

Focal values

Computation of general focal values based on the code in

O. A. Kuznetsova, An example of symbolic computation of Lyapunov quantities in Maple, in Proceedings of BICA'12, 2012, 195–198.

```
> m := 2; #number of focal values
      m := 2 (1.1)
```

```
> n := 2*m+1; #degree of taylor polynomials
      n := 5 (1.2)
```

```
> # Taylor polynomial def of functions f and g
```

Taylor polynomials

```
> f(x,y):=add(add(f[i,k-i]*x^i*y^(k-i),i=0..k),k=2..n);
f(x,y) := x^5 f_{5,0} + x^4 y f_{4,1} + x^3 y^2 f_{3,2} + x^2 y^3 f_{2,3} + x y^4 f_{1,4} + y^5 f_{0,5} + x^4 f_{4,0}
      + x^3 y f_{3,1} + x^2 y^2 f_{2,2} + x y^3 f_{1,3} + y^4 f_{0,4} + x^3 f_{3,0} + x^2 y f_{2,1} + x y^2 f_{1,2} + y^3 f_{0,3}
      + x^2 f_{2,0} + x y f_{1,1} + y^2 f_{0,2} (1.3)
```

```
> g(x,y):=add(add(g[i,k-i]*x^i*y^(k-i),i=0..k),k=2..n);
g(x,y) := x^5 g_{5,0} + x^4 y g_{4,1} + x^3 y^2 g_{3,2} + x^2 y^3 g_{2,3} + x y^4 g_{1,4} + y^5 g_{0,5} + x^4 g_{4,0}
      + x^3 y g_{3,1} + x^2 y^2 g_{2,2} + x y^3 g_{1,3} + y^4 g_{0,4} + x^3 g_{3,0} + x^2 y g_{2,1} + x y^2 g_{1,2}
      + y^3 g_{0,3} + x^2 g_{2,0} + x y g_{1,1} + y^2 g_{0,2} (1.4)
```

Transformation to the polar coordinates

```
> f(r,theta):=subs(x=r*cos(theta),y=r*sin(theta),f(x,y));
> g(r,theta):=subs(x=r*cos(theta),y=r*sin(theta),g(x,y));
> dr:=f(r,theta)*cos(theta)+g(r,theta)*sin(theta);
dt:=simplify(1-f(r,theta)*sin(theta)/r+g(r,theta)*cos(theta)/r);
> drdt := dr/dt;
```

```
> f(theta,r[0]):=0;
```

```
> R(theta,r) := 0;
```

```
> for i from 1 to n do
  f(theta,r[0]):=f(theta,r[0])+u[i](theta)*r[0]^i;
  R[i](theta):=subs(r=0,diff(drdt,[r$ i])/i!);
  R(theta,r):=R(theta,r)+R[i](theta)*r^i;
end:
> E:=diff(f(theta,r[0]),theta)-subs(r=f(theta,r[0]),R(theta,r));
> ic := Vector[row]([1,0,0,0,0]):
```

```
> for i from 1 to n do
  S:=int(diff(u[i](theta),theta)-coeff(E,r[0],i),theta);
  u[i](theta):=simplify(S-subs(theta=0,S)+ic[i]);
end:
```

Focal values:

```
> for i from 1 to m do
  L[i] := (simplify@subs) (theta=2*Pi, u[2*i+1] (theta));
end;
```

$$L_1 := \frac{1}{4} \pi f_{1,2} + \frac{1}{4} \pi g_{2,1} + \frac{3}{4} \pi f_{3,0} + \frac{3}{4} \pi g_{0,3} - \frac{1}{4} \pi g_{0,2} g_{1,1} + \frac{1}{4} \pi f_{0,2} f_{1,1} \\ + \frac{1}{4} \pi f_{1,1} f_{2,0} - \frac{1}{4} \pi g_{1,1} g_{2,0} - \frac{1}{2} f_{2,0} g_{2,0} \pi + \frac{1}{2} \pi f_{0,2} g_{0,2}$$

$$L_2 := \frac{35}{96} \pi f_{0,2} f_{0,3} f_{1,1} + \frac{13}{96} \pi f_{0,2} f_{1,1} f_{2,1} + \frac{1}{96} g_{2,0} \pi f_{0,2} f_{1,1}^2 - \frac{3}{32} g_{2,0} \pi \quad (1.5)$$

$$f_{1,1}^2 f_{2,0} + \frac{1}{96} g_{2,0} \pi f_{1,1} f_{1,2} + \frac{31}{96} g_{2,0} \pi f_{1,1} f_{3,0} + \frac{5}{32} g_{2,0} \pi f_{1,1} g_{0,3} \\ + \frac{17}{96} g_{2,0} \pi f_{1,1} g_{2,1} + \frac{1}{16} \pi f_{0,2} f_{1,1} g_{2,0}^2 - \frac{1}{12} \pi f_{1,1} f_{2,0} g_{2,0}^2 \\ - \frac{17}{96} \pi f_{1,1} g_{1,1} g_{2,0}^2 - \frac{1}{32} \pi f_{1,1}^2 g_{1,1} g_{2,0} - \frac{7}{48} \pi f_{2,0} g_{1,2} g_{2,0} \\ + \frac{35}{48} \pi f_{2,0} g_{2,0} g_{3,0} + \frac{1}{32} \pi f_{2,1} g_{1,1} g_{2,0} + \frac{13}{96} \pi g_{1,1} g_{1,2} g_{2,0} \\ + \frac{35}{96} \pi g_{1,1} g_{2,0} g_{3,0} + \frac{3}{8} \pi^2 f_{2,0} g_{1,1} g_{2,0}^2 + \frac{1}{4} \pi f_{0,2} f_{1,1} g_{0,2} g_{2,0} \\ + \frac{1}{24} \pi f_{1,1} f_{2,0} g_{0,2} g_{2,0} - \frac{11}{48} \pi f_{1,1} g_{0,2} g_{1,1} g_{2,0} - \frac{3}{8} \pi^2 f_{0,2} f_{1,1} f_{2,0} g_{2,0} \\ - \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{1,1} g_{2,0} - \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{1,1} g_{2,0} + \frac{1}{16} \pi f_{0,2} f_{1,1} f_{2,0} g_{1,1} \\ + \frac{3}{8} \pi^2 f_{0,2} f_{1,1} f_{2,0} g_{0,2} - \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{0,2} g_{1,1} - \frac{3}{4} \pi^2 f_{0,2} f_{2,0} g_{0,2} g_{2,0} \\ - \frac{3}{8} \pi^2 f_{0,2} g_{0,2} g_{1,1} g_{2,0} - \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{0,2} g_{1,1} + \frac{3}{8} \pi^2 f_{2,0} g_{0,2} g_{1,1} g_{2,0} \\ - \frac{11}{72} \pi f_{0,2} f_{2,0} g_{0,2} g_{1,1} - \frac{1}{36} \pi f_{0,2} f_{2,0} g_{1,1} g_{2,0} + \frac{3}{32} \pi^2 f_{1,2}^2 + \frac{27}{32} \pi^2 f_{3,0}^2 \\ + \frac{27}{32} \pi^2 g_{0,3}^2 + \frac{3}{32} \pi^2 g_{2,1}^2 + \frac{1}{8} \pi f_{1,4} + \frac{1}{8} \pi f_{3,2} + \frac{5}{8} \pi f_{5,0} + \frac{5}{8} \pi g_{0,5} \\ + \frac{1}{8} \pi g_{2,3} + \frac{1}{8} \pi g_{4,1} - \frac{1}{12} \pi f_{2,2} g_{2,0} - \frac{5}{6} \pi f_{4,0} g_{2,0} - \frac{1}{8} \pi g_{1,3} g_{2,0} \\ - \frac{5}{24} \pi g_{2,0} g_{3,1} + \frac{1}{4} \pi f_{2,0} g_{0,2}^3 + \frac{1}{12} \pi f_{1,3} g_{1,1} + \frac{5}{24} \pi g_{0,4} g_{1,1} \\ - \frac{1}{24} \pi g_{1,1} g_{2,2} - \frac{1}{8} \pi g_{1,1} g_{4,0} + \frac{3}{8} \pi^2 f_{2,0}^2 g_{2,0}^2 + \frac{3}{32} \pi^2 g_{1,1}^2 g_{2,0}^2 \\ - \frac{1}{32} \pi g_{1,2} f_{3,0} + \frac{15}{32} \pi f_{0,3} g_{0,3} + \frac{5}{32} \pi f_{0,3} f_{1,2} + \frac{1}{32} \pi f_{0,3} g_{2,1}$$

$$\begin{aligned}
& + \frac{3}{32} \pi f_{0,3} f_{3,0} - \frac{13}{48} \pi g_{2,0}^3 g_{1,1} - \frac{13}{24} \pi g_{2,0}^3 f_{2,0} + \frac{1}{16} \pi f_{1,2} g_{2,0}^2 \\
& + \frac{13}{16} \pi f_{3,0} g_{2,0}^2 + \frac{3}{16} \pi g_{0,3} g_{2,0}^2 + \frac{13}{48} \pi g_{2,0}^2 g_{2,1} + \frac{7}{96} \pi f_{0,2} f_{1,1}^3 + \frac{7}{96} \pi \\
& f_{1,1}^3 f_{2,0} + \frac{7}{96} \pi f_{1,1}^2 f_{1,2} + \frac{13}{96} \pi f_{1,1}^2 f_{3,0} + \frac{3}{32} \pi f_{1,1}^2 g_{0,3} + \frac{1}{32} \pi f_{1,1}^2 g_{2,1} \\
& + \frac{35}{72} \pi f_{0,2}^3 g_{0,2} - \frac{13}{288} \pi g_{0,2} g_{1,1}^3 + \frac{1}{24} \pi f_{1,2} g_{0,2}^2 + \frac{2}{3} \pi f_{3,0} g_{0,2}^2 + \frac{7}{8} \pi \\
& g_{0,2}^2 g_{0,3} + \frac{5}{12} \pi g_{0,2}^2 g_{2,1} + \frac{3}{8} \pi^2 f_{0,2}^2 g_{0,2}^2 + \frac{3}{32} \pi^2 g_{0,2}^2 g_{1,1}^2 + \frac{35}{144} \pi f_{0,2}^3 f_{1,1} \\
& + \frac{13}{72} \pi f_{1,1} f_{2,0}^3 + \frac{1}{8} \pi f_{0,4} f_{1,1} + \frac{1}{24} \pi f_{1,1} f_{2,2} - \frac{5}{24} \pi f_{1,1} f_{4,0} \\
& - \frac{1}{12} \pi f_{1,1} g_{3,1} + \frac{3}{32} \pi^2 f_{0,2}^2 f_{1,1}^2 + \frac{3}{32} \pi^2 f_{1,1}^2 f_{2,0}^2 + \frac{1}{288} \pi f_{1,2} g_{1,1}^2 \\
& + \frac{1}{96} \pi f_{3,0} g_{1,1}^2 + \frac{5}{96} \pi g_{0,3} g_{1,1}^2 + \frac{13}{288} \pi g_{1,1}^2 g_{2,1} - \frac{1}{4} \pi f_{2,0}^3 g_{0,2} \\
& + \frac{1}{4} \pi f_{0,4} g_{0,2} - \frac{1}{6} \pi f_{2,2} g_{0,2} - \frac{11}{12} \pi f_{4,0} g_{0,2} - \frac{3}{8} \pi g_{0,2} g_{1,3} \\
& - \frac{7}{24} \pi g_{0,2} g_{3,1} + \frac{9}{16} \pi^2 f_{1,2} f_{3,0} + \frac{9}{16} \pi^2 f_{1,2} g_{0,3} + \frac{3}{16} \pi^2 f_{1,2} g_{2,1} \\
& + \frac{27}{16} \pi^2 f_{3,0} g_{0,3} + \frac{9}{16} \pi^2 f_{3,0} g_{2,1} + \frac{9}{16} \pi^2 g_{0,3} g_{2,1} + \frac{35}{144} \pi f_{0,2}^2 f_{1,2} \\
& + \frac{5}{48} \pi f_{0,2}^2 f_{3,0} + \frac{35}{48} \pi f_{0,2}^2 g_{0,3} + \frac{5}{144} \pi f_{0,2}^2 g_{2,1} + \frac{7}{12} \pi f_{0,2} g_{0,2}^3 - \frac{7}{24} \pi \\
& g_{0,2}^3 g_{1,1} + \frac{5}{24} \pi f_{0,2} f_{1,3} + \frac{1}{8} \pi f_{0,2} f_{3,1} + \frac{5}{6} \pi f_{0,2} g_{0,4} + \frac{1}{12} \pi f_{0,2} g_{2,2} \\
& + \frac{11}{36} \pi f_{1,2} f_{2,0}^2 + \frac{13}{24} \pi f_{2,0}^2 f_{3,0} + \frac{1}{3} \pi f_{2,0}^2 g_{0,3} - \frac{5}{72} \pi f_{2,0}^2 g_{2,1} \\
& + \frac{7}{24} \pi f_{1,3} f_{2,0} + \frac{3}{8} \pi f_{2,0} f_{3,1} + \frac{11}{12} \pi f_{2,0} g_{0,4} + \frac{1}{6} \pi f_{2,0} g_{2,2} \\
& - \frac{1}{4} \pi f_{2,0} g_{4,0} - \frac{13}{36} \pi f_{2,0}^3 g_{2,0} - \frac{13}{288} \pi g_{1,1}^3 g_{2,0} + \frac{13}{288} \pi f_{0,2} g_{1,1}^2 g_{2,0} \\
& + \frac{3}{16} \pi f_{2,0}^2 g_{1,1} g_{2,0} + \frac{3}{32} \pi f_{2,0} g_{1,1}^2 g_{2,0} - \frac{1}{12} \pi f_{0,2} f_{2,1} g_{2,0} \\
& - \frac{1}{12} \pi f_{0,2} g_{1,2} g_{2,0} - \frac{1}{16} \pi f_{0,3} f_{2,0} g_{2,0} - \frac{1}{32} \pi f_{0,3} g_{1,1} g_{2,0} \\
& - \frac{17}{48} \pi f_{2,0} f_{2,1} g_{2,0} + \frac{35}{72} \pi f_{0,2}^2 f_{2,0} g_{0,2} + \frac{7}{96} \pi f_{0,2} f_{1,1}^2 g_{0,2} - \frac{1}{18} \pi f_{0,2} \\
& f_{2,0}^2 g_{0,2} + \frac{3}{32} \pi f_{0,2} g_{0,2} g_{1,1}^2 - \frac{1}{32} \pi f_{1,1}^2 f_{2,0} g_{0,2} - \frac{1}{32} \pi f_{1,1}^2 g_{0,2} g_{1,1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{36} \pi f_{2,0}^2 g_{0,2} g_{1,1} + \frac{41}{288} \pi f_{2,0} g_{0,2} g_{1,1}^2 - \frac{17}{24} \pi f_{2,0} g_{0,2} g_{2,0}^2 \\
& - \frac{5}{8} \pi g_{0,2} g_{1,1} g_{2,0}^2 + \frac{35}{48} \pi f_{0,2} f_{0,3} g_{0,2} - \frac{7}{48} \pi f_{0,2} f_{2,1} g_{0,2} \\
& - \frac{17}{48} \pi f_{0,2} g_{0,2} g_{1,2} - \frac{1}{16} \pi f_{0,2} g_{0,2} g_{3,0} + \frac{1}{96} \pi f_{0,3} g_{0,2} g_{1,1} \\
& + \frac{1}{96} \pi f_{1,1} f_{1,2} g_{0,2} + \frac{31}{96} \pi f_{1,1} f_{3,0} g_{0,2} + \frac{5}{32} \pi f_{1,1} g_{0,2} g_{0,3} \\
& + \frac{17}{96} \pi f_{1,1} g_{0,2} g_{2,1} + \frac{7}{48} \pi f_{1,2} g_{0,2} g_{2,0} - \frac{1}{96} \pi f_{2,1} g_{0,2} g_{1,1} \\
& + \frac{61}{48} \pi f_{3,0} g_{0,2} g_{2,0} + \frac{11}{16} \pi g_{0,2} g_{0,3} g_{2,0} + \frac{17}{96} \pi g_{0,2} g_{1,1} g_{1,2} \\
& + \frac{31}{96} \pi g_{0,2} g_{1,1} g_{3,0} + \frac{9}{16} \pi g_{0,2} g_{2,0} g_{2,1} + \frac{7}{48} \pi f_{0,2} f_{1,1} g_{0,2}^2 \\
& + \frac{1}{12} \pi f_{1,1} f_{2,0} g_{0,2}^2 - \frac{5}{96} \pi f_{1,1} g_{0,2}^2 g_{1,1} - \frac{1}{6} \pi f_{2,0} g_{0,2}^2 g_{2,0} - \frac{31}{48} \pi \\
& g_{0,2}^2 g_{1,1} g_{2,0} - \frac{3}{8} \pi^2 f_{0,2} g_{0,2}^2 g_{1,1} + \frac{11}{24} \pi f_{0,2} g_{0,2}^2 g_{2,0} - \frac{3}{8} \pi^2 f_{1,1} f_{2,0}^2 g_{2,0} \\
& + \frac{3}{16} \pi^2 f_{0,2} f_{1,1} f_{1,2} + \frac{9}{16} \pi^2 f_{0,2} f_{1,1} f_{3,0} + \frac{9}{16} \pi^2 f_{0,2} f_{1,1} g_{0,3} \\
& + \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{2,1} + \frac{3}{16} \pi^2 f_{1,1} f_{1,2} f_{2,0} + \frac{9}{16} \pi^2 f_{1,1} f_{2,0} f_{3,0} \\
& + \frac{9}{16} \pi^2 f_{1,1} f_{2,0} g_{0,3} + \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{2,1} + \frac{35}{72} \pi f_{0,2}^2 f_{1,1} f_{2,0} + \frac{35}{288} \pi \\
& f_{0,2}^2 f_{1,1} g_{1,1} + \frac{61}{144} \pi f_{0,2} f_{1,1} f_{2,0}^2 + \frac{1}{288} \pi f_{0,2} f_{1,1} g_{1,1}^2 - \frac{17}{288} \pi f_{1,1} f_{2,0}^2 g_{1,1} \\
& + \frac{1}{288} \pi f_{1,1} f_{2,0} g_{1,1}^2 + \frac{1}{32} \pi f_{0,2} f_{1,1} g_{1,2} - \frac{1}{32} \pi f_{0,2} f_{1,1} g_{3,0} \\
& + \frac{31}{96} \pi f_{0,3} f_{1,1} f_{2,0} + \frac{1}{12} \pi f_{0,3} f_{1,1} g_{1,1} + \frac{17}{96} \pi f_{1,1} f_{2,0} f_{2,1} \\
& - \frac{1}{96} \pi f_{1,1} f_{2,0} g_{1,2} + \frac{1}{96} \pi f_{1,1} f_{2,0} g_{3,0} + \frac{1}{12} \pi f_{1,1} g_{1,1} g_{3,0} + \frac{3}{16} \pi^2 f_{0,2} \\
& f_{1,1}^2 f_{2,0} + \frac{3}{8} \pi^2 f_{0,2}^2 f_{1,1} g_{0,2} + \frac{3}{16} \pi^2 g_{0,2} g_{1,1}^2 g_{2,0} + \frac{3}{8} \pi^2 f_{0,2} f_{1,2} g_{0,2} \\
& + \frac{9}{8} \pi^2 f_{0,2} f_{3,0} g_{0,2} + \frac{9}{8} \pi^2 f_{0,2} g_{0,2} g_{0,3} + \frac{3}{8} \pi^2 f_{0,2} g_{0,2} g_{2,1} \\
& - \frac{3}{16} \pi^2 f_{1,2} g_{0,2} g_{1,1} - \frac{9}{16} \pi^2 f_{3,0} g_{0,2} g_{1,1} - \frac{9}{16} \pi^2 g_{0,2} g_{0,3} g_{1,1} \\
& - \frac{3}{16} \pi^2 g_{0,2} g_{1,1} g_{2,1} + \frac{1}{8} \pi f_{0,2} g_{0,2} g_{2,0}^2 + \frac{1}{3} \pi f_{0,3} f_{2,0} g_{0,2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{12} \pi f_{2,0} f_{2,1} g_{0,2} - \frac{5}{12} \pi f_{2,0} g_{0,2} g_{1,2} + \frac{1}{3} \pi f_{2,0} g_{0,2} g_{3,0} \\
& + \frac{35}{288} \pi f_{0,2} f_{1,2} g_{1,1} - \frac{1}{96} \pi f_{0,2} f_{3,0} g_{1,1} + \frac{5}{32} \pi f_{0,2} g_{0,3} g_{1,1} \\
& - \frac{13}{288} \pi f_{0,2} g_{1,1} g_{2,1} + \frac{65}{144} \pi f_{0,2} f_{1,2} f_{2,0} + \frac{17}{48} \pi f_{0,2} f_{2,0} f_{3,0} \\
& + \frac{15}{16} \pi f_{0,2} f_{2,0} g_{0,3} + \frac{5}{144} \pi f_{0,2} f_{2,0} g_{2,1} + \frac{19}{288} \pi f_{1,2} f_{2,0} g_{1,1} \\
& - \frac{17}{96} \pi f_{2,0} f_{3,0} g_{1,1} - \frac{1}{96} \pi f_{2,0} g_{0,3} g_{1,1} - \frac{29}{288} \pi f_{2,0} g_{1,1} g_{2,1} \\
& - \frac{3}{8} \pi^2 f_{1,2} f_{2,0} g_{2,0} - \frac{3}{16} \pi^2 f_{1,2} g_{1,1} g_{2,0} - \frac{9}{8} \pi^2 f_{2,0} f_{3,0} g_{2,0} \\
& - \frac{9}{8} \pi^2 f_{2,0} g_{0,3} g_{2,0} - \frac{3}{8} \pi^2 f_{2,0} g_{2,0} g_{2,1} - \frac{9}{16} \pi^2 f_{3,0} g_{1,1} g_{2,0} \\
& - \frac{9}{16} \pi^2 g_{0,3} g_{1,1} g_{2,0} - \frac{3}{16} \pi^2 g_{1,1} g_{2,0} g_{2,1} - \frac{5}{72} \pi f_{0,2}^2 f_{2,0} g_{2,0} - \frac{5}{144} \pi \\
& f_{0,2}^2 g_{1,1} g_{2,0} - \frac{17}{72} \pi f_{0,2} f_{2,0}^2 g_{2,0} - \frac{3}{32} \pi g_{0,3} g_{3,0} - \frac{1}{32} \pi g_{3,0} f_{1,2} \\
& - \frac{5}{32} \pi g_{3,0} g_{2,1} - \frac{15}{32} \pi g_{3,0} f_{3,0} + \frac{1}{32} \pi f_{2,1} g_{0,3} + \frac{3}{32} \pi f_{2,1} f_{1,2} \\
& - \frac{1}{32} \pi f_{2,1} g_{2,1} + \frac{5}{32} \pi f_{2,1} f_{3,0} - \frac{5}{32} \pi g_{0,3} g_{1,2} + \frac{1}{32} \pi g_{1,2} f_{1,2} \\
& - \frac{3}{32} \pi g_{1,2} g_{2,1}
\end{aligned}$$

```

> L[1]:=(1/4)*Pi*f[1, 2]+(1/4)*Pi*g[2, 1]+(3/4)*Pi*f[3, 0]+(3/4)*
Pi*g[0, 3]-(1/4)*Pi*g[0, 2]*g[1, 1]+(1/4)*Pi*f[0, 2]*f[1, 1]+
(1/4)*Pi*f[1, 1]*f[2, 0]-(1/4)*Pi*g[1, 1]*g[2, 0]-(1/2)*f[2, 0]*g
[2, 0]*Pi+(1/2)*Pi*f[0, 2]*g[0, 2]:
> L[2]:=(1/12)*Pi*f[1, 3]*g[1, 1]-(1/8)*Pi*g[1, 3]*g[2, 0]-(1/12)*
Pi*f[2, 2]*g[2, 0]-(5/6)*Pi*f[4, 0]*g[2, 0]+(3/32)*Pi^2*f[1, 2]
^2+(27/32)*Pi^2*f[3, 0]^2+(27/32)*Pi^2*g[0, 3]^2+(3/32)*Pi^2*g[2,
1]^2+(1/8)*Pi*f[1, 4]+(1/8)*Pi*f[3, 2]+(5/8)*Pi*f[5, 0]+(5/8)*Pi*
g[0, 5]+(1/8)*Pi*g[2, 3]+(1/8)*Pi*g[4, 1]-(1/36)*Pi*f[0, 2]*f[2,
0]*g[1, 1]*g[2, 0]+(35/96)*Pi*g[1, 1]*g[2, 0]*g[3, 0]+(3/8)*Pi^2*
f[2, 0]*g[1, 1]*g[2, 0]^2+(3/8)*Pi^2*f[2, 0]^2*g[2, 0]^2-(5/24)*
Pi*g[2, 0]*g[3, 1]+(5/24)*Pi*g[0, 4]*g[1, 1]+(1/4)*Pi*f[2, 0]*g
[0, 2]^3-(1/24)*Pi*g[1, 1]*g[2, 2]-(1/8)*Pi*g[1, 1]*g[4, 0]-
(1/32)*Pi*g[1, 2]*f[3, 0]+(3/32)*Pi^2*g[1, 1]^2*g[2, 0]^2+(1/32)*
Pi*f[0, 3]*g[2, 1]+(5/32)*Pi*f[0, 3]*f[1, 2]+(15/32)*Pi*f[0, 3]*g
[0, 3]-(13/288)*Pi*g[1, 1]^3*g[2, 0]+(11/16)*Pi*g[0, 2]*g[0, 3]*g
[2, 0]+(11/24)*Pi*f[0, 2]*g[0, 2]^2*g[2, 0]+(5/24)*Pi*f[0, 2]*f
[1, 3]-(13/48)*Pi*g[2, 0]^3*g[1, 1]+(3/32)*Pi*f[0, 3]*f[3, 0]-
(13/24)*Pi*g[2, 0]^3*f[2, 0]+(1/16)*Pi*f[1, 2]*g[2, 0]^2+(13/16)*
Pi*f[3, 0]*g[2, 0]^2+(7/96)*Pi*f[1, 1]^3*f[2, 0]+(3/16)*Pi*g[0,
3]*g[2, 0]^2+(13/48)*Pi*g[2, 0]^2*g[2, 1]+(7/96)*Pi*f[0, 2]*f[1,
1]^3+(7/96)*Pi*f[1, 1]^2*f[1, 2]-(9/16)*Pi^2*f[3, 0]*g[0, 2]*g[1,
1]-(13/36)*Pi*f[2, 0]^3*g[2, 0]-(1/4)*Pi*f[2, 0]*g[4, 0]-(15/32)*
Pi*g[3, 0]*f[3, 0]-(11/12)*Pi*f[4, 0]*g[0, 2]-(1/6)*Pi*f[2, 2]*g

```

$[0, 2]+(35/144)*\text{Pi}^*f[0, 2]^2*f[1, 2]-(1/32)*\text{Pi}^*f[0, 2]*f[1, 1]*g$
 $[3, 0]-(17/288)*\text{Pi}^*f[1, 1]*f[2, 0]^2*g[1, 1]+(1/288)*\text{Pi}^*f[1, 1]*f$
 $[2, 0]*g[1, 1]^2+(3/8)*\text{Pi}^2*f[0, 2]^2*f[1, 1]*g[0, 2]+(3/16)*$
 $\text{Pi}^2*f[0, 2]*f[1, 1]^2*f[2, 0]+(5/32)*\text{Pi}^*f[2, 1]*f[3, 0]-(5/12)*$
 $\text{Pi}^*f[2, 0]*f[2, 1]*g[0, 2]+(7/12)*\text{Pi}^*f[0, 2]*g[0, 2]^3-(5/32)*\text{Pi}^*$
 $g[3, 0]*g[2, 1]-(1/32)*\text{Pi}^*g[3, 0]*f[1, 2]+(17/96)*\text{Pi}^*g[0, 2]*g[1,$
 $1]*g[1, 2]+(31/96)*\text{Pi}^*g[0, 2]*g[1, 1]*g[3, 0]+(1/12)*\text{Pi}^*f[1, 1]*g$
 $[1, 1]*g[3, 0]+(1/96)*\text{Pi}^*f[1, 1]*f[2, 0]*g[3, 0]+(35/48)*\text{Pi}^*f[0,$
 $2]^2*g[0, 3]+(9/16)*\text{Pi}^2*f[1, 1]*f[2, 0]*g[0, 3]+(9/16)*\text{Pi}^2*f[1,$
 $1]*f[2, 0]*f[3, 0]-(5/12)*\text{Pi}^*f[2, 0]*g[0, 2]*g[1, 2]+(1/96)*\text{Pi}^*f$
 $[1, 1]*f[1, 2]*g[0, 2]+(1/96)*\text{Pi}^*f[0, 3]*g[0, 2]*g[1, 1]-(1/16)*$
 $\text{Pi}^*f[0, 2]*g[0, 2]*g[3, 0]+(1/3)*\text{Pi}^*f[0, 3]*f[2, 0]*g[0, 2]+(1/8)$
 $*\text{Pi}^*f[0, 2]*g[0, 2]*g[2, 0]^2-(17/48)*\text{Pi}^*f[0, 2]*g[0, 2]*g[1, 2]+$
 $(35/48)*\text{Pi}^*f[0, 2]*f[0, 3]*g[0, 2]-(5/8)*\text{Pi}^*g[0, 2]*g[1, 1]*g[2,$
 $0]^2-(1/4)*\text{Pi}^*f[2, 0]^3*g[0, 2]+(1/4)*\text{Pi}^*f[0, 4]*g[0, 2]-(17/24)*$
 $\text{Pi}^*f[2, 0]*g[0, 2]*g[2, 0]^2+(1/12)*\text{Pi}^*f[1, 1]*f[2, 0]*g[0, 2]^2+$
 $(13/288)*\text{Pi}^*g[1, 1]^2*g[2, 1]+(65/144)*\text{Pi}^*f[0, 2]*f[1, 2]*f[2, 0]$
 $-(13/288)*\text{Pi}^*f[0, 2]*g[1, 1]*g[2, 1]+(9/16)*\text{Pi}^2*f[3, 0]*g[2, 1]+$
 $(1/32)*\text{Pi}^*g[1, 2]*f[1, 2]+(1/288)*\text{Pi}^*f[0, 2]*f[1, 1]*g[1, 1]^2+$
 $(5/96)*\text{Pi}^*g[0, 3]*g[1, 1]^2+(1/288)*\text{Pi}^*f[1, 2]*g[1, 1]^2+(1/96)*$
 $\text{Pi}^*f[3, 0]*g[1, 1]^2+(3/32)*\text{Pi}^2*f[1, 1]^2*f[2, 0]^2+(3/32)*\text{Pi}^2*$
 $f[0, 2]^2*f[1, 1]^2-(1/12)*\text{Pi}^*f[1, 1]*g[3, 1]+(3/32)*\text{Pi}^*f[2, 1]*f$
 $[1, 2]+(5/32)*\text{Pi}^*f[0, 2]*g[0, 3]*g[1, 1]+(41/288)*\text{Pi}^*f[2, 0]*g[0,$
 $2]*g[1, 1]^2+(1/36)*\text{Pi}^*f[2, 0]^2*g[0, 2]*g[1, 1]+(27/16)*\text{Pi}^2*f$
 $[3, 0]*g[0, 3]-(7/48)*\text{Pi}^*f[0, 2]*f[2, 1]*g[0, 2]+(35/48)*\text{Pi}^*f[2,$
 $0]*g[2, 0]*g[3, 0]+(1/32)*\text{Pi}^*f[2, 1]*g[1, 1]*g[2, 0]+(13/96)*\text{Pi}^*g$
 $[1, 1]*g[1, 2]*g[2, 0]-(17/96)*\text{Pi}^*f[1, 1]*g[1, 1]*g[2, 0]^2-$
 $(1/32)*\text{Pi}^*f[1, 1]^2*g[1, 1]*g[2, 0]-(7/48)*\text{Pi}^*f[2, 0]*g[1, 2]*g$
 $[2, 0]+(31/96)*g[2, 0]*\text{Pi}^*f[1, 1]*f[3, 0]+(5/32)*g[2, 0]*\text{Pi}^*f[1,$
 $1]*g[0, 3]+(17/96)*g[2, 0]*\text{Pi}^*f[1, 1]*g[2, 1]+(1/16)*\text{Pi}^*f[0, 2]*f$
 $[1, 1]*g[2, 0]^2-(1/12)*\text{Pi}^*f[1, 1]*f[2, 0]*g[2, 0]^2+(35/96)*\text{Pi}^*f$
 $[0, 2]*f[0, 3]*f[1, 1]+(13/96)*\text{Pi}^*f[0, 2]*f[1, 1]*f[2, 1]+(1/96)*$
 $g[2, 0]*\text{Pi}^*f[0, 2]*f[1, 1]^2-(3/32)*g[2, 0]*\text{Pi}^*f[1, 1]^2*f[2, 0]+$
 $(1/96)*g[2, 0]*\text{Pi}^*f[1, 1]*f[1, 2]-(3/8)*\text{Pi}^2*f[0, 2]*g[0, 2]^2*g$
 $[1, 1]+(17/48)*\text{Pi}^*f[0, 2]*f[2, 0]*f[3, 0]+(5/48)*\text{Pi}^*f[0, 2]^2*f$
 $[3, 0]-(3/16)*\text{Pi}^2*g[1, 1]*g[2, 0]*g[2, 1]-(31/48)*\text{Pi}^*g[0, 2]^2*g$
 $[1, 1]*g[2, 0]+(7/24)*\text{Pi}^*f[1, 3]*f[2, 0]-(3/8)*\text{Pi}^2*f[1, 1]*f[2,$
 $0]^2*g[2, 0]-(1/32)*\text{Pi}^*f[2, 1]*g[2, 1]-(5/72)*\text{Pi}^*f[0, 2]^2*f[2,$
 $0]*g[2, 0]+(31/96)*\text{Pi}^*f[0, 3]*f[1, 1]*f[2, 0]-(1/6)*\text{Pi}^*f[2, 0]*g$
 $[0, 2]^2*g[2, 0]-(9/16)*\text{Pi}^2*g[0, 3]*g[1, 1]*g[2, 0]-(9/16)*\text{Pi}^2*$
 $f[3, 0]*g[1, 1]*g[2, 0]-(1/96)*\text{Pi}^*f[2, 1]*g[0, 2]*g[1, 1]+(7/48)*$
 $\text{Pi}^*f[1, 2]*g[0, 2]*g[2, 0]+(19/288)*\text{Pi}^*f[1, 2]*f[2, 0]*g[1, 1]-$
 $(5/72)*\text{Pi}^*f[2, 0]^2*g[2, 1]+(61/48)*\text{Pi}^*f[3, 0]*g[0, 2]*g[2, 0]+$
 $(1/6)*\text{Pi}^*f[2, 0]*g[2, 2]+(11/12)*\text{Pi}^*f[2, 0]*g[0, 4]+(3/8)*\text{Pi}^*f[2,$
 $0]*f[3, 1]+(17/96)*\text{Pi}^*f[1, 1]*g[0, 2]*g[2, 1]-(1/32)*\text{Pi}^*f[1, 1]$
 $^2*g[0, 2]*g[1, 1]-(3/16)*\text{Pi}^2*g[0, 2]*g[1, 1]*g[2, 1]-(3/16)*$
 $\text{Pi}^2*f[1, 2]*g[1, 1]*g[2, 0]-(1/96)*\text{Pi}^*f[0, 2]*f[3, 0]*g[1, 1]+$
 $(35/288)*\text{Pi}^*f[0, 2]*f[1, 2]*g[1, 1]+(61/144)*\text{Pi}^*f[0, 2]*f[1, 1]*f$
 $[2, 0]^2+(35/288)*\text{Pi}^*f[0, 2]^2*f[1, 1]*g[1, 1]+(5/144)*\text{Pi}^*f[0, 2]$
 $*f[2, 0]*g[2, 1]+(15/16)*\text{Pi}^*f[0, 2]*f[2, 0]*g[0, 3]+(7/48)*\text{Pi}^*f$
 $[0, 2]*f[1, 1]*g[0, 2]^2+(9/16)*\text{Pi}^*g[0, 2]*g[2, 0]*g[2, 1]-(5/96)$
 $*\text{Pi}^*f[1, 1]*g[0, 2]^2*g[1, 1]-(17/72)*\text{Pi}^*f[0, 2]*f[2, 0]^2*g[2,$
 $0]-(1/16)*\text{Pi}^*f[0, 3]*f[2, 0]*g[2, 0]-(1/12)*\text{Pi}^*f[0, 2]*g[1, 2]*g$
 $[2, 0]-(1/12)*\text{Pi}^*f[0, 2]*f[2, 1]*g[2, 0]+(3/32)*\text{Pi}^*f[2, 0]*g[1,$
 $1]^2*g[2, 0]+(3/16)*\text{Pi}^*f[2, 0]^2*g[1, 1]*g[2, 0]+(13/288)*\text{Pi}^*f[0,$
 $2]*g[1, 1]^2*g[2, 0]+(17/96)*\text{Pi}^*f[1, 1]*f[2, 0]*f[2, 1]+(1/32)*$
 $\text{Pi}^*f[2, 1]*g[0, 3]+(3/16)*\text{Pi}^2*f[1, 2]*g[2, 1]-(1/32)*\text{Pi}^*f[1, 1]$
 $^2*f[2, 0]*g[0, 2]+(9/16)*\text{Pi}^2*f[1, 2]*g[0, 3]+(9/16)*\text{Pi}^2*f[0,$
 $2]*f[1, 1]*f[3, 0]+(3/32)*\text{Pi}^*f[0, 2]*g[0, 2]*g[1, 1]^2+(1/3)*\text{Pi}^*f$
 $[2, 0]*g[0, 2]*g[3, 0]-(9/8)*\text{Pi}^2*f[2, 0]*f[3, 0]*g[2, 0]+(9/8)*$
 $\text{Pi}^2*f[0, 2]*g[0, 2]*g[0, 3]-(3/32)*\text{Pi}^*g[0, 3]*g[3, 0]+(3/16)*$
 $\text{Pi}^2*f[0, 2]*f[1, 1]*f[1, 2]+(13/24)*\text{Pi}^*f[2, 0]^2*f[3, 0]+(9/16)*$

```

Pi^2*f[1, 2]*f[3, 0]+(9/8)*Pi^2*f[0, 2]*f[3, 0]*g[0, 2]+(1/12)*
Pi*f[0, 3]*f[1, 1]*g[1, 1]-(1/32)*Pi*f[0, 3]*g[1, 1]*g[2, 0]+
(3/8)*Pi^2*f[0, 2]*f[1, 2]*g[0, 2]-(5/24)*Pi*f[1, 1]*f[4, 0]+
(1/24)*Pi*f[1, 1]*f[2, 2]+(1/8)*Pi*f[0, 4]*f[1, 1]+(13/72)*Pi*f
[1, 1]*f[2, 0]^3+(35/144)*Pi*f[0, 2]^3*f[1, 1]+(3/32)*Pi^2*g[0,
2]^2*g[1, 1]^2+(3/8)*Pi^2*f[0, 2]^2*g[0, 2]^2+(5/12)*Pi*g[0, 2]
^2*g[2, 1]-(7/24)*Pi*g[0, 2]^3*g[1, 1]-(17/96)*Pi*f[2, 0]*f[3, 0]
*g[1, 1]-(17/48)*Pi*f[2, 0]*f[2, 1]*g[2, 0]-(1/18)*Pi*f[0, 2]*f
[2, 0]^2*g[0, 2]+(7/96)*Pi*f[0, 2]*f[1, 1]^2*g[0, 2]+(35/72)*Pi*f
[0, 2]^2*f[2, 0]*g[0, 2]+(3/16)*Pi^2*g[0, 2]*g[1, 1]^2*g[2, 0]-
(5/32)*Pi*g[0, 3]*g[1, 2]+(5/144)*Pi*f[0, 2]^2*g[2, 1]-(7/24)*Pi*
g[0, 2]*g[3, 1]+(3/16)*Pi^2*f[1, 1]*f[1, 2]*f[2, 0]+(1/12)*Pi*f
[0, 2]*g[2, 2]+(1/8)*Pi*f[0, 2]*f[3, 1]+(35/72)*Pi*f[0, 2]^2*f[1,
1]*f[2, 0]-(5/144)*Pi*f[0, 2]^2*g[1, 1]*g[2, 0]+(3/16)*Pi^2*f[0,
2]*f[1, 1]*g[2, 1]+(9/16)*Pi^2*f[0, 2]*f[1, 1]*g[0, 3]+(5/32)*Pi*
f[1, 1]*g[0, 2]*g[0, 3]-(3/8)*Pi^2*f[1, 2]*f[2, 0]*g[2, 0]-
(29/288)*Pi*f[2, 0]*g[1, 1]*g[2, 1]-(9/16)*Pi^2*g[0, 2]*g[0, 3]*g
[1, 1]-(1/96)*Pi*f[2, 0]*g[0, 3]*g[1, 1]+(1/4)*Pi*f[0, 2]*f[1, 1]
*g[0, 2]*g[2, 0]+(1/24)*Pi*f[1, 1]*f[2, 0]*g[0, 2]*g[2, 0]-
(11/48)*Pi*f[1, 1]*g[0, 2]*g[1, 1]*g[2, 0]-(3/8)*Pi^2*f[0, 2]*f
[1, 1]*f[2, 0]*g[2, 0]-(3/16)*Pi^2*f[0, 2]*f[1, 1]*g[1, 1]*g[2,
0]-(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[1, 1]*g[2, 0]+(1/16)*Pi*f[0, 2]*
f[1, 1]*f[2, 0]*g[1, 1]+(3/8)*Pi^2*f[0, 2]*f[1, 1]*f[2, 0]*g[0,
2]-(3/16)*Pi^2*f[0, 2]*f[1, 1]*g[0, 2]*g[1, 1]-(3/4)*Pi^2*f[0, 2]
*f[2, 0]*g[0, 2]*g[2, 0]-(3/8)*Pi^2*f[0, 2]*g[0, 2]*g[1, 1]*g[2,
0]-(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[0, 2]*g[1, 1]+(3/8)*Pi^2*f[2, 0]
*g[0, 2]*g[1, 1]*g[2, 0]-(11/72)*Pi*f[0, 2]*f[2, 0]*g[0, 2]*g[1,
1]+(31/96)*Pi*f[1, 1]*f[3, 0]*g[0, 2]+(11/36)*Pi*f[1, 2]*f[2, 0]
^2+(1/3)*Pi*f[2, 0]^2*g[0, 3]-(3/8)*Pi*g[0, 2]*g[1, 3]-(3/32)*Pi*
g[1, 2]*g[2, 1]+(1/32)*Pi*f[0, 2]*f[1, 1]*g[1, 2]+(5/6)*Pi*f[0,
2]*g[0, 4]+(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[2, 1]+(9/16)*Pi^2*g[0,
3]*g[2, 1]-(9/8)*Pi^2*f[2, 0]*g[0, 3]*g[2, 0]-(1/96)*Pi*f[1, 1]*f
[2, 0]*g[1, 2]-(3/8)*Pi^2*f[2, 0]*g[2, 0]*g[2, 1]+(7/8)*Pi*g[0,
2]^2*g[0, 3]+(2/3)*Pi*f[3, 0]*g[0, 2]^2+(35/72)*Pi*f[0, 2]^3*g[0,
2]-(13/288)*Pi*g[0, 2]*g[1, 1]^3+(1/24)*Pi*f[1, 2]*g[0, 2]^2+
(3/32)*Pi*f[1, 1]^2*g[0, 3]+(1/32)*Pi*f[1, 1]^2*g[2, 1]+(13/96)*
Pi*f[1, 1]^2*f[3, 0]-(3/16)*Pi^2*f[1, 2]*g[0, 2]*g[1, 1]+(3/8)*
Pi^2*f[0, 2]*g[0, 2]*g[2, 1]:

```

Coordinate change

Equation (4)

```
> ff:=x^(a1)*y^(b1)-1;
```

$$ff := x^{a1} y^{b1} - 1 \quad (2.1)$$

```
> gg:=K*(1 - x^a3*y^b3);
```

$$gg := K (1 - x^{a3} y^{b3}) \quad (2.2)$$

Equilibrium (1,1)

```
> subs({x=1,y=1},ff);
```

$$0 \quad (2.3)$$

```
> subs({x=1,y=1},gg);
```

$$0 \quad (2.4)$$

```
> Jac:=VectorCalculus[Jacobian]([ff,gg],[x,y]);
```

$$(2.5)$$

$$Jac := \begin{bmatrix} \frac{x^{a1} a1 y^{b1}}{x} & \frac{x^{a1} y^{b1} b1}{y} \\ -\frac{K x^{a3} a3 y^{b3}}{x} & -\frac{K x^{a3} y^{b3} b3}{y} \end{bmatrix} \quad (2.5)$$

> **J:=simplify(subs({x=1,y=1},Jac),symbolic);**

$$J := \begin{bmatrix} a1 & b1 \\ -K a3 & -K b3 \end{bmatrix} \quad (2.6)$$

> **tr:=LinearAlgebra[Trace](J);**

$$tr := -K b3 + a1 \quad (2.7)$$

> **alsubs:=a1=solve(tr,a1);**

$$alsubs := a1 = K b3 \quad (2.8)$$

> **detJ:=factor(subs(alsubs,LinearAlgebra[Determinant](J)));**

$$detJ := -K (K b3^2 - a3 b1) \quad (2.9)$$

> **wsubs:=w=sqrt(detJ);**

$$wsubs := w = \sqrt{-K (K b3^2 - a3 b1)} \quad (2.10)$$

> **fff:=subs(alsubs,subs({x=1+x,y=1-a1*x/b1 - y*w/b1},ff/w));**

$$fff := \frac{(1+x)^{K b3} \left(1 - \frac{K b3 x}{b1} - \frac{y w}{b1}\right)^{b1} - 1}{w} \quad (2.11)$$

> **ggg:=subs(alsubs,subs({x=1+x,y=1-a1*x/b1 - y*w/b1},-a1*ff/w^2-b1*gg/w^2));**

$$ggg := -\frac{K b3 \left((1+x)^{K b3} \left(1 - \frac{K b3 x}{b1} - \frac{y w}{b1}\right)^{b1} - 1 \right)}{w^2} - \frac{b1 K \left(1 - (1+x)^{a3} \left(1 - \frac{K b3 x}{b1} - \frac{y w}{b1}\right)^{b3}\right)}{w^2} \quad (2.12)$$

Linear part after coordinate change

> **simplify(coeftayl(fff,[x,y]=[0,0],[1,0]));**

$$0 \quad (2.13)$$

> **simplify(coeftayl(fff,[x,y]=[0,0],[0,1]));**

$$-1 \quad (2.14)$$

> **simplify(coeftayl(ggg,[x,y]=[0,0],[1,0]));**

$$-\frac{K (K b3^2 - a3 b1)}{w^2} \quad (2.15)$$

> **simplify(subs(wsubs,%));**

$$1 \quad (2.16)$$

> **simplify(coeftayl(ggg,[x,y]=[0,0],[0,1]));**

$$0 \quad (2.17)$$

Higher taylor coefficients after coordinate change


```

> for k from 2 to 5 do
  for i from 0 to k do
    ggco[i,k-i]:=simplify(coeftayl(ggg,[x,y]=[0,0],[i,k-i]));
    ffco[i,k-i]:=simplify(coeftayl(fff,[x,y]=[0,0],[i,k-i]));
  end:
end:
> subf:=seq(seq(f[i,k-i]=ffco[i,k-i],i=0..k),k=2..5):
> subg:=seq(seq(g[i,k-i]=ggco[i,k-i],i=0..k),k=2..5):

```

Case distinctions for first and second focal value

```

> L1:=simplify(subs({subf,subg},L[1])):

```

```

> L1:=simplify(subs(wsubs,L1));

```

$$L1 := -\frac{1}{8} \frac{\pi ((a^3 + b^3) K - a^3 - 1) b l - K a^3 (b^3 - 1) b^3 K}{\sqrt{-K (K b^3^2 - a^3 b l)} b l} \quad (3.1)$$

L1=0, case (a)

```

> subs(b3=0,detJ);

```

$$b l K a^3 \quad (3.2)$$

L1=0, case (b)

```

> b1sol:=solve(L1,b1);

```

$$b1sol := \frac{K a^3 (b^3 - 1)}{K a^3 + K b^3 - a^3 - 1} \quad (3.3)$$

```

> DD:=-denom(b1sol);

```

$$DD := -K a^3 - K b^3 + a^3 + 1 \quad (3.4)$$

L1=0, case (c)

```

> factor(subs(b3=1,DD));

```

$$-(a^3 + 1) (K - 1) \quad (3.5)$$

L2 in case (b)

```

> b1subs:=b1=b1sol;

```

$$b1subs := b l = \frac{K a^3 (b^3 - 1)}{K a^3 + K b^3 - a^3 - 1} \quad (3.6)$$

```

> L2:=simplify(subs({subf,subg},L[2])):

```

```

> L2:=simplify(subs(wsubs,L2)):

```

```

> L2:=simplify(subs(b1subs,L2));

```

$$L2 := \frac{1}{288} ((a^3 + b^3)^2 (K - 1) (K b^3 - 1) (K b^3 - a^3 - 1) (K b^3 - K - a^3 - 1) K \pi b^3) \left(\sqrt{-\frac{((-b^3 + 1) a^3 + K b^3^2 - b^3) K^2 (a^3 + b^3)}{(K - 1) a^3 + K b^3 - 1}} (b^3 - 1) (K a^3 + K b^3 - a^3 - 1) \right) \quad (3.7)$$

sqrt(det J):

```

> simplify(subs(b1subs,wsubs));

```

$$w = \sqrt{-\frac{K^2 (a^3 + b^3) (K b^3^2 - a^3 b^3 + a^3 - b^3)}{K a^3 + K b^3 - a^3 - 1}} \quad (3.8)$$

$$a^3 + b^3 = 0$$

$$\text{> subs}(a^3 = -b^3, DD);$$

$$-b^3 + 1 \quad (3.9)$$

$$\text{> subs}(b^3 = -a^3, b1\text{subs});$$

$$b1 = K a^3 \quad (3.10)$$

$$1 + a^3 - b^3 K = 0$$

$$\text{> solve}(1 + a^3 - b^3 * K, b^3);$$

$$\frac{a^3 + 1}{K} \quad (3.11)$$

$$\text{> simplify}(\text{subs}(b^3 = \%, DD));$$

$$-K a^3 \quad (3.12)$$

$$b^3 K = 1$$

$$\text{> simplify}(\text{subs}(b^3 = 1/K, b1\text{subs}));$$

$$b1 = -1 \quad (3.13)$$

$$K = 1$$

$$\text{> simplify}(\text{subs}(K = 1, b1\text{subs}));$$

$$b1 = a^3 \quad (3.14)$$

$$1 + a^3 + K - b^3 K = 0$$

$$\text{> solve}(1 + a^3 + K - b^3 * K, b^3);$$

$$\frac{K + a^3 + 1}{K} \quad (3.15)$$

$$\text{> simplify}(\text{subs}(b^3 = \%, b1\text{subs}));$$

$$b1 = \frac{a^3}{K} \quad (3.16)$$

$$\text{> K=solve}(\text{subs}(\{a^3 = b1 * K\}, 1 + a^3 + K - b^3 * K), K);$$

$$K = -\frac{1}{b1 - b^3 + 1} \quad (3.17)$$

Case (c1)

$$\text{> subs}(\{b^3 = 1, a^3 = -1\}, \text{detJ});$$

$$-K (K + b1) \quad (3.18)$$

L2 in case (c2)

$$\text{> L2:=simplify}(\text{subs}(\{\text{subf}, \text{subg}\}, L[2]));$$

$$\text{> L2:=simplify}(\text{subs}(w\text{subs}, L2));$$

$$\text{> L2:=simplify}(\text{subs}(\{b^3 = 1, K = 1\}, L2));$$

$$L2 := \frac{1}{288} \frac{(b1 + 1) (a^3 + 1) (a^3 - b1) \pi a^3}{\sqrt{a^3 b1 - 1} b1} \quad (3.19)$$