset{*}{\to}_{\Sigma_X}$, hence statement 1. holds for Σ_X if it holds for Σ. If statement 2. holds for Σ_X, then by $\overset{*}{\to}_\Sigma \subseteq \overset{*}{\to}_{\Sigma_X}$ and the fact that $K\langle M\rangle_{\mathrm{irr}}$ does not change it also holds for Σ. Finally, Lemma 17 implies that Σ_X and the restriction of $\leq$ to $\langle X\rangle$ satisfy the assumptions of Theorem 6, which concludes the proof. $\square$

Note that for $W, \tilde{W} \in \langle Z\rangle$ having a common specialization, i.e. $S(W) \cap S(\tilde{W}) \neq \emptyset$, there does not necessarily exist $V \in \langle Z\rangle$ such that $S(V) = S(W) \cap S(\tilde{W})$. In general, the intersection of two modules is given by

$$M_W \cap M_{\tilde{W}} = \bigoplus_{V\in S(W)\cap S(\tilde{W})} M_V = \bigotimes_{k=1}^{n} \bigoplus_{x\in S(w_k)\cap S(\tilde{w}_k)} M_x,$$

where $W = w_1 \dots w_n$ and $\tilde{W} = \tilde{w}_1 \dots \tilde{w}_n$.

Example 21. *Consider alphabets $X = \{x_1, x_2, x_3\}$ and $Z = X \cup \{y_1, y_2\}$ with bimodules $M_{y_1} = M_{x_1} \oplus M_{x_3}$ and $M_{y_2} = M_{x_2} \oplus M_{x_3}$. The words $W = x_1y_2y_1$ and $\tilde{W} = y_1y_2y_2$ in $\langle Z\rangle$ satisfy $S(W) \cap S(\tilde{W}) = \{x_1x_2x_3, x_1x_3x_3\} \neq \emptyset$. We have $M_W \cap M_{\tilde{W}} = M_{x_1} \otimes M_{y_2} \otimes M_{x_3}$. So, in this case, there even exists a word $V = x_1y_2x_3$ that satisfies $S(V) = S(W) \cap S(\tilde{W})$ and $M_V = M_W \cap M_{\tilde{W}}$.* $\square$

Example 22. *Consider alphabets* $X = \{x_1, x_2, x_3, x_4\}$ *and* $Z = X \cup \{y_1, y_2\}$ *with* $S(y_i) = X \setminus \{x_{5-i}\}$. *The words* $W = y_1$ *and* $\tilde{W} = y_2$ *satisfy* $S(W) \cap S(\tilde{W}) = \{x_1, x_2\} \neq \emptyset$ *and there is no word* V *with* $S(V) = S(W) \cap S(\tilde{W})$. □

In order to describe the intersection of modules in terms of words again it will be convenient to also consider another partial order $\preceq$ on $\langle Z \rangle$, which is induced by the natural partial order, given by set inclusion, on all sets of the form $S(W) \subseteq \langle X \rangle$. In other words, we have $V \preceq W$ in $\langle Z \rangle$ if and only if $S(V) \subseteq S(W)$, which holds if and only if M_V is contained in M_W.

In addition, for a set $S \subseteq \langle Z \rangle$ we define the K-bimodule

$$M_S := \sum_{W \in S} M_W \subseteq K\langle M \rangle \tag{7}$$

with M_S being the trivial bimodule $\{0\}$ if S is empty. We also define $lb(S) := \{V \in \langle Z \rangle \mid V \preceq W \text{ for all } W \in S\}$ as the set of all lower bounds of S with respect to the partial order $\preceq$. Note that this implies

$$\bigcap_{W \in S} M_W = M_{lb(S)} = M_{lb(S) \cap \langle X \rangle}$$

where we have $lb(S) \cap \langle X \rangle = \bigcap_{W \in S} S(W)$. If $\preceq$ satisfies the ascending chain condition, it is enough to consider only maximal elements of $lb(S)$ for $\bigcap_{W \in S} M_W = M_{lb(S)}$.

Example 23. *Consider alphabets* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *and* $Z = X \cup \{y_1, y_2, y_3, z_1, z_2\}$ *with* $S(y_i) = \{x_i, x_{i+1}\}$ *and* $S(z_i) = X \setminus \{x_{7-i}\}$. *The words* $W = z_1$ *and* $\tilde{W} = z_2$ *satisfy* $S(W) \cap S(\tilde{W}) = \{x_1, x_2, x_3, x_4\} \neq \emptyset$ *and there is no word* V *with* $S(V) = S(W) \cap S(\tilde{W})$. *We have* $lb(W, \tilde{W}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$ *and the maximal elements of* $lb(W, \tilde{W})$ *are* y_1, y_2, y_3. *As explained above, we have* $M_W \cap M_{\tilde{W}} = M_{lb(W,\tilde{W})} = M_{lb(W,\tilde{W}) \cap \langle X \rangle} = M_{\{y_1,y_2,y_3\}}$. *In this example, we can even find words such that the intersection is a direct sum of as few modules as possible:* $M_W \cap M_{\tilde{W}} = M_{y_1} \oplus M_{y_3}$. □

3.1. Multi-level setting

Our two-level tensor setting presented at ISSAC 2016 (Hossein Poor et al., 2016a, Sec. 4) can be generalized to obtain a multi-level tensor setting, which in turn is a special case of the setting presented above. We briefly describe how the multi-level tensor setting looks like. To this end, we first recall when one direct sum decomposition of M is a refinement of another.

For two families of K-bimodules with $M = \bigoplus_{x \in X} M_x = \bigoplus_{y \in Y} M_y$, we say that $(M_x)_{x \in X}$ is a refinement of $(M_y)_{y \in Y}$ if there exists a partition $(X_y)_{y \in Y}$ of X such that

1. $X_y = \{x\}$ for all $y \in X \cap Y$ and
2. $M_y = \bigoplus_{x \in X_y} M_x$ for all $y \in Y$.

For the multi-level setting we consider a family of alphabets $(X_i)_{i \in I}$ each corresponding to a direct sum decomposition $M = \bigoplus_{x \in X_i} M_x$, the "levels". On the index set I we can define a partial order $\leq$ such that $i \leq j$ if and only if $(M_x)_{x \in X_i}$ is a refinement of $(M_x)_{x \in X_j}$. We require that the set I has a least element $0 \in I$ w.r.t. $\leq$, i.e. there exists a finest level that is a refinement of all levels. Defining $X := X_0$ and $Z := \bigcup_{i \in I} X_i$ we easily recognize this as a special case of the above tensor setting with specialization.

Conversely, each instance of the tensor setting with specialization can be viewed as multi-level by completing each M_z, $z \in Z \setminus X$, into a level of its own: $M = M_z \oplus M_{\bar{z}}$ with $M_{\bar{z}} :=$

$\bigoplus_{x \in X \setminus S(z)} M_x$. The resulting order $\preceq$ on $I := \{0\} \cup (Z \setminus X)$ may be far from total, it may even be trivial.

The multi-level setting is worth mentioning mainly because of the following property. If $\preceq$ is a total order on I, i.e. if all levels are nested, then for any $W, \tilde{W} \in \langle Z \rangle$ with $S(W) \cap S(\tilde{W}) \neq \emptyset$, there exists (at least one) $V \in \langle Z \rangle$ such that $S(V) = S(W) \cap S(\tilde{W})$, i.e. $M_V = M_W \cap M_{\tilde{W}}$.

3.2. Computational Aspects

Many properties that we discussed for Bergman's tensor setting also hold for the tensor setting with specialization we introduced above. For instance, determining ambiguities and irreducible words is done just on the level of words. In the following, we discuss the differences of the two settings.

The main computational benefit of Theorem 20 compared to Theorem 6 lies in the fact that for the confluence criterion we only need to check ambiguities of Σ over the alphabet Z and no computations with Σ_X are needed. Computing with the refined reduction system over X instead, generally would lead to a higher number of ambiguities, since one reduction rule in Σ can give rise to many reduction rules in Σ_X. Only for determination of irreducible words we restrict to $\langle X \rangle$.

If we formulate our reduction system Σ over the alphabet Z, instead of using some $\tilde{\Sigma}$ over the smaller alphabet X for the same reduction ideals $I_{\tilde{\Sigma}} = I_{\Sigma}$, we may be able to considerably reduce the size of the reduction system. This may happen in two different ways. First, assume a partition of X such that some homomorphisms in $\tilde{\Sigma}$ are defined by the same formula and the homomorphisms differ only by the choice of their domain and the corresponding words are obtained as specializations from some template. Then the corresponding reduction rules from $\tilde{\Sigma}$ could be merged into one reduction rule in Σ. This is exactly what happens for $\tilde{\Sigma} = \Sigma_X$. Second, also extending the domain of some homomorphism from $\tilde{\Sigma}$ may contribute to obtaining a smaller reduction system Σ. So usually we will have $\tilde{\Sigma} \subset \Sigma_X$.

The package `TenReS` also provides routines for generating all overlap and inclusion ambiguities with specialization together with their corresponding S-polynomials. For a detailed comparison of Bergman's setting and our generalization for the example of IDOs see (Hossein Poor et al., 2016a).

4. Integro-differential operators

Integro-differential operators over a field of constants were introduced in (Rosenkranz, 2005; Rosenkranz and Regensburger, 2008) to study algebraic and algorithmic aspects of linear ordinary boundary problems. The construction made use of a parametrized Gröbner basis in infinitely many variables coming from a basis of the coefficient algebra; see also the survey (Rosenkranz et al., 2012) for an automated confluence proof and (Regensburger, 2016) for related references. For polynomial coefficients, also generalized Weyl algebras (Bavula, 2013), skew polynomials (Regensburger et al., 2009), and noncommutative Gröbner bases (Quadrat and Regensburger, 2017) have been used to study them. In this section, we apply the tensor setting with specialization introduced above to the construction of normal forms for integro-differential operators (IDOs) over an arbitrary integro-differential ring. First, we define an integro-differential ring analogous to the definition of an integro-differential algebra in (Rosenkranz et al., 2012; Guo et al., 2014).

Definition 24. *Let (R, ∂) be a differential ring such that $\partial R = R$. Moreover, let $\int\colon R \to R$ be a bimodule homomorphism over the ring of constants in R, such that*

$$\partial {\textstyle\int} f = f \tag{8}$$

for all $f \in R$. We call $(R, \partial, \int)$ an integro-differential ring *if the* evaluation

$$\mathrm{E}f := f - {\textstyle\int} \partial f \tag{9}$$

is multiplicative, i.e. for all $f, g \in R$ we have

$$\mathrm{E}fg = (\mathrm{E}f)\mathrm{E}g.$$

The following lemma shows that in any integro-differential ring, the evaluation E maps to the constants such that it acts as the identity on them, in particular, it is also a homomorphism of rings with unit element. Moreover, the ring R can be decomposed as direct sum of constant and non-constant "functions".

Lemma 25. *Let $(R, \partial, \int)$ be an integro-differential ring with constants K. Then, we have* $\mathrm{E}1 = 1$*, $\mathrm{E}f \in K$ for all $f \in R$, and*

$$R = K \oplus {\textstyle\int} R,$$

as direct sum of K-bimodules.

Proof. We first compute $\mathrm{E}1 = 1 - \int \partial 1 = 1$ and $\partial \mathrm{E} f = \partial(f - \int \partial f) = \partial f - \partial f = 0$. For any $f \in R$, we have $f = \mathrm{E}f + f - \mathrm{E}f = \mathrm{E}f + \int \partial f$ and hence $R = K + \int R$. Let $f \in K \cap \int R$ and $g \in R$ such that $f = \int g$. Then $0 = \partial f = \partial \int g = g$, which implies $f = 0$. ☐

For the rest of this section, we fix an arbitrary integro-differential ring $(R, \partial, \int)$ and we denote its ring of constants by K. By an operator, we understand in the following a K-bimodule homomorphism from R to R. For example, the operations $\partial, \int, \mathrm{E}$ can be viewed as operators.

Following Lemma 25, we consider the direct sum decomposition $R = K \oplus \int R$ and the corresponding K-bimodules

$$M_{\mathsf{K}} = K \quad \text{and} \quad M_{\tilde{\mathsf{R}}} = {\textstyle\int} R \tag{10}$$

(indexed by the symbols K and $\tilde{\mathsf{R}}$). Note that the elements of M_{K} and $M_{\tilde{\mathsf{R}}}$ are not interpreted as functions but as left multiplication operators $g \mapsto fg$ induced by those functions. For studying boundary value problems algebraically, we also need to deal with other multiplicative "functionals" on R with the same properties as E, so we consider the set

$$\Phi := \{\varphi\colon R \to K \mid \varphi \text{ is a } K\text{-bimodule homomorphism with } \varphi fg = (\varphi f)\varphi g \text{ and } \varphi 1 = 1\}. \tag{11}$$

Instead, one can also consider Φ as a proper subset (containing E) of the full set defined above. This amounts to working with a smaller ring of operators later. For the operators ∂, $\int$, E, and $\varphi \in \tilde{\Phi}$ with $\tilde{\Phi} = \Phi \setminus \{\mathrm{E}\}$, we consider the free left K-modules

$$M_{\mathsf{D}} = K\partial, \quad M_{\mathsf{I}} = K{\textstyle\int}, \quad M_{\mathsf{E}} = K\mathrm{E}, \quad M_{\tilde{\Phi}} = K\tilde{\Phi} \tag{12}$$

generated by them (indexed by the symbols D, I, E, and $\tilde{\Phi}$). We view these modules as K-bimodules with right multiplication defined by

$$c\alpha \cdot d = cd\alpha$$

where $\alpha \in \{\partial, \int, \mathrm{E}\} \cup \tilde{\Phi}$ and $c, d \in K$, since the generators of these modules correspond to left K-linear operators. We define two alphabets

$$X = \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{D}, \mathsf{I}, \mathsf{E}, \tilde{\Phi}\} \quad \text{and} \quad Z = X \cup \{\mathsf{R}, \Phi\}, \tag{13}$$

with the K-bimodules $(M_x)_{x \in X}$ defined in Eqs. (10) and (12) as well as

$$M_{\mathsf{R}} = M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}} \quad \text{and} \quad M_{\Phi} = M_{\mathsf{E}} \oplus M_{\tilde{\Phi}}. \tag{14}$$

Now, we define the module M by

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus M_{\mathsf{I}} \oplus M_{\Phi}, \tag{15}$$

which turns $(M_z)_{z \in Z}$ into a decomposition with specialization.

In order to compute with these operators, we need to collect identities they satisfy in form of a reduction system. To this end, we first list identities following immediately from their definitions (like multiplicativity of functionals, K-linearity, and the Leibniz rule) and some of their consequences that hold in R. For all $f, g \in R$ and $\varphi, \psi \in \Phi$:

$\varphi f g = (\varphi f)\varphi g$	$\partial \int g = g$
$\psi \varphi g = \varphi g$	$\int \partial g = g - \mathrm{E} g$
$\mathrm{E} \int g = 0$	$\int f \varphi g = (\int f)\varphi g$
$\partial f g = f \partial g + (\partial f) g$	$\int f \partial g = f g - \int (\partial f) g - (\mathrm{E} f)\mathrm{E} g$
$\partial \varphi g = 0$	$\int f \int g = (\int f) \int g - \int (\int f) g$

The identities that do not follow immediately from the definitions are $\mathrm{E}\int g = 0$, integration by parts

$$\int f \partial g = f g - \int (\partial f) g - (\mathrm{E} f)\mathrm{E} g,$$

and the Rota-Baxter identity

$$\int f \int g = (\int f) \int g - \int (\int f) g$$

for the integral. They can either be verified directly or we obtain them in Section 6 as a consequence of S-polynomial computations. All identities listed above correspond to identities for operators acting on $g \in R$. The reduction system Σ over the alphabet $\langle Z \rangle$ is given by Table 1, defined in terms of all $f, g \in R$ and $\varphi, \psi \in \Phi$.

In analogy to the definition of reduction homomorphisms in Section 2, the informal definitions in Table 1 have to be made formal. For instance,

$$\beta_{\mathsf{ID}}(\int, \partial) := \epsilon - \mathrm{E}$$

is extended to a balanced map on $M_{\mathsf{I}} \times M_{\mathsf{D}}$ via

$$\beta_{\mathsf{ID}}(c\int, d\partial) := cd\beta_{\mathsf{ID}}(\int, \partial)$$

and similarly

$$\beta_{\mathsf{IR\Phi}}(\int, f, \varphi) := \int f \otimes \varphi$$

with $\varphi \in \Phi$ is extended to a balanced map on $M_{\mathsf{I}} \times M_{\mathsf{R}} \times M_{\Phi}$ by

$$\beta_{\mathsf{IR\Phi}}(c\int, f, \sum_i c_i \varphi_i) := \sum_i \beta_{\mathsf{IR\Phi}}(\int, c f c_i, \varphi_i).$$

K	$1 \mapsto \epsilon$
RR	$f \otimes g \mapsto fg$
ΦR	$\varphi \otimes f \mapsto (\varphi f)\varphi$
ΦΦ	$\psi \otimes \varphi \mapsto \varphi$
EI	$\mathrm{E} \otimes \int \mapsto 0$
DR	$\partial \otimes f \mapsto f \otimes \partial + \partial f$
DΦ	$\partial \otimes \varphi \mapsto 0$
DI	$\partial \otimes \int \mapsto \epsilon$
IΦ	$\int \otimes \varphi \mapsto \int 1 \otimes \varphi$
ID	$\int \otimes \partial \mapsto \epsilon - \mathrm{E}$
II	$\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1$
IRΦ	$\int \otimes f \otimes \varphi \mapsto \int f \otimes \varphi$
IRD	$\int \otimes f \otimes \partial \mapsto f - \int \otimes \partial f - (\mathrm{E}f)\mathrm{E}$
IRI	$\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f$

Table 1: Reduction rules for IDOs

Definition 26. *Let $(R, \partial, \int)$ be an integro-differential ring with constants K. We call*

$$R\langle \partial, \textstyle\int, \Phi\rangle := K\langle M\rangle / J$$

the ring of integro-differential operators, *where J is the two-sided reduction ideal induced by the reduction system obtained from Table 1.*

In order to compute in $R\langle \partial, \int, \Phi\rangle$ we want to analyze the reduction system defined by Table 1 according to Theorem 20 above and determine normal forms of tensors. Following the definition in Eq. (6), the refined reduction system Σ_X is obtained, according to Eq. (14), by splitting rules whose words contain R or Φ into "smaller" rules using $S(\mathsf{R}) = \{\mathsf{K}, \tilde{\mathsf{R}}\}$ and $S(\Phi) = \{\mathsf{E}, \tilde{\Phi}\}$. For example, the reduction rule $(\Phi\mathsf{R}, h) \in \Sigma$ is split into the rules $(W, h|_{M_W}) \in \Sigma_X$ where $W \in S(\Phi\mathsf{R}) = \{\mathsf{EK}, \mathsf{E}\tilde{\mathsf{R}}, \tilde{\Phi}\mathsf{K}, \tilde{\Phi}\tilde{\mathsf{R}}\}$.

Theorem 27. *Let $(R, \partial, \int)$ be an integro-differential ring with constants K and let Φ be the set of multiplicative K-bimodule homomorphisms given by Eq. (11). Let M be defined by Eqs. (14) and (15) and let the reduction system Σ be defined by Table 1.*

Then every $t \in K\langle M\rangle$ has a unique normal form $t\downarrow_\Sigma$, which is given by a sum of pure tensors of the form

$$f \otimes \varphi \otimes \partial^{\otimes j} \quad \text{or} \quad f \otimes \varphi \otimes \textstyle\int \otimes g$$

where $j \in \mathbb{N}_0$, each of $f, g \in M_{\tilde{\mathsf{R}}}$ and $\varphi \in \Phi$ may be absent, and $\varphi \otimes \int$ does not specialize to $\mathrm{E} \otimes \int$. Moreover,

$$R\langle \partial, \textstyle\int, \Phi\rangle \cong K\langle M\rangle_{irr}$$

as K-rings, where the multiplication on $K\langle M\rangle_{irr}$ is defined by $s \cdot t := (s \otimes t)\downarrow_\Sigma$.

Proof. We consider the alphabets X and Z given by Eq. (13). This turns $(M_z)_{z \in Z}$ into a decomposition with specialization for the module M, see Definition 12. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ, it is sufficient to require the order to satisfy

$$\mathsf{DR} > \mathsf{RD},\ \mathsf{IRD} > \mathsf{E},\ \mathsf{ID} > \mathsf{E},\ \mathsf{I} > \tilde{\mathsf{R}}.$$

For instance, we could use a degree-lexicographic order with $\mathsf{I} > \mathsf{D} > \Phi > \mathsf{R}$ on $\langle\{\mathsf{R}, \mathsf{D}, \mathsf{I}, \Phi\}\rangle \subseteq \langle Z\rangle$ or other degree-lexicographic orders with $\mathsf{D} > \mathsf{R}$ and $\mathsf{I} > \mathsf{R}$. We extend it to a monoid partial order on $\langle Z\rangle$ based on Definition 16 in order to make it consistent with specialization. Then by the package `TenReS` we verify that all ambiguities of Σ are resolvable, see Section 4.1. Hence by Theorem 20 every element of $K\langle M\rangle$ has a unique normal form and $K\langle M\rangle/I_\Sigma \cong K\langle M\rangle_{\text{irr}}$ as K-rings.

It remains to determine the explicit form of elements in $K\langle M\rangle_{\text{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\text{irr}}$ in $\langle X\rangle$. Irreducible words containing only the letters K and $\tilde{\mathsf{R}}$ have to avoid the subwords K and $S(\mathsf{RR}) = \{\mathsf{KK}, \mathsf{K}\tilde{\mathsf{R}}, \tilde{\mathsf{R}}\mathsf{K}, \tilde{\mathsf{R}}\tilde{\mathsf{R}}\}$, hence only the words ϵ and $\tilde{\mathsf{R}}$ are left. The irreducible words containing only E and $\tilde{\Phi}$ are exactly ϵ, E, and $\tilde{\Phi}$, since they have to avoid the subwords $S(\Phi\Phi) = \{\mathsf{EE}, \mathsf{E}\tilde{\Phi}, \tilde{\Phi}\mathsf{E}, \tilde{\Phi}\tilde{\Phi}\}$. Altogether, we see that the irreducible words containing only the letters K, $\tilde{\mathsf{R}}$, E, and $\tilde{\Phi}$ are given by the set $\{\epsilon, \tilde{\mathsf{R}}, \mathsf{E}, \tilde{\Phi}, \tilde{\mathsf{R}}\mathsf{E}, \tilde{\mathsf{R}}\tilde{\Phi}\}$, since they also have to avoid the subwords $S(\Phi\mathsf{R}) = \{\mathsf{EK}, \mathsf{E}\tilde{\mathsf{R}}, \tilde{\Phi}\mathsf{K}, \tilde{\Phi}\tilde{\mathsf{R}}\}$. Allowing also the letter D, we have to avoid the subwords coming from $S(\mathsf{DR}) = \{\mathsf{DK}, \mathsf{D}\tilde{\mathsf{R}}\}$ and $S(\mathsf{D}\Phi) = \{\mathsf{DE}, \mathsf{D}\tilde{\Phi}\}$. Therefore, we can only append words D^j with $j \in \mathbb{N}_0$ to the irreducible words determined so far, in order to obtain all elements of $\langle X\rangle_{\text{irr}}$ not containing the letter I. Finally, we also consider the letter I. Since subwords EI and DI have to be avoided, the first occurrence of I in an irreducible word can only be preceded by ϵ, $\tilde{\mathsf{R}}$, $\tilde{\Phi}$, or $\tilde{\mathsf{R}}\tilde{\Phi}$. We also have to avoid the subwords $S(\mathsf{I}\Phi) = \{\mathsf{IE}, \mathsf{I}\tilde{\Phi}\}$, ID, and II, so any letter immediately following I has to be $\tilde{\mathsf{R}}$. In addition, we have to avoid the subwords $S(\mathsf{IR}\Phi) = \{\mathsf{IKE}, \mathsf{IK}\tilde{\Phi}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{E}, \mathsf{I}\tilde{\mathsf{R}}\tilde{\Phi}\}$, $S(\mathsf{IRD}) = \{\mathsf{IKD}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{D}\}$, and $S(\mathsf{IRI}) = \{\mathsf{IKI}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{I}\}$, so the letter I cannot be followed by a subword of length greater than one. Altogether, the elements of $\langle X\rangle_{\text{irr}}$ are of the form

$$\tilde{\mathsf{R}}V\mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}\tilde{\Phi}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\Phi}$, and $V \in S(\Phi) = \{\mathsf{E}, \tilde{\Phi}\}$ may be absent. The normal forms follow from Eq. (3). □

Note that the formulae given in Table 1 above to define the reduction system for the tensor ring are the same as the formulae presented in Table 2 in (Hossein Poor et al., 2016a) for the tensor algebra with commutative K. Here we use these formulae to define K-bimodule homomorphisms via balanced maps instead of defining K-module homomorphisms via multilinear maps. The same ambiguities need to be considered for checking confluence and we obtain the same structure of normal forms. Differences arise only from R now being a K-ring instead of a K-algebra.

4.1. Computational aspects

In the following, we briefly discuss computational details of the tensor setting with specialization for integro-differential operators. Applying `TenReS` to the reduction system Σ, in total 52 ambiguities and corresponding S-polynomials are generated. Among them, there are 4 ambiguities for which the corresponding S-polynomials are zero anyway, for instance

$$\mathrm{SP}(\mathsf{D}\underline{\Phi}, \underline{\mathsf{E}}\mathsf{I}) = 0 \otimes \textstyle\int - \partial \otimes 0 = 0.$$

The S-polynomials of 48 remaining ambiguities are reduced to zero by applying automatically the implementation of rules from Σ, identities in R and identities in M_{D}, M_{I} and M_Φ. The complete computation is included in the example files of the package. Here we consider a few concrete instances of ambiguities. For example, we use the definition of E in R in the reduction of the following S-Polynomial

$$\begin{aligned}\mathrm{SP}(\mathsf{IR}\underline{\mathsf{D}},\underline{\mathsf{D}}\Phi) &= (f-\textstyle\int\otimes\partial f-(\mathrm{E}f)\mathrm{E})\otimes\varphi-\textstyle\int\otimes f\otimes 0\to_{r_{\mathsf{IR}\Phi}} f\otimes\varphi-(\textstyle\int\partial f)\otimes\varphi-(\mathrm{E}f)\mathrm{E}\otimes\varphi\\ &= f\otimes\varphi-(f-\mathrm{E}f)\otimes\varphi-(\mathrm{E}f)\mathrm{E}\otimes\varphi=\mathrm{E}f\otimes\varphi-(\mathrm{E}f)\mathrm{E}\otimes\varphi\to_{r_{\Phi\Phi}}\mathrm{E}f\otimes\varphi-(\mathrm{E}f)\varphi\to_{r_{\mathsf{K}}} 0.\end{aligned}$$

As another example, we use the definition of the right multiplication in the K-bimodule M_{I} in the following reduction

$$\begin{aligned}\mathrm{SP}(\mathsf{I}\underline{\Phi},\underline{\Phi}\mathsf{R}) &= (\textstyle\int 1\otimes\varphi)\otimes f-\textstyle\int\otimes(\varphi f)\varphi\to_{r_{\mathsf{I}\Phi}}\textstyle\int 1\otimes\varphi\otimes f-\varphi f(\textstyle\int 1\otimes\varphi)\to_{r_{\Phi\mathsf{R}}}\textstyle\int 1\otimes(\varphi f)\varphi-\varphi f(\textstyle\int 1\otimes\varphi)\\ &= (\textstyle\int 1\varphi f)\otimes\varphi-\varphi f(\textstyle\int 1\otimes\varphi)=(\varphi f)\textstyle\int 1\otimes\varphi-\varphi f(\textstyle\int 1\otimes\varphi)=\varphi f(\textstyle\int 1\otimes\varphi)-\varphi f(\textstyle\int 1\otimes\varphi)=0.\end{aligned}$$

There are 41 ambiguities without specialization. The remaining 11 ambiguities consist of 4 overlap ambiguities with specialization and 7 inclusion ambiguities with specialization. For example,

$$\mathrm{SP}(\mathsf{IR}\underline{\Phi},\underline{\mathsf{E}}\mathsf{I})=(\textstyle\int f\otimes\mathrm{E})\otimes\textstyle\int-\textstyle\int\otimes f\otimes 0\to_{r_{\mathsf{EI}}} 0,$$

and

$$\mathrm{SP}(\underline{\mathsf{K}},\mathsf{D}\underline{\mathsf{R}})=\partial\otimes\epsilon\otimes\epsilon-1\otimes\partial\ \to_{r_{\mathsf{K}}}\partial-\partial=0.$$

We emphasize again that the confluence criterion of Theorem 20 directly works with the reduction system Σ, no computations with the refined reduction system Σ_X over X are needed.

5. Integro-differential operators with linear substitutions

In this section, we apply our tensor setting with specialization to extend the ring of integro-differential operators by adding linear substitution operators. An important motivation for studying this ring comes from the work by Quadrat (2015). In this paper, such operators and their commutation rules are used for an algorithmic approach to Artstein's integral transformation of linear differential systems with delayed inputs to linear differential system without delays. IDOs with linear substitutions also address the univariate case in (Rosenkranz et al., 2015), where algebraic aspects of multivariate integration with linear substitutions are studied. Moreover, they provide an algebraic setting for dealing with delay differential equations and the corresponding initial and boundary problems in general.

A delay differential equation is an ordinary differential equation in which the derivative at a certain time depends on the solution at prior times; see, for example, (Hale and Verduyn Lunel, 1993; Smith, 2011). A general first-order constant delay equation has the form

$$y'(x)=f(x,y(x),y(x-b_1),y(x-b_2),\ldots,y(x-b_n))$$

where the time delays b_j for $1\le j\le n$ are positive constants. A homogeneous linear first-order time-delay equation with one constant delay has the form

$$y'(x)=A(x)y(x)+B(x)y(x-b).$$

The chain rule and integration by substitution from calculus describe the interaction of linear substitutions $f(ax-b)$ with differentiation and integration. More formally, let $\sigma_{a,b}$ denote the linear substitution operator mapping a smooth function $f(x)$ to $f(ax-b)$ for a nonzero constant a and an arbitrary constant b. Then

$$\partial_x \sigma_{a,b} f(x) = af'(ax-b) = a\sigma_{a,b}\partial_x f(x)$$

and

$$\int_0^x \sigma_{a,b} f(t)\,dt = \int_0^x f(at-b)\,dt = \frac{1}{a}\int_{-b}^{ax-b} f(t)\,dt = \frac{1}{a}\sigma_{a,b}\int_0^x f(t)\,dt - \frac{1}{a}\mathrm{E}\sigma_{a,b}\int_0^x f(t)\,dt.$$

Following these identities, we want to define an integro-differential ring with linear substitutions. In what follows, $C = K \cap \mathcal{Z}(R)$ denotes the ring of elements of K which commute with all elements of R and C^* denotes its group of units. In order to find a proper algebraic setting, we will add an axiomatization of linear substitution operations to an integro-differential ring.

Definition 28. *Let $(R, \partial, \int)$ be an integro-differential ring with constants K and let*

$$S := \{\sigma_{a,b} \mid a \in C^*,\ b \in C\}$$

where $\sigma_{a,b}\colon R \to R$ are multiplicative K-bimodule homomorphisms on R fixing the constants K such that

$$\sigma_{1,0} f = f, \quad \sigma_{a,b}\sigma_{c,d} f = \sigma_{ac,bc+d} f \tag{16}$$

and

$$\partial\sigma_{a,b} f = a\sigma_{a,b}\partial f \tag{17}$$

for all $a, c \in C^*$, $b, d \in C$ and $f \in R$. Then we call $(R, \partial, \int, S)$ an integro-differential ring with linear substitutions*.*

Remark 29. *The set S along with composition can be considered as a group of K-bimodule homomorphisms on R. The neutral element is $\sigma_{1,0}$ and the inverse for $\sigma_{a,b} \in S$ is given by*

$$\sigma_{a,b}^{-1} = \sigma_{a^{-1},-ba^{-1}}.$$

So the elements in S actually are automorphisms.

As in analysis, integration by substitution is a consequence of the chain rule and the fundamental theorem of calculus.

Lemma 30. *Let $(R, \partial, \int, S)$ be an integro-differential ring with linear substitutions. For all $\sigma_{a,b} \in S$ and $f \in R$,*

$$\int\sigma_{a,b} f = a^{-1}(\mathrm{id} - \mathrm{E})\sigma_{a,b}\int f. \tag{18}$$

Proof. We first apply $\int$ to Eq. (17). So

$$\int\partial\sigma_{a,b} f = \int a\sigma_{a,b}\partial f = a\int\sigma_{a,b}\partial f.$$

By Eq. (9), we substitute $\int\partial\sigma_{a,b} f$ with $(\mathrm{id} - \mathrm{E})\sigma_{a,b} f$ and multiply the resulting equation by a^{-1}. This gives the identity

$$\int\sigma_{a,b}\partial f = a^{-1}(\mathrm{id} - \mathrm{E})\sigma_{a,b} f,$$

which implies Eq. (18) by just replacing f with $\int f$. ☐

In the sequel, we fix an integro-differential ring with linear substitutions $(R, \partial, \int, S)$ with constants K and evaluation $\mathrm{E} = \mathrm{id} - \int\partial$. We consider the modules M_{K}, $M_{\tilde{\mathsf{R}}}$, M_{D}, M_{I}, M_{E}, $M_{\tilde{\Phi}}$, M_{R}, and M_{Φ} which are introduced in Eqs. (10), (12), and (14). In addition, we add the free left K-module

$$M_{\mathsf{G}} := KS.$$

We also view it as a K-bimodule with the right multiplication defined by $c\sigma_{a,b} \cdot d = cd\sigma_{a,b}$ with $c, d \in K$. It has the direct sum decomposition

$$M_{\mathsf{G}} = M_{\mathsf{N}} \oplus M_{\tilde{\mathsf{G}}}$$

such that $M_{\mathsf{N}} := K\sigma_{1,0}$ is the K-bimodule generated by the trivial substitution $\sigma_{1,0} = \mathrm{id}$ and $M_{\tilde{\mathsf{G}}} := K\tilde{S}$ is the K-bimodule generated by all linear substitutions in $\tilde{S} = S \setminus \{\sigma_{1,0}\}$. Therefore we take the alphabets

$$X := \{\mathsf{K}, \tilde{\mathsf{R}}, \mathsf{D}, \mathsf{I}, \mathsf{E}, \tilde{\Phi}, \mathsf{N}, \tilde{\mathsf{G}}\}, \quad Z := X \cup \{\mathsf{R}, \Phi, \mathsf{G}\}. \tag{19}$$

With the K-bimodules

$$M_{\mathsf{R}} = M_{\mathsf{K}} \oplus M_{\tilde{\mathsf{R}}}, \quad M_{\Phi} = M_{\mathsf{E}} \oplus M_{\tilde{\Phi}}, \quad M_{\mathsf{G}} = M_{\mathsf{N}} \oplus M_{\tilde{\mathsf{G}}}, \tag{20}$$

we define

$$M := M_{\mathsf{R}} \oplus M_{\mathsf{D}} \oplus M_{\mathsf{I}} \oplus M_{\Phi} \oplus M_{\mathsf{G}}. \tag{21}$$

Then $(M_z)_{z \in Z}$ is a decomposition with specialization.

In addition to the identities of IDOs that we collected in Section 4, the identities for IDOs with linear substitutions include additional identities involving the substitution operators. Again, we first collect some identities involving substitution operations that hold in R. For all $f, g \in R$, $\varphi \in \Phi$ and $\sigma_{a,b}, \sigma_{c,d} \in S$ we have:

$\sigma_{1,0} g = g$	$\sigma_{a,b}\sigma_{c,d} g = \sigma_{ac,bc+d} g$
$\sigma_{a,b} f g = (\sigma_{a,b} f)(\sigma_{a,b} g)$	$\partial\sigma_{a,b} g = a\sigma_{a,b}\partial g$
$\sigma_{a,b}\varphi g = \varphi g$	$\int f\sigma_{a,b} g = a^{-1}(\mathrm{id} - \mathrm{E})\sigma_{a,b}\int(\sigma_{a,b}^{-1} f) g$

The only identity above that does not follow immediately from Definition 28 is

$$\int f\sigma_{a,b} g = a^{-1}(\mathrm{id} - \mathrm{E})\sigma_{a,b}\int(\sigma_{a,b}^{-1} f) g.$$

It can be verified by replacing f with $(\sigma_{a,b}^{-1} f) g$ in Lemma 30 and then using multiplicativity of $\sigma_{a,b}$. Corresponding reduction rules to these identities in R are listed in Table 2.

In order to obtain our reduction system Σ over the alphabet $\langle Z \rangle$, we consider reduction rules of the Table 1 along with the reduction rules of the Table 2 simultaneously.

Definition 31. *Let $(R, \partial, \int, S)$ be an integro-differential ring with linear substitutions. We call*

$$R\langle \partial, \textstyle\int, \Phi, S \rangle := K\langle M \rangle / J$$

the ring of integro-differential operators with linear substitutions*, where J is the two-sided reduction ideal induced by the reduction system obtained from adjoining Table 2 to Table 1.*

N	$\sigma_{1,0} \mapsto \epsilon$
GR	$\sigma_{a,b} \otimes f \mapsto \sigma_{a,b} f \otimes \sigma_{a,b}$
GΦ	$\sigma_{a,b} \otimes \varphi \mapsto \varphi$
GG	$\sigma_{a,b} \otimes \sigma_{c,d} \mapsto \sigma_{ac,bc+d}$
DG	$\partial \otimes \sigma_{a,b} \mapsto a\sigma_{a,b} \otimes \partial$
IG	$\int \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int$
IRG	$\int \otimes f \otimes \sigma_{a,b} \mapsto a^{-1}(\epsilon - \mathrm{E}) \otimes \sigma_{a,b} \otimes \int \otimes \sigma_{a,b}^{-1} f$

Table 2: New reduction rules for IDOs with linear substitutions

Similar to the previous example, the refined reduction system Σ_X is obtained, according to Eq. (20), by splitting rules whose words contain R, Φ or G into "smaller" rules using $S(\mathsf{R}) = \{\mathsf{K}, \tilde{\mathsf{R}}\}$, $S(\Phi) = \{\mathsf{E}, \tilde{\Phi}\}$ and $S(\mathsf{G}) = \{\mathsf{N}, \tilde{\mathsf{G}}\}$. Following Theorem 20, we determine normal forms of tensors in $R\langle \partial, \int, \Phi, S\rangle$.

Theorem 32. *Let $(R, \partial, \int, S)$ be an integro-differential ring with linear substitutions and let M be as in Eqs. (21) and (20) and let the reduction system Σ be defined by Tables 1 and 2. Then every $t \in K\langle M\rangle$ has a unique normal form given by a sum of pure tensors*

$$f \otimes \varphi \otimes \sigma_{a,b} \otimes \partial^{\otimes j} \qquad \text{or} \qquad f \otimes \varphi \otimes \sigma_{a,b} \otimes \int \otimes g,$$

where $j \in \mathbb{N}_0$, each of $f, g \in M_{\tilde{\mathsf{R}}}$, $\varphi \in \Phi$ and $\sigma_{a,b} \in \tilde{S}$ may be absent, and $\varphi \otimes \sigma_{a,b} \otimes \int$ does not specialize to $\mathrm{E} \otimes \int$. Moreover, with defining the multiplication $s \cdot t := (s \otimes t)\downarrow_\Sigma$ on $K\langle M\rangle_{irr}$

$$R\langle \partial, \int, \Phi, S\rangle \cong K\langle M\rangle_{irr}.$$

Proof. We consider the alphabets X and Z as defined in Eq. (19). Then $(M_z)_{z\in Z}$ is a decomposition with specialization for the module M, see Definition 12. For defining a Noetherian monoid partial order $\leq$ on $\langle Z\rangle$ that is compatible with Σ, it is sufficient to require the order to satisfy

$$\mathsf{DR} > \mathsf{RD},\ \mathsf{IRD} > \mathsf{E},\ \mathsf{ID} > \mathsf{E},\ \mathsf{I} > \tilde{\mathsf{R}},\ \mathsf{GR} > \mathsf{RG},\ \mathsf{DG} > \mathsf{GD},\ \mathsf{IG} > \mathsf{EGI},\ \mathsf{IRG} > \mathsf{EGIR}.$$

For instance, on $\langle Y\rangle$ with $Y = \{\mathsf{R}, \mathsf{D}, \mathsf{I}, \Phi, \mathsf{G}\}$ we first define a monoid order by

$$V \leq W :\Leftrightarrow \tilde{V} \prec \tilde{W} \text{ or } \tilde{V} = \tilde{W} \text{ and } V \preceq W,$$

where $\tilde{V}$ and $\tilde{W}$ are obtained by removing all occurrences of Φ, cf. Remark 4, and $\preceq$ is the degree-lexicographic order with $\mathsf{I} \succ \mathsf{D} \succ \mathsf{G} \succ \Phi \succ \mathsf{R}$ on $\langle Y\rangle$. Then, we extend $\leq$ to a monoid partial order on $\langle Z\rangle$ based on Definition 16 in order to make it consistent with specialization.

Then by the package `TenReS` we verify that all ambiguities of Σ are resolvable, see Section 5.1. Hence by Theorem 20 every element of $K\langle M\rangle$ has a unique normal form and $K\langle M\rangle/I_\Sigma \cong K\langle M\rangle_{\mathrm{irr}}$ as K-rings.

It remains to determine the explicit form of elements in $K\langle M\rangle_{\mathrm{irr}}$. To do so, we determine the set of irreducible words $\langle X\rangle_{\mathrm{irr}}$ in $\langle X\rangle$. Note that $\Sigma_{\mathrm{IDO}} \subset \Sigma$, where Σ_{IDO} is given by Table 1. Hence the irreducible words w.r.t. Σ are among the irreducible words w.r.t. Σ_{IDO}. In Theorem 27, we already determined the irreducible words that do not contain the letters N and $\tilde{\mathsf{G}}$ to be of the form

$$\tilde{\mathsf{R}} V \mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}\tilde{\Phi}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\Phi}$, and $V \in S(\Phi)$ may be absent.

The irreducible words containing only N and $\tilde{\mathsf{G}}$ are exactly ϵ and $\tilde{\mathsf{G}}$, since they have to avoid the subwords N and $S(\mathsf{GG}) = \{\mathsf{NN}, \mathsf{N}\tilde{\mathsf{G}}, \tilde{\mathsf{G}}\mathsf{N}, \tilde{\mathsf{G}}\tilde{\mathsf{G}}\}$. The irreducible words in $\langle X\rangle_{\mathrm{irr}}$ also have to avoid subwords from $S(\mathsf{GR})$, $S(\mathsf{G}\Phi)$, $S(\mathsf{DG})$, $S(\mathsf{IG})$, and $S(\mathsf{IRG})$. Hence they are of the form

$$\tilde{\mathsf{R}}V\tilde{\mathsf{G}}\mathsf{D}^j \quad \text{or} \quad \tilde{\mathsf{R}}V\tilde{\mathsf{G}}\mathsf{I}\tilde{\mathsf{R}},$$

where $j \in \mathbb{N}_0$ and each of $\tilde{\mathsf{R}}$, $\tilde{\mathsf{G}}$, and $V \in S(\Phi)$ may be absent and $V\tilde{\mathsf{G}}\mathsf{I}$ does not specialize to EI. The normal forms follow from Eq. (3). □

5.1. Computational aspects

In the following, we shortly mention some computational details of the tensor setting with specialization for integro-differential operators with linear substitutions. Applying `TenReS` to the reduction system Σ given by Tables 1 and 2, in total 87 ambiguities and corresponding S-polynomials are generated. All ambiguities are resolvable and the automatic verification can be found in the example files of the package. There are 66 ambiguities without specialization. For instance,

$$\mathrm{SP}(\mathsf{IR}\underline{\Phi}, \underline{\mathsf{E}}\mathsf{I}) = (\textstyle\int f \otimes \mathrm{E}) \otimes \int - \int \otimes f \otimes 0 \to_{r_{\mathsf{EI}}} \int f \otimes 0 = 0,$$

and

$$\begin{aligned}\mathrm{SP}(\mathsf{I}\underline{\mathsf{G}}, \underline{\mathsf{G}}\mathsf{R}) &= (a^{-1}\sigma_{a,b} \otimes \textstyle\int - a^{-1}\mathrm{E} \otimes \sigma_{a,b} \otimes \int) \otimes f - \int \otimes (\sigma_{a,b} f \otimes \sigma_{a,b}) \\ &= a^{-1}\sigma_{a,b} \otimes \textstyle\int \otimes f - a^{-1}\mathrm{E} \otimes \sigma_{a,b} \otimes \int \otimes f - \int \otimes \sigma_{a,b} f \otimes \sigma_{a,b} \to_{r_{\mathsf{IRG}}} 0.\end{aligned}$$

The remaining 21 ambiguities consist of 5 overlap ambiguities with specialization and 16 inclusion ambiguities with specialization. They all involve the following three reduction rules (over X)

$$(\mathsf{K}, 1 \mapsto \epsilon), \ (\mathsf{EI}, \mathrm{E} \otimes \textstyle\int \mapsto 0), \ (\mathsf{N}, \sigma_{1,0} \mapsto \epsilon)$$

and their S-polynomials can be reduced to zero. For example,

$$\mathrm{SP}(\underline{\mathsf{N}}, \mathsf{D}\underline{\mathsf{G}}) = \partial \otimes \epsilon - \sigma_{1,0} \otimes \partial \to_{r_{\mathsf{N}}} \partial - \partial = 0,$$

and

$$\mathrm{SP}(\underline{\mathsf{N}}, \mathsf{IR}\underline{\mathsf{G}}) = \textstyle\int \otimes f - (\epsilon - \mathrm{E}) \otimes \sigma_{1,0} \otimes \int \otimes f \to_{r_{\mathsf{N}}} \mathrm{E} \otimes \sigma_{1,0} \otimes \int \otimes f \to_{r_{\mathsf{N}}} \mathrm{E} \otimes \int \otimes f \to_{r_{\mathsf{EI}}} 0.$$

6. Completion of tensor reduction systems

For computing in the quotient ring $K\langle M\rangle/I_\Sigma$, we would like to compute with a system of representatives. By Theorem 6, the irreducible tensors $K\langle M\rangle_{\mathrm{irr}}$ are such a system if the tensor reduction system is confluent. If the reduction system is not confluent, we want to construct a confluent one that generates the same reduction ideal of Eq. (4).

Like Buchberger's algorithm (Buchberger, 1965) and Knuth-Bendix completion (Knuth and Bendix, 1970), the completion process involves adding new rules corresponding to non-resolvable ambiguities (S-polynomials resp. critical pairs); see also (Buchberger, 1987). Obstructions for general algorithms are inherited from the noncommutative polynomial algebra case (Mora, 1994), e.g., deciding existence of finite Gröbner bases and the undecidability of the word problem. Unlike noncommutative Gröbner basis computations and Knuth-Bendix completion, where we have semi-decision algorithms, the method we describe for completing tensor reduction systems involves also non-algorithmic steps. One of the main difficulties is to define a new reduction homomorphism based on the S-polynomials of a non-resolvable ambiguity. Since for verification of confluence, a compatible semigroup partial order is sufficient, one can also start the completion process with a compatible semigroup partial order instead of a total one. Extending this order in a compatible way may not always be possible.

Before we discuss aspects of the completion process for tensor reduction systems more formally below, we have a look at a few concrete non-resolvable ambiguities. We start with the following reduction rules for integro-differential operators that follow immediately from the definition:

$$\begin{aligned}\Sigma_0 = \{&(\mathsf{K}, 1 \mapsto \epsilon), (\mathsf{RR}, f \otimes g \mapsto fg), (\Phi\mathsf{R}, \varphi \otimes f \mapsto (\varphi f)\varphi), (\Phi\Phi, \psi \otimes \varphi \mapsto \varphi)\\ &(\mathsf{DR}, \partial \otimes f \mapsto f \otimes \partial + \partial f), (\mathsf{D}\Phi, \partial \otimes \varphi \mapsto 0), (\mathsf{DI}, \partial \otimes \textstyle\int \mapsto \epsilon), (\mathsf{ID}, \textstyle\int \otimes \partial \mapsto \epsilon - \mathrm{E})\}\end{aligned}$$

On $\langle Z\rangle$ we define a partial order $\leq$ based on the length of words with the additional property that $\mathsf{DR} > \mathsf{RD}$. Generating from it the minimal partial order that is consistent with specialization means that we also have to define $\mathsf{DK} > \mathsf{KD}$, $\mathsf{DK} > \tilde{\mathsf{R}}\mathsf{D}$, $\mathsf{D}\tilde{\mathsf{R}} > \mathsf{KD}$, and $\mathsf{D}\tilde{\mathsf{R}} > \tilde{\mathsf{R}}\mathsf{D}$. In order to obtain the minimal semigroup partial order generated by that, we not only have to define $A\mathsf{DR}B > A\mathsf{RD}B$ for any $A, B \in \langle Z\rangle$, but also for all $k \geq 2$ the general condition $A_1\mathsf{DR}A_2\mathsf{DR}\ldots\mathsf{DR}A_k > A_1\mathsf{RD}A_2\mathsf{RD}\ldots\mathsf{RD}A_k$ for all $A_i \in \langle Z\rangle$ along with all 2^{2k-2} specializations $\mathsf{R} \in \{\mathsf{K}, \tilde{\mathsf{R}}\}$. The resulting semigroup partial order $\leq$ is compatible with Σ_0.

The rules r_{DI} and r_{ID} have two overlap ambiguities with each other, one is resolvable and one is not. The latter has S-polynomial

$$\mathrm{SP}(\mathsf{I}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{I}) = (\epsilon - \mathrm{E}) \otimes \textstyle\int - \int \otimes\, \epsilon = -\mathrm{E} \otimes \int.$$

This trivially gives rise to the new rule

$$(\mathsf{EI}, \mathrm{E} \otimes \textstyle\int \mapsto 0).$$

The rules r_{ID} and r_{DR} have a non-resolvable overlap ambiguity with S-polynomials

$$\mathrm{SP}(\mathsf{I}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{R}) = (\epsilon - \mathrm{E}) \otimes f - \textstyle\int \otimes (f \otimes \partial + \partial f) \to_{r_{\Phi\mathsf{R}}} f - (\mathrm{E}f)\mathrm{E} - \int \otimes f \otimes \partial - \int \otimes \partial f.$$

While we could reduce further, by using r_{K} for example, we will not be able to reduce to zero for all $f \in R$. Based on the expression above, however, we can introduce a new rule

$$(\mathsf{IRD}, \textstyle\int \otimes f \otimes \partial \mapsto f - (\mathrm{E}f)\mathrm{E} - \int \otimes \partial f)$$

that allows to reduce all the S-polynomials of the overlap ambiguity of r_{ID} and r_{DR} to zero. This rule gives rise to a non-resolvable overlap ambiguity with r_{DI} among others. The corresponding S-polynomials can be reduced to

$$\mathrm{SP}(\mathsf{IR}\underline{\mathsf{D}}, \underline{\mathsf{D}}\mathsf{I}) = (f - (\mathrm{E}f)\mathrm{E} - \textstyle\int \otimes \partial f) \otimes \int - \int \otimes f \otimes \epsilon \to_{r_{\mathsf{EI}}} f \otimes \int - \int \otimes \partial f \otimes \int - \int \otimes f.$$

We would like to have a new reduction homomorphism on M_{IRI} that reduces $\int \otimes \partial f \otimes \int$ to $f \otimes \int - \int \otimes f$. Replacing f by $\int f$, we arrive at the definition

$$(\mathsf{IRI}, \textstyle\int \otimes f \otimes \int \mapsto \int f \otimes \int - \int \otimes \int f).$$

Finally, we consider the inclusion ambiguity (with specialization) of this new rule with r_{K}, which has irreducible S-polynomials

$$\mathrm{SP}(\underline{\mathsf{K}}, \mathsf{I}\underline{\mathsf{R}}\mathsf{I}) = \textstyle\int \otimes \epsilon \otimes \int - (\int 1 \otimes \int - \int \otimes \int 1) = \int \otimes \int - \int 1 \otimes \int + \int \otimes \int 1.$$

At this point, the leading term is not determined by our partial order above. We decide to have the new rule

$$(\mathsf{II}, \textstyle\int \otimes \int \mapsto \int 1 \otimes \int - \int \otimes \int 1)$$

and extend $\leq$ accordingly to have it compatible with the new rule. Similarly, the overlap ambiguity of r_{IRD} and $r_{\mathsf{D}\Phi}$ gives rise to the rule $r_{\mathsf{IR}\Phi}$, which in turn has an inclusion ambiguity with r_{K} giving rise to $r_{\mathsf{I}\Phi}$. Thereby we obtain the reduction system given in Table 1. The whole completion process for both Table 1 and 2 can be found in the example files of the `TenReS` package.

In the following, we discuss these issues more formally. For a better overview we consider three different tensor settings starting with the special case of a total order for Bergman's original setting, which already covers most issues that may arise during the completion process. Incrementally we discuss the problems arising in more general situations below. After that we illustrate some of those problems by revisiting the computations done for Σ_0 above.

Bergman's tensor setting with a total order. Based on the direct sum decomposition (2) into word modules M_W we define the support of a tensor $t \in K\langle M\rangle$ by

$$\mathrm{supp}(t) := \{W \in \langle X\rangle \mid \pi_W(t) \neq 0\}, \tag{22}$$

where π_W denotes the canonical projection onto the direct summand M_W of $K\langle M\rangle$. For each non-resolvable ambiguity, the following points have to be considered.

- We apply a sequence of reductions uniformly to the bimodule generated by S-polynomials to obtain a new bimodule S_{red} generated by reduced S-polynomials. It is not necessary to have $S_{\mathrm{red}} \subseteq K\langle M\rangle_{\mathrm{irr}}$.

- Among all possible supports $\mathrm{supp}(S_{\mathrm{red}}) = \{\mathrm{supp}(t) \mid t \in S_{\mathrm{red}}\}$ we pick some nonempty support $S \in \mathrm{supp}(S_{\mathrm{red}})$, e.g. a maximal element of $\mathrm{supp}(S_{\mathrm{red}})$ w.r.t. $\subseteq$. The total order $\leq$ determines a maximal element $W \in S$, determining the "leading term" of the corresponding tensors in S_{red}.

- A new homomorphism h should be defined on M_W that allows to reduce $t \in S_{\mathrm{red}} \cap M_S$ with $\pi_W(t) \neq 0$ to zero, where M_S is defined in Eq. (7) as the sum of all modules M_V with $V \in S$. In addition, h has to be defined such that $\mathrm{id} - h$ maps M_W into I_Σ, i.e. the reduction ideal stays the same $I_\Sigma = I_{\Sigma\cup\{(W,h)\}}$. To discuss this we consider the subbimodule N of S_{red} generated by all $t \in S_{\mathrm{red}} \cap M_S$ with $\pi_W(t) \neq 0$. This bimodule N is contained in $S_{\mathrm{red}} \cap M_S$, but they are not necessarily equal. If $\pi_W : N \to M_W$ is bijective, then it is natural to define h via $h(\pi_W(t)) = \pi_W(t) - t$. Such a homomorphism may not exist for two reasons.
  - If there are distinct $t_1, t_2 \in N$ with $\pi_W(t_1) = \pi_W(t_2)$, then we cannot have $h(\pi_W(t_1)) = \pi_W(t_1) - t_1$ and $h(\pi_W(t_2)) = \pi_W(t_2) - t_2$ at the same time. In that case, we need to be content with some homomorphism $g\colon M_W \to N$ such that $h(t) = t - g(t)$ and $\pi_W \circ g = \mathrm{id}$. As a consequence $t_1 - t_2 \in S_{\mathrm{red}}$ may still not be reducible to zero with $\Sigma \cup \{(W, h)\}$.

- If there is a $t \in M_W$ that is not in $\pi_W(N)$, then it is not clear how to define h on all of M_W so that $\leq$ is still compatible with $\Sigma \cup \{(W,h)\}$, in particular $\pi_W(h(M_W)) = \{0\}$, without violating $I_\Sigma = I_{\Sigma\cup\{(W,h)\}}$. Instead of N, considering the larger bimodule $\tilde{N} := S_{\text{red}} \cap \bigoplus_{V\leq W} M_V$ might satisfy $\pi_W(\tilde{N}) = M_W$. If not, it may be necessary to split some modules M_x, $x \in X$, further in order to turn $\pi_W(N)$ or $\pi_W(\tilde{N})$ into a word module M_V over some new alphabet X.

- Finally, we include the new reduction rule (W,h) into Σ. If $\operatorname{supp}(S_{\text{red}}) \neq \{\emptyset, S\}$, then it may happen that the new rule is not sufficient to reduce all elements of S_{red} to zero. In that case, we need to check resolvability of the current ambiguity again.

Bergman's tensor setting with a partial order. The only new issue that appears with a partial order $\leq$ on words that is not a total order, is that the "leading term" of tensors in S_{red} may not be determined by the order. If the selected support $S \in \operatorname{supp}(S_{\text{red}})$ does not have a greatest element already, we need to choose a word $W \in S$ so that we can extend the semigroup partial order in a compatible way, i.e. W becomes the greatest element of S. Such a choice is not guaranteed to exist.

Tensor setting with specialization. The first thing to note is that we cannot have a total order on $\langle Z\rangle$ that is consistent with specialization (as long as $Z \neq X$). All points of the above discussion apply also to decompositions of M with specialization except that $\operatorname{supp}(S_{\text{red}})$ should now be defined as $\operatorname{supp}(S_{\text{red}}) = \bigcup\{\operatorname{supp}(t) \mid t \in S_{\text{red}}\}$ where for a particular tensor t we now define $\operatorname{supp}(t)$ as the set of "all possible supports"

$$\operatorname{supp}(t) := \{S \subseteq \langle Z\rangle \mid t \in M_S, \forall W, \tilde{W} \in S : \pi_W(t) \neq 0 \wedge S(W) \cap S(\tilde{W}) = \emptyset\}.$$

Other than that, no new fundamental obstacles arise in this setting. We just add a few remarks.

It can be advantageous to pick supports with words associated to bigger modules in order to construct reduction homomorphisms h with larger domains. Also, it can be useful to introduce additional letters to the alphabet Z in order to collect some of the bimodules appearing in the process. We illustrate some of the points discussed formally by revisiting the concrete ambiguities treated above.

The first and simplest case above was the overlap ambiguity of r_{ID} and r_{DI} with words I, D, and I. All S-polynomials are irreducible w.r.t. Σ_0 and the bimodule generated by them has $\operatorname{supp}(S_{\text{red}}) = \{\emptyset, \{\mathsf{EI}\}, \{\Phi\mathsf{I}\}\}$. Picking $S = \{\Phi\mathsf{I}\}$ and $W = \Phi\mathsf{I}$ would lead to $\pi_W|_{S_{\text{red}}}$ not being surjective onto M_W. So the choice $S = \{\mathsf{EI}\}$ and $W = \mathsf{EI}$ is preferable and we can define the homomorphism $h\colon M_W \to K\langle M\rangle$ of r_{EI} by $h(\pi_W(t)) = \pi_W(t) - t = 0$ in this case.

For the overlap ambiguity of r_{ID} and r_{DR} we applied the reduction $h_{\epsilon,r_{\Phi\mathsf{R}},\epsilon}$ to all S-polynomials. The bimodule generated by them now has

$$\begin{aligned}\operatorname{supp}(S_{\text{red}}) = \{\emptyset, &\{\mathsf{K},\mathsf{E},\mathsf{IKD}\}, \{\tilde{\mathsf{R}},\mathsf{I\tilde{R}D},\mathsf{IK}\}, \{\tilde{\mathsf{R}},\mathsf{I\tilde{R}D},\mathsf{I\tilde{R}}\}, \{\tilde{\mathsf{R}},\mathsf{I\tilde{R}D},\mathsf{IR}\},\\ &\{\mathsf{R},\mathsf{E},\mathsf{IRD},\mathsf{IK}\}, \{\mathsf{R},\mathsf{E},\mathsf{IRD},\mathsf{I\tilde{R}}\}, \{\mathsf{R},\mathsf{E},\mathsf{IRD},\mathsf{IR}\}, \dots\}.\end{aligned}$$

The chosen partial order $\leq$ determines a greatest element of most $S \in \operatorname{supp}(S_{\text{red}})$. Picking $S \in \operatorname{supp}(S_{\text{red}})$ with the largest M_S gives $S = \{\mathsf{R},\Phi,\mathsf{IRD},\mathsf{IR}\}$ and $W = \mathsf{IRD}$ so that $\pi_W\colon S_{\text{red}} \to M_W$ is bijective. This allows for a straightforward definition of r_{IRD} again.

A more interesting case is the overlap ambiguity of r_{IRD} and r_{DR} with words IR, D, R. After applying the reduction $h_{\epsilon,r_{\mathsf{EI}},\epsilon}$ to all S-polynomials the bimodule generated by them has

$$\operatorname{supp}(S_{\text{red}}) = \{\emptyset, \{\mathsf{KI}, \mathsf{IK}\}, \{\tilde{\mathsf{R}}\mathsf{I}, \mathsf{IKI}, \mathsf{I}\tilde{\mathsf{R}}\}, \{\mathsf{RI}, \mathsf{IKI}, \mathsf{IR}\}, \{\tilde{\mathsf{R}}\mathsf{I}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{I}, \mathsf{I}\tilde{\mathsf{R}}\}, \{\mathsf{RI}, \mathsf{I}\tilde{\mathsf{R}}\mathsf{I}, \mathsf{IR}\}, \{\tilde{\mathsf{R}}\mathsf{I}, \mathsf{IRI}, \mathsf{I}\tilde{\mathsf{R}}\},$$
$$\{\mathsf{RI}, \mathsf{IRI}, \mathsf{IR}\}, \{\mathsf{RI}, \mathsf{IR}\}, \{\mathsf{KI}, \tilde{\mathsf{R}}\mathsf{I}, \mathsf{IKI}, \mathsf{IK}, \mathsf{I}\tilde{\mathsf{R}}\}, \dots\}.$$

Picking again one $S \in \operatorname{supp}(S_{\text{red}})$ with the largest M_S gives $S = \{\mathsf{RI}, \mathsf{IRI}, \mathsf{IR}\}$ and $W = \mathsf{IRI}$. Now $N = S_{\text{red}}$ and $\pi_W \colon S_{\text{red}} \to M_W$ is surjective but not injective. We choose the bimodule homomorphism $g \colon M_W \to S_{\text{red}}$ to be defined by $g(\int \otimes f \otimes \int) = \int \otimes f \otimes \int + \int \otimes \int f - \int f \otimes \int$. It satisfies $\pi_W \circ g = \text{id}$ and we define the homomorphism $h \colon M_W \to K\langle M \rangle$ of r_{IRI} by $h := \text{id} - g$. While $h_{\epsilon,r_{\mathsf{IRI}},\epsilon}$ does not map S_{red} to $\{0\}$, the image contains only elements of the form $c \otimes \int - \int \otimes c$ with $c \in K$, which are reducible to zero by Σ_0.

The last ambiguity dealt with explicitly above is the inclusion ambiguity (with specialization) of r_{IRI} and r_{K}. Its S-polynomials are irreducible and we have

$$\operatorname{supp}(S_{\text{red}}) = \{\emptyset, \{\mathsf{II}, \tilde{\mathsf{R}}\mathsf{I}, \mathsf{I}\tilde{\mathsf{R}}\}, \{\mathsf{II}, \tilde{\mathsf{R}}\mathsf{I}, \mathsf{IR}\}, \{\mathsf{II}, \mathsf{RI}, \mathsf{I}\tilde{\mathsf{R}}\}, \{\mathsf{II}, \mathsf{RI}, \mathsf{IR}\}\}.$$

As pointed out already, the partial order does not determine a greatest element within any of the possible supports. Since $\pi_W \colon S_{\text{red}} \to M_W$ is not surjective except for $W = \mathsf{II}$, we would have to split $M_{\tilde{\mathsf{R}}}$ further in order to define a new reduction rule on $\pi_W(S_{\text{red}})$ in all other cases. So we choose $W = \mathsf{II}$ and extend the semigroup partial order such that $\mathsf{II} > \tilde{\mathsf{R}}\mathsf{I}$ and $\mathsf{II} > \mathsf{I}\tilde{\mathsf{R}}$.

7. Concluding remarks

A ring of operators may not be finitely presented by generators and relations, it may not even be finitely generated. The tensor setting nonetheless often allows to have a finite decomposition of the module M of basic operators together with a finite reduction system. Reduction rules need to be defined by homomorphisms due to non-uniqueness of the representation of tensors. In addition, homomorphisms collect families of relations into one reduction rule. If a reduction system is confluent, the normal forms are unique as tensors while tensors themselves do not have unique representations in terms of pure tensors. Both the theoretical concepts and the concrete formulae for the reduction systems in the examples presented essentially are the same when working in the tensor algebra or in the tensor ring.

In comparison to Bergman's tensor setting, our tensor setting with specialization allows more flexibility in defining a reduction system for a given ring of operators. This is achieved by relaxing the restriction that the submodules of M that are used for defining the reduction homomorphisms have to form a direct sum. As a consequence, reduction systems can be smaller and reduction is more efficient by avoiding unnecessary splitting.

Already when we compare quotients of the tensor algebra with quotients of the free algebra we note some important differences. All computations in quotients of the free algebra happen on two levels: polynomial arithmetic in the free algebra and polynomial reduction modulo the ideal. Computations in the K-algebra $K\langle M \rangle$ actually take place on three levels. The additional level are computations in the module M and its submodules M_z. Analogous to the free algebra there are computations in $K\langle M \rangle$ coming from the properties of the tensor product and the reduction system that acts by applying the reduction homomorphisms.

Depending on the choice of the module M and its decomposition, certain identities of operators either are dealt with by the reduction system or only within the module M. One extreme case occurs when M already is the whole K-ring of operators. Then the reduction system only consists of the rules $1 \mapsto \epsilon$ and $m_1 \otimes m_2 \mapsto m_1m_2$ which do not expose any structure of the ring of operators. Another extreme case occurs when M is some module that generates the ring of operators and all M_z are cyclic. Then the reduction system has to encode all identities among those generators, which makes it harder to have a finite reduction system. For instance, any confluent reduction system for IDOs with polynomial coefficients $K[x]\langle\partial, \int, \mathrm{E}\rangle$, $\mathbb{Q} \subseteq K$, is infinite if M is just generated by x, ∂, $\int$, and E. In between those two extreme cases there is the opportunity to encode only part of the identities by the reduction system and "hide" the remaining ones inside the modules M_z. For instance, following the construction of $K[x]\langle\partial, \int, \mathrm{E}\rangle$, $\mathbb{Q} \subseteq K$, given in Section 4 the module M consists of $K[x]$ and the modules generated by ∂, $\int$, and E and the confluent reduction system given in Table 1 with $R = K[x]$ is finite. Finiteness of this reduction system can be understood by recalling that reduction rules can collect many identities of the same form into one reduction homomorphism.

In principle, if M is a free module, one could reformulate each reduction rule in terms of reduction rules on individual basis elements and work in the free algebra without making use of tensors. Consequently, computations with the reduction system would then have to use basis expansion in each step. In the tensor setting, however, we do not need to fix a basis of the module M. It is enough to work with the decomposition into modules M_z, which also enables working with non-free modules. This even allows to consider arbitrary modules M that are not concrete but carry a certain algebraic structure. For example, the reduction systems and the computations for checking their confluence in Sections 4 and 5 do not rely on a concrete integro-differential ring R.

Based on the normal forms, a confluent reduction system for a ring of operators enables to automatize many computations and proofs involving these operators. The confluent reduction systems given for IDOs and IDOs with linear substitutions can be used e.g. to prove the Taylor formula, to compute Green's operators of linear ordinary boundary problems, or to support computations in Artstein's reduction of linear time-delay systems. Since K is neither required to be a field nor commutative, we can directly consider operators with matrix coefficients to model systems. Elements in R can even model matrices of generic size. The tensor setting can also be used to model other rings of operators. For example, we already have results for IDOs with more general types of functionals or a discrete analog of IDOs.

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