In[∘]:=
```
<< "TenReS.m"
```

```
Package TenReS version 0.2.4
written by Clemens G. Raab
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```

Out[∘]=

Symbol

A few definitions need to be made by the user. Type
?CoeffQ, ?Specialization, and ?CyclicModules for more information.

Basic definitions

In[∘]:=
```
Specialization = {ℛ → {𝒦, ℛ̃}, Φ → {"E", Φ̃}};
```

In[∘]:=
```
CyclicModules = {{𝒦, 1}, {"E", Eval}, {D, Diff}, {"I", Int}};
```

Bimodule structure

In[∘]:=
```
Prod[a___, b : Diff | Int | Eval, (c_?MemberQ𝒦) Prod[], d___] :=
  Prod[a, c Prod[], b, d]
Prod[a___, b_?MemberQΦ, (c_?MemberQ𝒦) Prod[], d___] := Prod[a, c Prod[], b, d]
Prod[a___, b_?MemberQℛ, (c_?MemberQ𝒦) Prod[], d___] := Prod[a, mul[b, c], d]
Prod[a___, (b_?MemberQ𝒦) Prod[], (c_?MemberQ𝒦) Prod[], d___] :=
  Prod[a, mul[b, c] × Prod[], d]
```

In[∘]:=
```
Prod[(a_?MemberQ𝒦) Prod[], b_?MemberQℛ, c___] := Prod[mul[a, b], c]
```

Membership checks

Check for constants:

In[∘]:=
```
MemberQ𝒦[f_] := (Diff[f] === 0)
```

Check for membership in the function algebra:

In[∘]:=
```
MemberQℛ[f_?MemberQ𝒦] := True
MemberQℛ[ℛ[_Integer]] := True
MemberQℛ[Diff[f_?MemberQℛ]] := True
MemberQℛ[Int[f_?MemberQℛ]] := True
MemberQℛ[f_mul] := And @@ (MemberQℛ /@ (List @@ f))
```

Check for functionals:

In[∘]:=
```
MemberQΦ[Eval] := True
MemberQΦ[Φ[_Integer]] := True
(*MemberQΦ[mul[f__/;MemberQ𝒦[mul[f]],φ_?MemberQΦ]]:=True*)
```

Check for membership in the other cyclic modules:

```
In[•]:= MemberQ_D[Diff] := True
```

```
In[•]:= MemberQ_"I"[Int] := True
```

```
In[•]:= MemberQ_"E"[Eval] := True
        MemberQ_"E"[mul[f__ /; MemberQ_𝒦[mul[f]], Eval]] := True
```

Function algebra

Multiplication

```
In[•]:= mul[a___, b_Plus, c___] := (mul[a, #, c] & /@ b)
        mul[a___, b_Integer, c__] := b mul[a, c]
        mul[a__, b_Integer, c___] := b mul[a, c]
        mul[a___, d_Integer * b_, c___] := d mul[a, b, c]
        mul[a___, mul[b__], c___] := mul[a, b, c]
        mul[a_] := a
```

Differentiation

```
In[•]:= Diff[f_?NumericQ] := 0
        Diff[(φ_?MemberQ_Φ)[f_]] := 0
        Diff[Int[f_?MemberQ_ℛ]] := f
        Diff[a_mul] := Sum[MapAt[Diff, a, i], {i, Length[a]}]
```

Integration

```
In[•]:= Int[f_?(MemberQ_𝒦[#] && (# =!= 1) &)] := mul[f, Int[1]]
        Int[mul[f__, c_?MemberQ_𝒦]] := mul[Int[mul[f]], c]
        Int[Diff[f_?MemberQ_ℛ]] := f - Eval[f]
```

```
In[•]:= Int /: mul[a___, Int[f_], Int[g_], b___] := mul[a, Int[mul[Int[f], g]], b] +
          mul[a, Int[mul[f, Int[g]]], b] + mul[a, Eval[Hold[mul[Int[f], Int[g]]]], b]
```

```
In[•]:= mul[a___, Int[1], c_?MemberQ_𝒦, b___] := mul[a, c, Int[1], b]
```

Induced evaluation

```
In[•]:= Eval[f_?MemberQ_𝒦] := f
        Eval[Int[f_?MemberQ_ℛ]] := 0
```

Functionals

```
In[•]:= Φ /: Φ[i_Integer][mul[f__, c_?MemberQ_𝒦]] := mul[Φ[i][mul[f]], c]
        Φ /: Φ[i_Integer][mul[c_?MemberQ_𝒦, f__]] := mul[c, Φ[i][mul[f]]]
```

```
In[•]:= Φ /: Φ[i_Integer][f_?(MemberQ_𝒦[#] && (# =!= 1) &)] := mul[f, Φ[i][1]]
        (*Φ/:Φ[i_Integer][f_?(MemberQ_𝒦[#]&&(#=!=1)&)]:=mul[Φ[i][1],f]*)
```

```
In[•]:= mul[a___, Except[Φ[_Integer][1], c_?MemberQ𝒦], Φ[i_Integer][1], b___] :=
         mul[a, Φ[i][1], c, b]
        (*mul[a___,Φ[i_Integer][1],Except[Φ[_Integer][1],c_?MemberQ𝒦],b___]:=
         mul[a,c,Φ[i][1],b]*)
        mul[a___, Φ[j_Integer][1], Φ[i_Integer][1], b___] /; j > i :=
         mul[a, Φ[i][1], Φ[j][1], b]
```

Defining reduction system

```
In[•]:= hℛℛ[f_, g_] := Prod[mul[f, g]]
        hDℛ[Diff, f_] := Prod[Diff[f]] + Prod[f, Diff]
        hDI[Diff, Int] := Prod[]
        hID[Int, Diff] := Prod[] - Prod[Eval]
        hDℛΦ[Diff, f_, φ_] := Prod[Diff[f], φ]
        hIℛΦ[Int, f_, φ_] := Prod[Int[f], φ]
        hΦℛΦ[φ_, f_, ψ_] := Prod[φ[f] × Prod[], ψ]

In[•]:= h𝒦[k_] := Prod[k Prod[]]

In[•]:= RedSys = {{{ℛ, ℛ}, hℛℛ}, {{D, ℛ}, hDℛ}, {{D, "I"}, hDI}, {{"I", D}, hID},
           {{D, ℛ, Φ}, hDℛΦ}, {{"I", ℛ, Φ}, hIℛΦ}, {{Φ, ℛ, Φ}, hΦℛΦ}, {{𝒦}, h𝒦}};
```

Complete reduction system

```
In[•]:= hEI[e_, Int] := 0
        hDΦ[Diff, φ_] := 0
        hIΦ[Int, φ_] := Prod[Int[1], φ]
        hΦΦ[φ_, ψ_] := Prod[φ[1] × Prod[], ψ]
        hIℛD[Int, f_, Diff] := Prod[f] - Prod[Eval, f] - Prod[Int, Diff[f]]
        hIℛI[Int, f_, Int] :=
         Prod[Int[f], Int] - Prod[Int, Int[f]] - Prod[Eval, Int[f], Int]
        hII[Int, Int] := Prod[Int[1], Int] - Prod[Int, Int[1]] - Prod[Eval, Int[1], Int]

In[•]:= RedSys = {{{𝒦}, h𝒦}, {{ℛ, ℛ}, hℛℛ}, {{D, ℛ}, hDℛ},
           {{D, Φ}, hDΦ}, {{D, "I"}, hDI}, {{Φ, ℛ, Φ}, hΦℛΦ}, {{Φ, Φ}, hΦΦ},
           {{"E", "I"}, hEI}, {{"I", ℛ, D}, hIℛD}, {{"I", ℛ, Φ}, hIℛΦ},
           {{"I", ℛ, "I"}, hIℛI}, {{"I", D}, hID}, {{"I", Φ}, hIΦ}, {{"I", "I"}, hII}};

In[•]:= CheckResolvability[RedSys, Count → True, Print → True]

        54 ambiguities in total

        3 ambiguities have all S-polynomials equal to zero

        1: Overlap[{ℛ, ℛ, ℛ}, {ℛ}, {ℛ}]

        {{ℛ, ℛ}, hℛℛ}

        {{ℛ, ℛ}, hℛℛ}

        2: Overlap[{D, ℛ, ℛ}, {ℛ}, {D}]
```

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

3: Overlap$[\{D, \Phi, \mathcal{R}, \Phi\}, \{\mathcal{R}, \Phi\}, \{D\}]$

$\{\{D, \Phi\}, h_{D\Phi}\}$

4: Overlap$[\{D, \Phi, \Phi\}, \{\Phi\}, \{D\}]$

$\{\{D, \Phi\}, h_{D\Phi}\}$

5: Overlap$[\{D, I, \mathcal{R}, D\}, \{\mathcal{R}, D\}, \{D\}]$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{D, \Phi\}, h_{D\Phi}\}$

$\{\{D, I\}, h_{DI}\}$

6: Overlap$[\{D, I, \mathcal{R}, \Phi\}, \{\mathcal{R}, \Phi\}, \{D\}]$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{D, \Phi\}, h_{D\Phi}\}$

7: Overlap$[\{D, I, \mathcal{R}, I\}, \{\mathcal{R}, I\}, \{D\}]$

$\{\{D, I\}, h_{DI}\}$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{D, \Phi\}, h_{D\Phi}\}$

$\{\{D, I\}, h_{DI}\}$

8: Overlap$[\{D, I, D\}, \{D\}, \{D\}]$

$\{\{D, \Phi\}, h_{D\Phi}\}$

9: Overlap$[\{D, I, \Phi\}, \{\Phi\}, \{D\}]$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{D, \Phi\}, h_{D\Phi}\}$

10: Overlap$[\{D, I, I\}, \{I\}, \{D\}]$

$\{\{D, I\}, h_{DI}\}$

$\{\{D, \mathcal{R}\}, h_{D\mathcal{R}}\}$

$\{\{D, \Phi\}, h_{D\Phi}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{D, I\}, h_{DI}\}$

11: Overlap$[\{\Phi, \mathcal{R}, \Phi, \mathcal{R}, \Phi\}, \{\mathcal{R}, \Phi\}, \{\Phi, \mathcal{R}\}]$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

12: Overlap$[\{\Phi, \mathcal{R}, \Phi, \Phi\}, \{\Phi\}, \{\Phi, \mathcal{R}\}]$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

13: Overlap[$\{\Phi, \Phi, \mathcal{R}, \Phi\}$, $\{\mathcal{R}, \Phi\}$, $\{\Phi\}$]

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

14: Overlap[$\{\Phi, \Phi, \Phi\}$, $\{\Phi\}$, $\{\Phi\}$]

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

15: Overlap[$\{E, I, \mathcal{R}, D\}$, $\{\mathcal{R}, D\}$, $\{E\}$]

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

$\{\{E, I\}, h_{EI}\}$

16: Overlap[$\{E, I, \mathcal{R}, \Phi\}$, $\{\mathcal{R}, \Phi\}$, $\{E\}$]

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

17: Overlap[$\{E, I, \mathcal{R}, I\}$, $\{\mathcal{R}, I\}$, $\{E\}$]

$\{\{E, I\}, h_{EI}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

18: Overlap[$\{E, I, D\}$, $\{D\}$, $\{E\}$]

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

19: Overlap[$\{E, I, \Phi\}$, $\{\Phi\}$, $\{E\}$]

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

20: Overlap[$\{E, I, I\}$, $\{I\}$, $\{E\}$]

$\{\{E, I\}, h_{EI}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

21: Overlap[$\{I, \mathcal{R}, D, \mathcal{R}\}$, $\{\mathcal{R}\}$, $\{I, \mathcal{R}\}$]

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

22: Overlap[$\{I, \mathcal{R}, D, \Phi\}$, $\{\Phi\}$, $\{I, \mathcal{R}\}$]

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

23: Overlap[$\{I, \mathcal{R}, D, I\}$, $\{I\}$, $\{I, \mathcal{R}\}$]

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{E, I\}, h_{EI}\}$

24: Overlap$[\{I, \mathcal{R}, \Phi, \mathcal{R}, \Phi\}, \{\mathcal{R}, \Phi\}, \{I, \mathcal{R}\}]$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

25: Overlap$[\{I, \mathcal{R}, \Phi, \Phi\}, \{\Phi\}, \{I, \mathcal{R}\}]$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

26: Overlap$[\{I, \mathcal{R}, I, \mathcal{R}, D\}, \{\mathcal{R}, D\}, \{I, \mathcal{R}\}]$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

27: Overlap$[\{I, \mathcal{R}, I, \mathcal{R}, \Phi\}, \{\mathcal{R}, \Phi\}, \{I, \mathcal{R}\}]$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

28: Overlap$[\{I, \mathcal{R}, I, \mathcal{R}, I\}, \{\mathcal{R}, I\}, \{I, \mathcal{R}\}]$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{E, I\}, h_{EI}\}$

29: Overlap[$\{I, \mathcal{R}, I, D\}, \{D\}, \{I, \mathcal{R}\}$]

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, D\}, h_{ID}\}$

$\{\{I, D\}, h_{ID}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

30: Overlap[$\{I, \mathcal{R}, I, \Phi\}, \{\Phi\}, \{I, \mathcal{R}\}$]

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

31: Overlap[$\{I, \mathcal{R}, I, I\}, \{I\}, \{I, \mathcal{R}\}$]

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, I\}, h_{II}\}$

```
{{I, I}, h_II}
{{I, ℛ, I}, h_IℛI}
{{ℛ, ℛ}, h_ℛℛ}
{{I, ℛ, Φ}, h_IℛΦ}
{{ℛ, ℛ}, h_ℛℛ}
{{I, ℛ, I}, h_IℛI}
{{ℛ, ℛ}, h_ℛℛ}
{{ℛ, ℛ}, h_ℛℛ}
{{Φ, ℛ, Φ}, h_ΦℛΦ}
{{𝒦}, h_𝒦}
{{𝒦}, h_𝒦}
{{𝒦}, h_𝒦}
{{E, I}, h_EI}
32: Overlap[{I, D, ℛ}, {ℛ}, {I}]
{{I, ℛ, D}, h_IℛD}
33: Overlap[{I, D, Φ}, {Φ}, {I}]
{{Φ, Φ}, h_ΦΦ}
34: Overlap[{I, D, I}, {I}, {I}]
{{E, I}, h_EI}
35: Overlap[{I, Φ, ℛ, Φ}, {ℛ, Φ}, {I}]
{{I, Φ}, h_IΦ}
{{Φ, ℛ, Φ}, h_ΦℛΦ}
36: Overlap[{I, Φ, Φ}, {Φ}, {I}]
{{Φ, Φ}, h_ΦΦ}
{{I, Φ}, h_IΦ}
37: Overlap[{I, I, ℛ, D}, {ℛ, D}, {I}]
{{I, Φ}, h_IΦ}
{{I, I}, h_II}
{{ℛ, ℛ}, h_ℛℛ}
{{I, ℛ, D}, h_IℛD}
{{I, ℛ, D}, h_IℛD}
{{ℛ, ℛ}, h_ℛℛ}
{{ℛ, ℛ}, h_ℛℛ}
{{ℛ, ℛ}, h_ℛℛ}
{{I, ℛ, D}, h_IℛD}
{{Φ, ℛ, Φ}, h_ΦℛΦ}
38: Overlap[{I, I, ℛ, Φ}, {ℛ, Φ}, {I}]
```

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

39: Overlap[$\{I, I, \mathcal{R}, I\}, \{\mathcal{R}, I\}, \{I\}$]

$\{\{I, I\}, h_{II}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{E, I\}, h_{EI}\}$

40: Overlap[$\{I, I, D\}, \{D\}, \{I\}$]

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{I, \mathcal{R}, D\}, h_{I\mathcal{R}D}\}$

$\{\{I, D\}, h_{ID}\}$

$\{\{I, D\}, h_{ID}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

41: Overlap[$\{I, I, \Phi\}, \{\Phi\}, \{I\}$]

$\{\{I, \mathcal{R}, \Phi\}, h_{I\mathcal{R}\Phi}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

42: Overlap[{I, I, I}, {I}, {I}]

$\{\{I, I\}, h_{II}\}$

$\{\{I, \mathcal{R}, I\}, h_{I\mathcal{R}I}\}$

$\{\{I, I\}, h_{II}\}$

$\{\{I, I\}, h_{II}\}$

$\{\{I, \Phi\}, h_{I\Phi}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\mathcal{R}, \mathcal{R}\}, h_{\mathcal{R}\mathcal{R}}\}$

$\{\{\Phi, \mathcal{R}, \Phi\}, h_{\Phi\mathcal{R}\Phi}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

$\{\{E, I\}, h_{EI}\}$

43: SpecialOverlap[{D, E, I}, {I}, {D}]

44: SpecialOverlap[{Φ, $\mathcal{R}$, E, I}, {I}, {Φ, $\mathcal{R}$}]

$\{\{E, I\}, h_{EI}\}$

45: SpecialOverlap[{Φ, E, I}, {I}, {Φ}]

$\{\{E, I\}, h_{EI}\}$

46: SpecialOverlap[{I, $\mathcal{R}$, E, I}, {I}, {I, $\mathcal{R}$}]

$\{\{E, I\}, h_{EI}\}$

47: SpecialOverlap[{I, E, I}, {I}, {I}]

$\{\{E, I\}, h_{EI}\}$

48: SpecialInclusion[{$\mathcal{K}$, $\mathcal{R}$}, {}, {$\mathcal{R}$}]

49: SpecialInclusion[{$\mathcal{R}$, $\mathcal{K}$}, {$\mathcal{R}$}, {}]

50: SpecialInclusion[{D, $\mathcal{K}$}, {D}, {}]

$\{\{\mathcal{K}\}, h_{\mathcal{K}}\}$

51: SpecialInclusion[{Φ, $\mathcal{K}$, Φ}, {Φ}, {Φ}]

$\{\{\Phi, \Phi\}, h_{\Phi\Phi}\}$

```
52: SpecialInclusion[{I, 𝒦, D}, {I}, {D}]
{{𝒦}, h𝒦}
{{𝒦}, h𝒦}
{{I, D}, hID}
53: SpecialInclusion[{I, 𝒦, Φ}, {I}, {Φ}]
{{I, Φ}, hIΦ}
54: SpecialInclusion[{I, 𝒦, I}, {I}, {I}]
{{I, I}, hII}
54 ambiguities are resolvable
```

Out[•]= { }

Include multiplicativity of some functionals

New basic definitions

```
In[•]:= Specialization = {ℛ → {𝒦, ℛ̃}, Φ → {"E", Φm, Φ̃}};
```

Membership checks

```
In[•]:= MemberQΦ[φ_?MemberQΦm] := True
In[•]:= MemberQΦm[Φm[_Integer]] := True
```

Function algebra

```
In[•]:= Φm /: (Φm[i_Integer])[f_mul] := Map[Φm[i], f]
        Φm /: (Φm[_Integer])[f_?MemberQ𝒦] := f
```

Additional reduction rule

```
In[•]:= hΦm ℛ[φ_, f_] := Prod[φ[f] × Prod[], φ]
In[•]:= AppendTo[RedSys, {{Φm, ℛ}, hΦm ℛ}]
Out[•]= {{{𝒦}, h𝒦}, {{ℛ, ℛ}, hℛℛ}, {{D, ℛ}, hDℛ}, {{D, Φ}, hDΦ}, {{D, I}, hDI},
  {{Φ, ℛ, Φ}, hΦℛΦ}, {{Φ, Φ}, hΦΦ}, {{E, I}, hEI}, {{I, ℛ, D}, hIℛD}, {{I, ℛ, Φ}, hIℛΦ},
  {{I, ℛ, I}, hIℛI}, {{I, D}, hID}, {{I, Φ}, hIΦ}, {{I, I}, hII}, {{Φm, ℛ}, hℛ Φm}}
In[•]:= CheckResolvability[RedSys, Count → True]
62 ambiguities in total
4 ambiguities have all S-polynomials equal to zero
62 ambiguities are resolvable
```

Out[•]= { }

Multiplicative induced evaluation

In[∘]:=
```
Specialization = {ℛ → {𝒦, ℛ̃}, Φ → {"E", Φm, Φ̃}, Φm → {"E"}};
```

In[∘]:=
```
MemberQ_Φm[Eval] := True
```

In[∘]:=
```
CheckResolvability[RedSys, Count → True]
```

```
62 ambiguities in total

4 ambiguities have all S-polynomials equal to zero

62 ambiguities are resolvable
```

Out[∘]= { }