Max-Plus Linear Algebra in Maple and Generalized Solutions for First-Order Ordinary BVPs via Max-Plus Interpolation

Georg Regensburger*

Abstract

If we consider the real numbers extended by minus infinity with the operations maximum and addition, we obtain the max-algebra or the max-plus semiring. The analog of linear algebra for these operations extended to matrices and vectors has been widely studied.

We outline some facts on semirings and max-plus linear algebra, in particular, the solution of max-plus linear systems. As an application, we discuss how to compute symbolically generalized solutions for nonlinear first-order ordinary boundary value problems (BVPs) by solving a corresponding max-plus interpolation problem. Finally, we present the Maple package `MaxLinearAlgebra` and illustrate the implementation and our application with some examples.

1 Semirings and Idempotent Mathematics

The *max-algebra* or *max-plus semiring* (also known as the schedule algebra) $\mathbb{R}_{\max}$ is the set $\mathbb{R} \cup \{-\infty\}$ with the operations

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \odot b = a + b.$$

So for example, $2 \oplus 3 = 3$ and $2 \odot 3 = 5$. Moreover, we have $a \oplus -\infty = a$ and $a \odot 0 = a$ so that $-\infty$ and 0 are respectively the neutral element for the addition and for the multiplication. Hence $\mathbb{R}_{\max}$ is indeed a *semiring*, a ring "without minus", or, more precisely, a triple $(S, \oplus, \odot)$ such that $(S, \oplus)$ is a commutative additive monoid with neutral element $\mathbf{0}$, $(S, \odot)$ is a multiplicative monoid with neutral element $\mathbf{1}$, we have distributivity from both sides, and $\mathbf{0} \odot a = a \odot \mathbf{0} = \mathbf{0}$.

Other examples of semirings are the natural numbers $\mathbb{N}$, the dual $\mathbb{R}_{\min}$ of $\mathbb{R}_{\max}$ (the set $\mathbb{R} \cup \{\infty\}$ and min instead of max), the ideals of a commutative ring with sum and intersection of ideals as operations or the square matrices over a semiring; see [Gol99] for the theory of semirings in general and applications.

The semirings $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are *semifields* with $a^{(-1)} = -a$. Moreover, they are *idempotent* semifields, that is, $a \oplus a = a$. Note that nontrivial rings cannot be idempotent since then we would have $1 + 1 = 1$ and so by subtracting one also $1 = 0$. Idempotent semirings are actually "as far away as possible" from being a ring because in such semirings $a \oplus b = \mathbf{0} \Rightarrow a = b = \mathbf{0}$. Hence zero is the only element with an additive inverse.

There is a *standard partial order* on idempotent semirings defined by $a \preceq b$ if $a \oplus b = b$. For $\mathbb{R}_{\max}$ this is the usual order on $\mathbb{R}$. Due to this order, the theory of idempotent semirings and modules is closely related to lattice theory. Moreover, it is a crucial ingredient for the development of *idempotent analysis* [KM97], which studies functions with values in an idempotent semiring. The idempotent analog of algebraic geometry over $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$ respectively is known as *tropical algebraic geometry* [RGST05]. For a recent survey on *idempotent mathematics* and an extensive bibliography we refer to [Lit05].

*Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences. E-mail: georg.regensburger@oeaw.ac.at.
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2 Max-Plus Linear Algebra

The analog of linear algebra for matrices over idempotent semirings and in particular for the max-algebra has been widely studied starting from the classical paper [Kle56]. The first comprehensive monograph on this topic is [CG79]. See for example the survey [GP97] for more references, historical remarks, and some typical applications of max-plus linear algebra ranging from language theory to optimization and control theory.

From now we consider only the max-algebra $\mathbb{R}_{\max}$, although the results remain valid for $\mathbb{R}_{\min}$ after the appropriate changes (for example, replacing $\leq$ with $\geq$ or $-\infty$ with ∞). Moreover, most of the results can be generalized to linearly ordered commutative groups with addition defined by the maximum, see for example [But94].

For matrices with entries in $\mathbb{R}_{\max}$ and compatible sizes we define

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \quad \text{and} \quad (A \odot B)_{ij} = \bigoplus_k A_{ik} \odot B_{kj} = \max_k (A_{ik} + B_{kj}).$$

Like in Linear Algebra matrices represent max-plus linear operators over max-plus semimodules and the matrix operartions correspond to the addition and composition of such operators.

The *identity matrix* is

$$I = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & -\infty & \dots & -\infty \\ -\infty & 0 & \dots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ -\infty & -\infty & \dots & 0 \end{pmatrix}.$$

More generally, we denote *diagonal matrices* with $\mathbf{0} = -\infty$ outside the diagonal by $\operatorname{diag}(a_1, \dots, a_n)$. A *permutation matrix* is a matrix obtained by permuting the rows and/or the columns of the identity matrix, and a *generalized permutation matrix* is the product of a diagonal matrix and a permutation matrix. It can be shown [CG79, GP97] that the only invertible matrices in the max-algebra are generalized permutation matrices. So in particular a matrix $A \in \mathbb{R}^{n \times n}$ is not invertible in $\mathbb{R}_{\max}$.

Many basic problems in max-plus linear algebra such as systems of linear equations, eigenvalue problems, linear independence and dimension are closely related to combinatorial problems and hence also the corresponding solution algorithms, see [But03]. For the application described in the next section we are interested in particular in solving linear systems over $\mathbb{R}_{\max}$, see Section 4.

3 Generalized Solutions for BVPs and Max-Plus Interpolation

We consider *boundary value problems* (BVPs) for implicit first-order nonlinear ordinary differential equations of the form

$$f(x, y'(x)) = 0, \tag{1}$$

which are known as (stationary) Hamilton-Jacobi equations. As a simple example, take

$$(y'(x))^2 = 1 \quad \text{with} \quad y(-1) = y(1) = 0. \tag{2}$$

Such BVPs usually do not have classical C^1 solutions, one has to define a suitable solution concept to ensure existence and uniqueness of solutions; see [MS92, KM97] for *generalized solutions* in the context idempotent analysis and the relation to *viscosity solutions* as in [CIL92] and for ordinary differential equations in [Li01].

We want to compute symbolically generalized solutions for BVPs assuming that we have a symbolic representation of some or all solutions for the differential equation. The approach is based on Maslov's *idempotent superposition principle*, which in our setting amounts to the following observation.

Suppose we are given two classical C^1 solutions $y_1(x), y_2(x)$ of (1). Then the *max-plus linear combination*

$$y(x) = \max(a_1 + y_1(x), a_2 + y_2(x))$$

for two constants $a_1, a_2 \in \mathbb{R}$ is a again a (generalized) solution, possibly nondifferentiable at some points.

So if we want to solve a BVP given by Equation (1) and two boundary conditions $y(x_1) = b_1$ and $y(x_2) = b_2$ with x_1, x_2 and b_1, b_2 in $\mathbb{R}$, we have to solve the system

$$\max(a_1 + y_1(x_1), a_2 + y_2(x_1)) = b_1$$
$$\max(a_1 + y_1(x_2), a_2 + y_2(x_2)) = b_2.$$

of max-plus linear equations.

More generally, we arrive at the following *max-plus interpolation problem*: *Given m points $x_1, \ldots, x_m$ with the corresponding values $b_1, \ldots, b_m$ in $\mathbb{R}$ and n functions $y_1(x), \ldots, y_n(x)$. Find a (or all) max-plus linear combinations $y(x)$ of $y_1(x), \ldots, y_n(x)$ such that $y(x_i) = b_i$.*

To solve this interpolation problem, we have to find a (or all) solutions of the max-plus linear system $A \odot x = b$ with the *interpolation matrix* $A_{ij} = (y_j(x_i))$ and $b = (b_1, \ldots, b_m)^T$.

4 Max-Plus Linear Systems

In this section, we outline how we can compute the solution set

$$S(A, b) = \{x \in \mathbb{R}^n \mid A \odot x = b\}$$

of a max-plus linear system for given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The method is known since the 1970s. Our presentation and notation is based on [But03], see also there for further details and references.

Note first that by multiplying the linear system $A \odot x = b$ with the invertible diagonal matrix $D = \mathrm{diag}(b_1^{-1}, \ldots, b_m^{-1}) = \mathrm{diag}(-b_1, \ldots, -b_m)$, we obtain an equivalent *normalized system* $D \odot A \odot x = D \odot b = 0$ (but not a homogenous system in the usual sense since $0 = \mathbf{1}$ in $\mathbb{R}_{\max}$).

So we can assume that we have have to solve a normalized system $A \odot x = 0$, which is in conventional notation the nonlinear system

$$\max_j (a_{ij} + x_j) = 0,$$

for $i = 1, \ldots, m$. We see immediately that if x is a solution, then

$$x_j \leq \min_i -a_{ij} = -\max_i a_{ij}$$

for $j = 1, \ldots, n$. Writing $\bar{x}_j = -\max_i a_{ij}$ for the negative of the jth column maximum, this gives in vector notation $x \leq \bar{x}$.

On the other hand, for x being a solution, we must also have in each row at least one column maximum that is attained by x_j. More precisely, let

$$M_j = \{k \mid a_{kj} = \max_i a_{ij}\}.$$

Then $x \in S(A) = S(A, 0)$ iff

$$x \leq \bar{x} \quad \text{and} \quad \bigcup_{j \in N_x} M_j = \{1, \ldots m\},$$

where $N_x = \{j \mid x_j = \bar{x}_j\}$. Hence $A \odot x = 0$ has a solution iff the *principal solution* $\bar{x}$ solves the system iff $\bigcup_j M_j = \{1, \ldots m\}$.

Since the principal solution can be computed in $\mathcal{O}(mn)$ operations, we can decide the *solvability* of a max-plus linear system with this complexity. With the above characterization of solutions one also sees that for deciding if the principal solution is the *unique* solution, we have to check that $\bar{x}$ is a solution and $\bigcup_{j \in N} M_j \neq \{1, \ldots m\}$ for any proper subset $N \subset \{1, \ldots n\}$. This amounts to a *minimal set covering problem*, which is well known to be NP-complete. For the max-plus interpolation problem this means that deciding if there exists a solution and computing it is fast but deciding uniqueness for larger problems is difficult.

Like in Linear Algebra the *number of solutions* $|S(A,b)|$ for a linear system is either 0, 1 or ∞. By contrast, even if a system $A \odot x = b$ has a unique solution for some right hand side b, one can always find a b such that there are respectively no and infinitely many solutions. More precisely,

$$T(A) = \{|S(A,b)| \mid b \in \mathbb{R}^m\} = \{0, 1, \infty\}.$$

Furthermore, the only other possible case is $T(A) = \{0, \infty\}$. For the max-plus interpolation problem this implies in particular that the solvability depends on b and there are always values b such that it is solvable.

Finally, we want to emphasize that unlike in Linear Algebra, a *general max-plus linear system* $A \odot x \oplus b = C \odot x \oplus d$ is not always equivalent to one of the form $A \odot x = b$. For several other important cases, like the spectral problem $A \odot x = \lambda \odot x$, the fixed point problem $x = A \odot x \oplus b$, or two-sided linear systems $A \odot x = B \odot x$, there also exist efficient solution methods, see [But03, GP97].

5 The MaxLinearAlgebra Package

To the best of our knowledge, the only package for max-plus computations in a computer algebra system is the Maple package MAX by Stéphane Gaubert. It implements basic scalar-matrix operations, rational operations in the so called minmax-algebra, and several other more specialized algorithms. The package works in Maple V up to R3 but not in newer versions, for details see `http://amadeus.inria.fr/gaubert/PAPERS/MAX.html`.

For numerical computations in the max-algebra, there is the Maxplus toolbox for Scilab, which is developed by the Maxplus INRIA working group. The current version is available at `http://www.scilab.org`. A toolbox for max-algebra in Excel and some MATLAB functions (e.g. for two-sided max-plus linear systems) by Peter Butkovič and his studends are available at `http://web.mat.bham.ac.uk/P.Butkovic/software`. Some additional software is available at `http://www-rocq.inria.fr/MaxplusOrg`.

Our Maple package `MaxLinearAlgebra` is based on the `LinearAlgebra` package introduced in Maple 6. We also use the `ListTools` and `combinat` package. The names correspond (wherever applicable) to the commands in Maple with a `Max` and `Min` prefix, respectively. We have implemented basic matrix operations and utility functions, solvability tests and solutions for max/min-plus linear systems, and max/min linear combinations and interpolation. The package could serve as framework for implementing other max-plus algorithms in Maple, some also based on the already implemented ones, as for example the computation of bases in $\mathbb{R}_{\max}$, see [CGB04].

For the application to BVPs we rely on Maple's `dsolve` command to compute symbolic solutions of differential equations. Using the identities

$$\max(a,b) = \frac{a+b+|a-b|}{2} \quad \text{and} \quad \min(a,b) = \frac{a+b-|a-b|}{2},$$

we can express max/min linear combinations and hence generalized solutions for BVPs with nested absolute values. This has advantages in particular for symbolic differentiation.

The package and a worksheet with examples for all functions, large linear systems, and BVPs are available at `http://gregensburger.com`. See also the next section for two examples.

6 Examples

We first consider the example (2). The differential equation has the two solutions $y_1(x) = x$ and $y_2(x) = -x$. After loading the package

```
>  with(MaxLinearAlgebra):
```

we compute the interpolation matrix

```
>  A:=InterpolationMatrix([x->x,x->-x],<-1,1>);
```

$$A := \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and solve the corresponding max-plus linear system

```
> linsolmax:=MaxLinearSolve(A);
```

$$linsolmax := [\begin{bmatrix} -1 \\ -1 \end{bmatrix}, [[1,2]]]$$

The first element is the principal solution and the second element describes the solution space, here we have a unique solution $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$. The generalized max solution is then

```
> MaxLinearCombination(linsolmax[1],[x,-x]);
```

$$\max(-1 + x, -1 - x)$$

or with absolute values

```
> MaxLinearCombinationAbs(linsolmax[1],[x,-x]);
```

$$-1 + |x|$$

As a second example, we consider the BVP

$$y'^3 - xy'^2 - y' + x \quad \text{with} \quad y(-1) = y(0) = y(1) = 0. \tag{3}$$

The differential equation has three solutions $y_1(x) = x$, $y_2(x) = -x$ and $y_3(x) = 1/2x^2$. The corresponding interpolation matrix is

```
> A:=InterpolationMatrix([x->x,x->-x,x->1/2*x^2],<-1,0,1>);
```

$$A := \begin{bmatrix} -1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 1 & -1 & 1/2 \end{bmatrix}$$

There is no max-plus solution

```
> IsMaxMinSolvable(A,ColumnMax(A));
```

false

but one min-plus solution that gives the generalized solution

```
> MinLinearCombinationAbs(linsolmin[1],[x,-x,1/2*x^2]);
```

$$1/2 - 1/2\,|x| + 1/4\,x^2 - 1/2\,\left|-1 + |x| + 1/2\,x^2\right|$$

for (3), and it looks like:

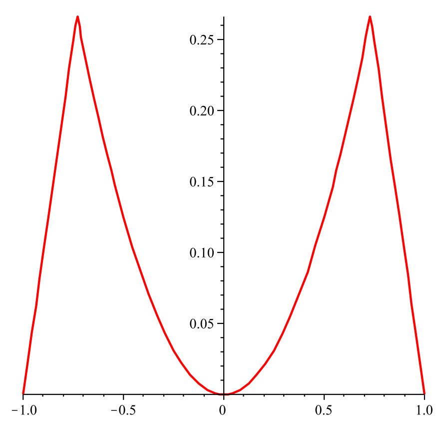

Figure 1: The generalized min-plus solution for (3).

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