

Focal values

Computation of general focal values based on the code in

O. A. Kuznetsova, An example of symbolic computation of Lyapunov quantities in Maple, in Proceedings of BICA'12, 2012, 195–198.

```
> m := 2; #number of focal values
      m := 2 (1.1)
```

```
> n := 2*m+1; #degree of taylor polynomials
      n := 5 (1.2)
```

```
> # Taylor polynomial def of functions f and g
```

Taylor polynomials

```
> f(x,y):=add(add(f[i,k-i]*x^i*y^(k-i),i=0..k),k=2..n);
f(x,y) := x^5 f_{5,0} + x^4 y f_{4,1} + x^3 y^2 f_{3,2} + x^2 y^3 f_{2,3} + x y^4 f_{1,4} + y^5 f_{0,5} + x^4 f_{4,0}
      + x^3 y f_{3,1} + x^2 y^2 f_{2,2} + x y^3 f_{1,3} + y^4 f_{0,4} + x^3 f_{3,0} + x^2 y f_{2,1} + x y^2 f_{1,2} + y^3 f_{0,3}
      + x^2 f_{2,0} + x y f_{1,1} + y^2 f_{0,2} (1.3)
```

```
> g(x,y):=add(add(g[i,k-i]*x^i*y^(k-i),i=0..k),k=2..n);
g(x,y) := x^5 g_{5,0} + x^4 y g_{4,1} + x^3 y^2 g_{3,2} + x^2 y^3 g_{2,3} + x y^4 g_{1,4} + y^5 g_{0,5} + x^4 g_{4,0}
      + x^3 y g_{3,1} + x^2 y^2 g_{2,2} + x y^3 g_{1,3} + y^4 g_{0,4} + x^3 g_{3,0} + x^2 y g_{2,1} + x y^2 g_{1,2}
      + y^3 g_{0,3} + x^2 g_{2,0} + x y g_{1,1} + y^2 g_{0,2} (1.4)
```

Transformation to the polar coordinates

```
> f(r,theta):=subs(x=r*cos(theta),y=r*sin(theta),f(x,y));
> g(r,theta):=subs(x=r*cos(theta),y=r*sin(theta),g(x,y));
> dr:=f(r,theta)*cos(theta)+g(r,theta)*sin(theta);
  dt:=simplify(1-f(r,theta)*sin(theta)/r+g(r,theta)*cos(theta)/r)
  :
> drdt := dr/dt:
```

```
> f(theta,r[0]):=0:
```

```
> R(theta,r) := 0:
```

```
> for i from 1 to n do
  f(theta,r[0]):=f(theta,r[0])+u[i](theta)*r[0]^i;
  R[i](theta):=subs(r=0,diff(drdt,[r$ i])/i!);
  R(theta,r):=R(theta,r)+R[i](theta)*r^i;
end:
> E:=diff(f(theta,r[0]),theta)-subs(r=f(theta,r[0]),R(theta,r));
> ic := Vector[row]([1,0,0,0,0]):
```

```
> for i from 1 to n do
  S:=int(diff(u[i](theta),theta)-coeff(E,r[0],i),theta);
  u[i](theta):=simplify(S-subs(theta=0,S)+ic[i]);
end:
```

Focal values:

```
> for i from 1 to m do
```

L[i] := (simplify@subs) (theta=2*Pi, u[2*i+1] (theta));
end;

$$\begin{aligned}
L_1 &:= \frac{1}{4} \pi f_{1,2} + \frac{1}{4} \pi g_{2,1} + \frac{3}{4} \pi f_{3,0} + \frac{3}{4} \pi g_{0,3} - \frac{1}{4} \pi g_{0,2} g_{1,1} + \frac{1}{4} \pi f_{0,2} f_{1,1} \\
&\quad + \frac{1}{4} \pi f_{1,1} f_{2,0} - \frac{1}{4} \pi g_{1,1} g_{2,0} - \frac{1}{2} f_{2,0} g_{2,0} \pi + \frac{1}{2} \pi f_{0,2} g_{0,2} \\
L_2 &:= -\frac{13}{288} \pi g_{1,1}^3 g_{2,0} + \frac{5}{24} \pi f_{0,2} f_{1,3} - \frac{13}{48} \pi g_{2,0}^3 g_{1,1} + \frac{3}{32} \pi f_{0,3} f_{3,0} - \frac{13}{24} \pi \\
&\quad g_{2,0}^3 f_{2,0} + \frac{1}{16} \pi f_{1,2} g_{2,0}^2 + \frac{13}{16} \pi f_{3,0} g_{2,0}^2 + \frac{7}{96} \pi f_{1,1}^3 f_{2,0} + \frac{3}{16} \pi g_{0,3} g_{2,0}^2 \\
&\quad + \frac{13}{48} \pi g_{2,0}^2 g_{2,1} + \frac{7}{96} \pi f_{0,2} f_{1,1}^3 + \frac{7}{96} \pi f_{1,1}^2 f_{1,2} - \frac{13}{36} \pi f_{2,0}^3 g_{2,0} \\
&\quad - \frac{1}{4} \pi f_{2,0} g_{4,0} - \frac{15}{32} \pi g_{3,0} f_{3,0} - \frac{11}{12} \pi f_{4,0} g_{0,2} - \frac{1}{6} \pi f_{2,2} g_{0,2} + \frac{35}{144} \pi \\
&\quad f_{0,2}^2 f_{1,2} + \frac{5}{32} \pi f_{2,1} f_{3,0} + \frac{7}{12} \pi f_{0,2} g_{0,2}^3 - \frac{5}{32} \pi g_{3,0} g_{2,1} - \frac{1}{32} \pi g_{3,0} f_{1,2} \\
&\quad + \frac{35}{48} \pi f_{0,2}^2 g_{0,3} - \frac{1}{4} \pi f_{2,0}^3 g_{0,2} + \frac{1}{4} \pi f_{0,4} g_{0,2} + \frac{13}{288} \pi g_{1,1}^2 g_{2,1} \\
&\quad + \frac{9}{16} \pi^2 f_{3,0} g_{2,1} + \frac{1}{32} \pi g_{1,2} f_{1,2} + \frac{5}{96} \pi g_{0,3} g_{1,1}^2 + \frac{1}{288} \pi f_{1,2} g_{1,1}^2 \\
&\quad + \frac{1}{96} \pi f_{3,0} g_{1,1}^2 + \frac{3}{32} \pi^2 f_{1,1}^2 f_{2,0} + \frac{3}{32} \pi^2 f_{0,2}^2 f_{1,1}^2 - \frac{1}{12} \pi f_{1,1} g_{3,1} \\
&\quad + \frac{3}{32} \pi f_{2,1} f_{1,2} + \frac{27}{16} \pi^2 f_{3,0} g_{0,3} + \frac{5}{48} \pi f_{0,2}^2 f_{3,0} + \frac{7}{24} \pi f_{1,3} f_{2,0} \\
&\quad - \frac{1}{32} \pi f_{2,1} g_{2,1} - \frac{5}{72} \pi f_{2,0}^2 g_{2,1} + \frac{1}{6} \pi f_{2,0} g_{2,2} + \frac{11}{12} \pi f_{2,0} g_{0,4} \\
&\quad + \frac{3}{8} \pi f_{2,0} f_{3,1} + \frac{1}{32} \pi f_{2,1} g_{0,3} + \frac{3}{16} \pi^2 f_{1,2} g_{2,1} + \frac{9}{16} \pi^2 f_{1,2} g_{0,3} \\
&\quad - \frac{3}{32} \pi g_{0,3} g_{3,0} + \frac{13}{24} \pi f_{2,0}^2 f_{3,0} + \frac{9}{16} \pi^2 f_{1,2} f_{3,0} - \frac{5}{24} \pi f_{1,1} f_{4,0} \\
&\quad + \frac{1}{24} \pi f_{1,1} f_{2,2} + \frac{1}{8} \pi f_{0,4} f_{1,1} + \frac{13}{72} \pi f_{1,1} f_{2,0}^3 + \frac{35}{144} \pi f_{0,2}^3 f_{1,1} + \frac{3}{32} \pi^2 g_{0,2}^2 \\
&\quad g_{1,1}^2 + \frac{3}{8} \pi^2 f_{0,2}^2 g_{0,2}^2 + \frac{5}{12} \pi g_{0,2}^2 g_{2,1} - \frac{7}{24} \pi g_{0,2}^3 g_{1,1} - \frac{5}{32} \pi g_{0,3} g_{1,2} \\
&\quad + \frac{5}{144} \pi f_{0,2}^2 g_{2,1} - \frac{7}{24} \pi g_{0,2} g_{3,1} + \frac{1}{12} \pi f_{0,2} g_{2,2} + \frac{1}{8} \pi f_{0,2} f_{3,1} \\
&\quad + \frac{11}{36} \pi f_{1,2} f_{2,0}^2 + \frac{1}{3} \pi f_{2,0}^2 g_{0,3} - \frac{3}{8} \pi g_{0,2} g_{1,3} - \frac{3}{32} \pi g_{1,2} g_{2,1} \\
&\quad + \frac{5}{6} \pi f_{0,2} g_{0,4} + \frac{9}{16} \pi^2 g_{0,3} g_{2,1} + \frac{7}{8} \pi g_{0,2}^2 g_{0,3} + \frac{2}{3} \pi f_{3,0} g_{0,2}^2 + \frac{35}{72} \pi
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
& f_{0,2}^3 g_{0,2} - \frac{13}{288} \pi g_{0,2} g_{1,1}^3 + \frac{1}{24} \pi f_{1,2} g_{0,2}^2 + \frac{3}{32} \pi f_{1,1}^2 g_{0,3} + \frac{1}{32} \pi f_{1,1}^2 g_{2,1} \\
& + \frac{13}{96} \pi f_{1,1}^2 f_{3,0} - \frac{1}{36} \pi f_{0,2} f_{2,0} g_{1,1} g_{2,0} + \frac{1}{4} \pi f_{0,2} f_{1,1} g_{0,2} g_{2,0} \\
& + \frac{1}{24} \pi f_{1,1} f_{2,0} g_{0,2} g_{2,0} - \frac{11}{48} \pi f_{1,1} g_{0,2} g_{1,1} g_{2,0} - \frac{3}{8} \pi^2 f_{0,2} f_{1,1} f_{2,0} g_{2,0} \\
& - \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{1,1} g_{2,0} - \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{1,1} g_{2,0} + \frac{1}{16} \pi f_{0,2} f_{1,1} f_{2,0} g_{1,1} \\
& + \frac{3}{8} \pi^2 f_{0,2} f_{1,1} f_{2,0} g_{0,2} - \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{0,2} g_{1,1} - \frac{3}{4} \pi^2 f_{0,2} f_{2,0} g_{0,2} g_{2,0} \\
& - \frac{3}{8} \pi^2 f_{0,2} g_{0,2} g_{1,1} g_{2,0} - \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{0,2} g_{1,1} + \frac{3}{8} \pi^2 f_{2,0} g_{0,2} g_{1,1} g_{2,0} \\
& - \frac{11}{72} \pi f_{0,2} f_{2,0} g_{0,2} g_{1,1} + \frac{35}{96} \pi g_{1,1} g_{2,0} g_{3,0} + \frac{3}{8} \pi^2 f_{2,0} g_{1,1} g_{2,0}^2 \\
& + \frac{11}{16} \pi g_{0,2} g_{0,3} g_{2,0} + \frac{11}{24} \pi f_{0,2} g_{0,2}^2 g_{2,0} - \frac{9}{16} \pi^2 f_{3,0} g_{0,2} g_{1,1} \\
& - \frac{1}{32} \pi f_{0,2} f_{1,1} g_{3,0} - \frac{17}{288} \pi f_{1,1} f_{2,0}^2 g_{1,1} + \frac{1}{288} \pi f_{1,1} f_{2,0} g_{1,1}^2 + \frac{3}{8} \pi^2 \\
& f_{0,2} f_{1,1} g_{0,2} + \frac{17}{96} \pi f_{1,1} g_{0,2} g_{2,1} - \frac{1}{32} \pi f_{1,1}^2 g_{0,2} g_{1,1} - \frac{3}{16} \pi^2 g_{0,2} g_{1,1} g_{2,1} \\
& - \frac{3}{16} \pi^2 f_{1,2} g_{1,1} g_{2,0} - \frac{1}{96} \pi f_{0,2} f_{3,0} g_{1,1} + \frac{35}{288} \pi f_{0,2} f_{1,2} g_{1,1} \\
& + \frac{61}{144} \pi f_{0,2} f_{1,1} f_{2,0}^2 + \frac{35}{288} \pi f_{0,2}^2 f_{1,1} g_{1,1} + \frac{5}{144} \pi f_{0,2} f_{2,0} g_{2,1} \\
& + \frac{15}{16} \pi f_{0,2} f_{2,0} g_{0,3} + \frac{7}{48} \pi f_{0,2} f_{1,1} g_{0,2}^2 + \frac{9}{16} \pi g_{0,2} g_{2,0} g_{2,1} - \frac{5}{96} \pi f_{1,1} \\
& g_{0,2} g_{1,1} - \frac{17}{72} \pi f_{0,2} f_{2,0}^2 g_{2,0} - \frac{1}{16} \pi f_{0,3} f_{2,0} g_{2,0} - \frac{1}{12} \pi f_{0,2} g_{1,2} g_{2,0} \\
& - \frac{1}{12} \pi f_{0,2} f_{2,1} g_{2,0} + \frac{3}{32} \pi f_{2,0} g_{1,1}^2 g_{2,0} + \frac{3}{16} \pi f_{2,0} g_{1,1} g_{2,0} + \frac{13}{288} \pi f_{0,2} \\
& g_{1,1} g_{2,0} + \frac{17}{96} \pi f_{1,1} f_{2,0} f_{2,1} - \frac{1}{32} \pi f_{1,1}^2 f_{2,0} g_{0,2} + \frac{9}{16} \pi^2 f_{0,2} f_{1,1} f_{3,0} \\
& + \frac{3}{32} \pi f_{0,2} g_{0,2} g_{1,1}^2 + \frac{1}{3} \pi f_{2,0} g_{0,2} g_{3,0} - \frac{9}{8} \pi^2 f_{2,0} f_{3,0} g_{2,0} \\
& + \frac{9}{8} \pi^2 f_{0,2} g_{0,2} g_{0,3} + \frac{3}{16} \pi^2 f_{0,2} f_{1,1} f_{1,2} + \frac{9}{8} \pi^2 f_{0,2} f_{3,0} g_{0,2} \\
& + \frac{1}{12} \pi f_{0,3} f_{1,1} g_{1,1} - \frac{1}{32} \pi f_{0,3} g_{1,1} g_{2,0} + \frac{3}{8} \pi^2 f_{0,2} f_{1,2} g_{0,2} \\
& - \frac{17}{96} \pi f_{2,0} f_{3,0} g_{1,1} - \frac{17}{48} \pi f_{2,0} f_{2,1} g_{2,0} - \frac{1}{18} \pi f_{0,2} f_{2,0}^2 g_{0,2} + \frac{7}{96} \pi f_{0,2}
\end{aligned}$$

$$\begin{aligned}
& f_{1,1}^2 g_{0,2} + \frac{35}{72} \pi f_{0,2}^2 f_{2,0} g_{0,2} + \frac{3}{16} \pi^2 g_{0,2} g_{1,1}^2 g_{2,0} + \frac{3}{16} \pi^2 f_{1,1} f_{1,2} f_{2,0} \\
& + \frac{35}{72} \pi f_{0,2}^2 f_{1,1} f_{2,0} - \frac{5}{144} \pi f_{0,2}^2 g_{1,1} g_{2,0} + \frac{3}{16} \pi^2 f_{0,2} f_{1,1} g_{2,1} \\
& + \frac{9}{16} \pi^2 f_{0,2} f_{1,1} g_{0,3} + \frac{5}{32} \pi f_{1,1} g_{0,2} g_{0,3} - \frac{3}{8} \pi^2 f_{1,2} f_{2,0} g_{2,0} \\
& - \frac{29}{288} \pi f_{2,0} g_{1,1} g_{2,1} - \frac{9}{16} \pi^2 g_{0,2} g_{0,3} g_{1,1} - \frac{1}{96} \pi f_{2,0} g_{0,3} g_{1,1} \\
& + \frac{31}{96} \pi f_{1,1} f_{3,0} g_{0,2} + \frac{1}{32} \pi f_{0,2} f_{1,1} g_{1,2} + \frac{3}{16} \pi^2 f_{1,1} f_{2,0} g_{2,1} \\
& - \frac{9}{8} \pi^2 f_{2,0} g_{0,3} g_{2,0} - \frac{1}{96} \pi f_{1,1} f_{2,0} g_{1,2} - \frac{3}{8} \pi^2 f_{2,0} g_{2,0} g_{2,1} \\
& - \frac{3}{16} \pi^2 f_{1,2} g_{0,2} g_{1,1} + \frac{3}{8} \pi^2 f_{0,2} g_{0,2} g_{2,1} + \frac{3}{16} \pi^2 f_{0,2} f_{1,1}^2 f_{2,0} \\
& - \frac{5}{12} \pi f_{2,0} f_{2,1} g_{0,2} + \frac{17}{96} \pi g_{0,2} g_{1,1} g_{1,2} + \frac{31}{96} \pi g_{0,2} g_{1,1} g_{3,0} \\
& + \frac{1}{12} \pi f_{1,1} g_{1,1} g_{3,0} + \frac{1}{96} \pi f_{1,1} f_{2,0} g_{3,0} + \frac{9}{16} \pi^2 f_{1,1} f_{2,0} g_{0,3} \\
& + \frac{9}{16} \pi^2 f_{1,1} f_{2,0} f_{3,0} - \frac{5}{12} \pi f_{2,0} g_{0,2} g_{1,2} + \frac{1}{96} \pi f_{1,1} f_{1,2} g_{0,2} \\
& + \frac{1}{96} \pi f_{0,3} g_{0,2} g_{1,1} - \frac{1}{16} \pi f_{0,2} g_{0,2} g_{3,0} + \frac{1}{3} \pi f_{0,3} f_{2,0} g_{0,2} + \frac{1}{8} \pi f_{0,2} g_{0,2} \\
& g_{2,0}^2 - \frac{17}{48} \pi f_{0,2} g_{0,2} g_{1,2} + \frac{35}{48} \pi f_{0,2} f_{0,3} g_{0,2} - \frac{5}{8} \pi g_{0,2} g_{1,1} g_{2,0}^2 \\
& - \frac{17}{24} \pi f_{2,0} g_{0,2} g_{2,0}^2 + \frac{1}{12} \pi f_{1,1} f_{2,0} g_{0,2}^2 + \frac{65}{144} \pi f_{0,2} f_{1,2} f_{2,0} \\
& - \frac{13}{288} \pi f_{0,2} g_{1,1} g_{2,1} + \frac{1}{288} \pi f_{0,2} f_{1,1} g_{1,1}^2 + \frac{5}{32} \pi f_{0,2} g_{0,3} g_{1,1} \\
& + \frac{41}{288} \pi f_{2,0} g_{0,2} g_{1,1}^2 + \frac{1}{36} \pi f_{2,0}^2 g_{0,2} g_{1,1} - \frac{7}{48} \pi f_{0,2} f_{2,1} g_{0,2} \\
& + \frac{35}{48} \pi f_{2,0} g_{2,0} g_{3,0} + \frac{1}{32} \pi f_{2,1} g_{1,1} g_{2,0} + \frac{13}{96} \pi g_{1,1} g_{1,2} g_{2,0} \\
& - \frac{17}{96} \pi f_{1,1} g_{1,1} g_{2,0}^2 - \frac{1}{32} \pi f_{1,1}^2 g_{1,1} g_{2,0} - \frac{7}{48} \pi f_{2,0} g_{1,2} g_{2,0} \\
& + \frac{31}{96} g_{2,0} \pi f_{1,1} f_{3,0} + \frac{5}{32} g_{2,0} \pi f_{1,1} g_{0,3} + \frac{17}{96} g_{2,0} \pi f_{1,1} g_{2,1} \\
& + \frac{1}{16} \pi f_{0,2} f_{1,1} g_{2,0}^2 - \frac{1}{12} \pi f_{1,1} f_{2,0} g_{2,0}^2 + \frac{35}{96} \pi f_{0,2} f_{0,3} f_{1,1} \\
& + \frac{13}{96} \pi f_{0,2} f_{1,1} f_{2,1} + \frac{1}{96} g_{2,0} \pi f_{0,2} f_{1,1}^2 - \frac{3}{32} g_{2,0} \pi f_{1,1}^2 f_{2,0}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{96} g_{2,0} \pi f_{1,1} f_{1,2} - \frac{3}{8} \pi^2 f_{0,2} g_{0,2}^2 g_{1,1} + \frac{17}{48} \pi f_{0,2} f_{2,0} f_{3,0} \\
& - \frac{3}{16} \pi^2 g_{1,1} g_{2,0} g_{2,1} - \frac{31}{48} \pi g_{0,2}^2 g_{1,1} g_{2,0} - \frac{3}{8} \pi^2 f_{1,1} f_{2,0}^2 g_{2,0} - \frac{5}{72} \pi \\
& f_{0,2}^2 f_{2,0} g_{2,0} + \frac{31}{96} \pi f_{0,3} f_{1,1} f_{2,0} - \frac{1}{6} \pi f_{2,0} g_{0,2}^2 g_{2,0} - \frac{9}{16} \pi^2 g_{0,3} g_{1,1} g_{2,0} \\
& - \frac{9}{16} \pi^2 f_{3,0} g_{1,1} g_{2,0} - \frac{1}{96} \pi f_{2,1} g_{0,2} g_{1,1} + \frac{7}{48} \pi f_{1,2} g_{0,2} g_{2,0} \\
& + \frac{19}{288} \pi f_{1,2} f_{2,0} g_{1,1} + \frac{61}{48} \pi f_{3,0} g_{0,2} g_{2,0} + \frac{1}{12} \pi f_{1,3} g_{1,1} - \frac{1}{8} \pi g_{1,3} g_{2,0} \\
& - \frac{1}{12} \pi f_{2,2} g_{2,0} - \frac{5}{6} \pi f_{4,0} g_{2,0} + \frac{3}{8} \pi^2 f_{2,0}^2 g_{2,0}^2 - \frac{5}{24} \pi g_{2,0} g_{3,1} \\
& + \frac{5}{24} \pi g_{0,4} g_{1,1} + \frac{1}{4} \pi f_{2,0} g_{0,2}^3 - \frac{1}{24} \pi g_{1,1} g_{2,2} - \frac{1}{8} \pi g_{1,1} g_{4,0} \\
& - \frac{1}{32} \pi g_{1,2} f_{3,0} + \frac{3}{32} \pi^2 g_{1,1}^2 g_{2,0}^2 + \frac{1}{32} \pi f_{0,3} g_{2,1} + \frac{5}{32} \pi f_{0,3} f_{1,2} \\
& + \frac{15}{32} \pi f_{0,3} g_{0,3} + \frac{3}{32} \pi^2 f_{1,2}^2 + \frac{27}{32} \pi^2 f_{3,0}^2 + \frac{27}{32} \pi^2 g_{0,3}^2 + \frac{3}{32} \pi^2 g_{2,1}^2 \\
& + \frac{1}{8} \pi f_{1,4} + \frac{1}{8} \pi f_{3,2} + \frac{5}{8} \pi f_{5,0} + \frac{5}{8} \pi g_{0,5} + \frac{1}{8} \pi g_{2,3} + \frac{1}{8} \pi g_{4,1}
\end{aligned}$$

```

> L[1]:=(1/4)*Pi*f[1, 2]+(1/4)*Pi*g[2, 1]+(3/4)*Pi*f[3, 0]+(3/4)*
Pi*g[0, 3]-(1/4)*Pi*g[0, 2]*g[1, 1]+(1/4)*Pi*f[0, 2]*f[1, 1]+
(1/4)*Pi*f[1, 1]*f[2, 0]-(1/4)*Pi*g[1, 1]*g[2, 0]-(1/2)*f[2, 0]*g
[2, 0]*Pi+(1/2)*Pi*f[0, 2]*g[0, 2]:
> L[2]:=(1/12)*Pi*f[1, 3]*g[1, 1]-(1/8)*Pi*g[1, 3]*g[2, 0]-(1/12)*
Pi*f[2, 2]*g[2, 0]-(5/6)*Pi*f[4, 0]*g[2, 0]+(3/32)*Pi^2*f[1, 2]
^2+(27/32)*Pi^2*f[3, 0]^2+(27/32)*Pi^2*g[0, 3]^2+(3/32)*Pi^2*g[2,
1]^2+(1/8)*Pi*f[1, 4]+(1/8)*Pi*f[3, 2]+(5/8)*Pi*f[5, 0]+(5/8)*Pi*
g[0, 5]+(1/8)*Pi*g[2, 3]+(1/8)*Pi*g[4, 1]-(1/36)*Pi*f[0, 2]*f[2,
0]*g[1, 1]*g[2, 0]+(35/96)*Pi*g[1, 1]*g[2, 0]*g[3, 0]+(3/8)*Pi^2*
f[2, 0]*g[1, 1]*g[2, 0]^2+(3/8)*Pi^2*f[2, 0]^2*g[2, 0]^2-(5/24)*
Pi*g[2, 0]*g[3, 1]+(5/24)*Pi*g[0, 4]*g[1, 1]+(1/4)*Pi*f[2, 0]*g
[0, 2]^3-(1/24)*Pi*g[1, 1]*g[2, 2]-(1/8)*Pi*g[1, 1]*g[4, 0]-
(1/32)*Pi*g[1, 2]*f[3, 0]+(3/32)*Pi^2*g[1, 1]^2*g[2, 0]^2+(1/32)*
Pi*f[0, 3]*g[2, 1]+(5/32)*Pi*f[0, 3]*f[1, 2]+(15/32)*Pi*f[0, 3]*g
[0, 3]-(13/288)*Pi*g[1, 1]^3*g[2, 0]+(11/16)*Pi*g[0, 2]*g[0, 3]*g
[2, 0]+(11/24)*Pi*f[0, 2]*g[0, 2]^2*g[2, 0]+(5/24)*Pi*f[0, 2]*f
[1, 3]-(13/48)*Pi*g[2, 0]^3*g[1, 1]+(3/32)*Pi*f[0, 3]*f[3, 0]-
(13/24)*Pi*g[2, 0]^3*f[2, 0]+(1/16)*Pi*f[1, 2]*g[2, 0]^2+(13/16)*
Pi*f[3, 0]*g[2, 0]^2+(7/96)*Pi*f[1, 1]^3*f[2, 0]+(3/16)*Pi*g[0,
3]*g[2, 0]^2+(13/48)*Pi*g[2, 0]^2*g[2, 1]+(7/96)*Pi*f[0, 2]*f[1,
1]^3+(7/96)*Pi*f[1, 1]^2*f[1, 2]-(9/16)*Pi^2*f[3, 0]*g[0, 2]*g[1,
1]-(13/36)*Pi*f[2, 0]^3*g[2, 0]-(1/4)*Pi*f[2, 0]*g[4, 0]-(15/32)*
Pi*g[3, 0]*f[3, 0]-(11/12)*Pi*f[4, 0]*g[0, 2]-(1/6)*Pi*f[2, 2]*g
[0, 2]+(35/144)*Pi*f[0, 2]^2*f[1, 2]-(1/32)*Pi*f[0, 2]*f[1, 1]*g
[3, 0]-(17/288)*Pi*f[1, 1]*f[2, 0]^2*g[1, 1]+(1/288)*Pi*f[1, 1]*f
[2, 0]*g[1, 1]^2+(3/8)*Pi^2*f[0, 2]^2*f[1, 1]*g[0, 2]+(3/16)*
Pi^2*f[0, 2]*f[1, 1]^2*f[2, 0]+(5/32)*Pi*f[2, 1]*f[3, 0]-(5/12)*
Pi*f[2, 0]*f[2, 1]*g[0, 2]+(7/12)*Pi*f[0, 2]*g[0, 2]^3-(5/32)*Pi*
g[3, 0]*g[2, 1]-(1/32)*Pi*g[3, 0]*f[1, 2]+(17/96)*Pi*g[0, 2]*g[1,

```

$1] * g[1, 2] + (31/96) * \text{Pi} * g[0, 2] * g[1, 1] * g[3, 0] + (1/12) * \text{Pi} * f[1, 1] * g$
 $[1, 1] * g[3, 0] + (1/96) * \text{Pi} * f[1, 1] * f[2, 0] * g[3, 0] + (35/48) * \text{Pi} * f[0,$
 $2] ^2 * g[0, 3] + (9/16) * \text{Pi} ^2 * f[1, 1] * f[2, 0] * g[0, 3] + (9/16) * \text{Pi} ^2 * f[1,$
 $1] * f[2, 0] * f[3, 0] - (5/12) * \text{Pi} * f[2, 0] * g[0, 2] * g[1, 2] + (1/96) * \text{Pi} * f$
 $[1, 1] * f[1, 2] * g[0, 2] + (1/96) * \text{Pi} * f[0, 3] * g[0, 2] * g[1, 1] - (1/16) *$
 $\text{Pi} * f[0, 2] * g[0, 2] * g[3, 0] + (1/3) * \text{Pi} * f[0, 3] * f[2, 0] * g[0, 2] + (1/8)$
 $* \text{Pi} * f[0, 2] * g[0, 2] * g[2, 0] ^2 - (17/48) * \text{Pi} * f[0, 2] * g[0, 2] * g[1, 2] +$
 $(35/48) * \text{Pi} * f[0, 2] * f[0, 3] * g[0, 2] - (5/8) * \text{Pi} * g[0, 2] * g[1, 1] * g[2,$
 $0] ^2 - (1/4) * \text{Pi} * f[2, 0] ^3 * g[0, 2] + (1/4) * \text{Pi} * f[0, 4] * g[0, 2] - (17/24) *$
 $\text{Pi} * f[2, 0] * g[0, 2] * g[2, 0] ^2 + (1/12) * \text{Pi} * f[1, 1] * f[2, 0] * g[0, 2] ^2 +$
 $(13/288) * \text{Pi} * g[1, 1] ^2 * g[2, 1] + (65/144) * \text{Pi} * f[0, 2] * f[1, 2] * f[2, 0]$
 $- (13/288) * \text{Pi} * f[0, 2] * g[1, 1] * g[2, 1] + (9/16) * \text{Pi} ^2 * f[3, 0] * g[2, 1] +$
 $(1/32) * \text{Pi} * g[1, 2] * f[1, 2] + (1/288) * \text{Pi} * f[0, 2] * f[1, 1] * g[1, 1] ^2 +$
 $(5/96) * \text{Pi} * g[0, 3] * g[1, 1] ^2 + (1/288) * \text{Pi} * f[1, 2] * g[1, 1] ^2 + (1/96) *$
 $\text{Pi} * f[3, 0] * g[1, 1] ^2 + (3/32) * \text{Pi} ^2 * f[1, 1] ^2 * f[2, 0] ^2 + (3/32) * \text{Pi} ^2 *$
 $f[0, 2] ^2 * f[1, 1] ^2 - (1/12) * \text{Pi} * f[1, 1] * g[3, 1] + (3/32) * \text{Pi} * f[2, 1] * f$
 $[1, 2] + (5/32) * \text{Pi} * f[0, 2] * g[0, 3] * g[1, 1] + (41/288) * \text{Pi} * f[2, 0] * g[0,$
 $2] * g[1, 1] ^2 + (1/36) * \text{Pi} * f[2, 0] ^2 * g[0, 2] * g[1, 1] + (27/16) * \text{Pi} ^2 * f$
 $[3, 0] * g[0, 3] - (7/48) * \text{Pi} * f[0, 2] * f[2, 1] * g[0, 2] + (35/48) * \text{Pi} * f[2,$
 $0] * g[2, 0] * g[3, 0] + (1/32) * \text{Pi} * f[2, 1] * g[1, 1] * g[2, 0] + (13/96) * \text{Pi} * g$
 $[1, 1] * g[1, 2] * g[2, 0] - (17/96) * \text{Pi} * f[1, 1] * g[1, 1] * g[2, 0] ^2 -$
 $(1/32) * \text{Pi} * f[1, 1] ^2 * g[1, 1] * g[2, 0] - (7/48) * \text{Pi} * f[2, 0] * g[1, 2] * g$
 $[2, 0] + (31/96) * g[2, 0] * \text{Pi} * f[1, 1] * f[3, 0] + (5/32) * g[2, 0] * \text{Pi} * f[1,$
 $1] * g[0, 3] + (17/96) * g[2, 0] * \text{Pi} * f[1, 1] * g[2, 1] + (1/16) * \text{Pi} * f[0, 2] * f$
 $[1, 1] * g[2, 0] ^2 - (1/12) * \text{Pi} * f[1, 1] * f[2, 0] * g[2, 0] ^2 + (35/96) * \text{Pi} * f$
 $[0, 2] * f[0, 3] * f[1, 1] + (13/96) * \text{Pi} * f[0, 2] * f[1, 1] * f[2, 1] + (1/96) *$
 $g[2, 0] * \text{Pi} * f[0, 2] * f[1, 1] ^2 - (3/32) * g[2, 0] * \text{Pi} * f[1, 1] ^2 * f[2, 0] +$
 $(1/96) * g[2, 0] * \text{Pi} * f[1, 1] * f[1, 2] - (3/8) * \text{Pi} ^2 * f[0, 2] * g[0, 2] ^2 * g$
 $[1, 1] + (17/48) * \text{Pi} * f[0, 2] * f[2, 0] * f[3, 0] + (5/48) * \text{Pi} * f[0, 2] ^2 * f$
 $[3, 0] - (3/16) * \text{Pi} ^2 * g[1, 1] * g[2, 0] * g[2, 1] - (31/48) * \text{Pi} * g[0, 2] ^2 * g$
 $[1, 1] * g[2, 0] + (7/24) * \text{Pi} * f[1, 3] * f[2, 0] - (3/8) * \text{Pi} ^2 * f[1, 1] * f[2,$
 $0] ^2 * g[2, 0] - (1/32) * \text{Pi} * f[2, 1] * g[2, 1] - (5/72) * \text{Pi} * f[0, 2] ^2 * f[2,$
 $0] * g[2, 0] + (31/96) * \text{Pi} * f[0, 3] * f[1, 1] * f[2, 0] - (1/6) * \text{Pi} * f[2, 0] * g$
 $[0, 2] ^2 * g[2, 0] - (9/16) * \text{Pi} ^2 * g[0, 3] * g[1, 1] * g[2, 0] - (9/16) * \text{Pi} ^2 *$
 $f[3, 0] * g[1, 1] * g[2, 0] - (1/96) * \text{Pi} * f[2, 1] * g[0, 2] * g[1, 1] + (7/48) *$
 $\text{Pi} * f[1, 2] * g[0, 2] * g[2, 0] + (19/288) * \text{Pi} * f[1, 2] * f[2, 0] * g[1, 1] -$
 $(5/72) * \text{Pi} * f[2, 0] ^2 * g[2, 1] + (61/48) * \text{Pi} * f[3, 0] * g[0, 2] * g[2, 0] +$
 $(1/6) * \text{Pi} * f[2, 0] * g[2, 2] + (11/12) * \text{Pi} * f[2, 0] * g[0, 4] + (3/8) * \text{Pi} * f[2,$
 $0] * f[3, 1] + (17/96) * \text{Pi} * f[1, 1] * g[0, 2] * g[2, 1] - (1/32) * \text{Pi} * f[1, 1]$
 $^2 * g[0, 2] * g[1, 1] - (3/16) * \text{Pi} ^2 * g[0, 2] * g[1, 1] * g[2, 1] - (3/16) *$
 $\text{Pi} ^2 * f[1, 2] * g[1, 1] * g[2, 0] - (1/96) * \text{Pi} * f[0, 2] * f[3, 0] * g[1, 1] +$
 $(35/288) * \text{Pi} * f[0, 2] * f[1, 2] * g[1, 1] + (61/144) * \text{Pi} * f[0, 2] * f[1, 1] * f$
 $[2, 0] ^2 + (35/288) * \text{Pi} * f[0, 2] ^2 * f[1, 1] * g[1, 1] + (5/144) * \text{Pi} * f[0, 2]$
 $* f[2, 0] * g[2, 1] + (15/16) * \text{Pi} * f[0, 2] * f[2, 0] * g[0, 3] + (7/48) * \text{Pi} * f$
 $[0, 2] * f[1, 1] * g[0, 2] ^2 + (9/16) * \text{Pi} * g[0, 2] * g[2, 0] * g[2, 1] - (5/96)$
 $* \text{Pi} * f[1, 1] * g[0, 2] ^2 * g[1, 1] - (17/72) * \text{Pi} * f[0, 2] * f[2, 0] ^2 * g[2,$
 $0] - (1/16) * \text{Pi} * f[0, 3] * f[2, 0] * g[2, 0] - (1/12) * \text{Pi} * f[0, 2] * g[1, 2] * g$
 $[2, 0] - (1/12) * \text{Pi} * f[0, 2] * f[2, 1] * g[2, 0] + (3/32) * \text{Pi} * f[2, 0] * g[1,$
 $1] ^2 * g[2, 0] + (3/16) * \text{Pi} * f[2, 0] ^2 * g[1, 1] * g[2, 0] + (13/288) * \text{Pi} * f[0,$
 $2] * g[1, 1] ^2 * g[2, 0] + (17/96) * \text{Pi} * f[1, 1] * f[2, 0] * f[2, 1] + (1/32) *$
 $\text{Pi} * f[2, 1] * g[0, 3] + (3/16) * \text{Pi} ^2 * f[1, 2] * g[2, 1] - (1/32) * \text{Pi} * f[1, 1]$
 $^2 * f[2, 0] * g[0, 2] + (9/16) * \text{Pi} ^2 * f[1, 2] * g[0, 3] + (9/16) * \text{Pi} ^2 * f[0,$
 $2] * f[1, 1] * f[3, 0] + (3/32) * \text{Pi} * f[0, 2] * g[0, 2] * g[1, 1] ^2 + (1/3) * \text{Pi} * f$
 $[2, 0] * g[0, 2] * g[3, 0] - (9/8) * \text{Pi} ^2 * f[2, 0] * f[3, 0] * g[2, 0] + (9/8) *$
 $\text{Pi} ^2 * f[0, 2] * g[0, 2] * g[0, 3] - (3/32) * \text{Pi} * g[0, 3] * g[3, 0] + (3/16) *$
 $\text{Pi} ^2 * f[0, 2] * f[1, 1] * f[1, 2] + (13/24) * \text{Pi} * f[2, 0] ^2 * f[3, 0] + (9/16) *$
 $\text{Pi} ^2 * f[1, 2] * f[3, 0] + (9/8) * \text{Pi} ^2 * f[0, 2] * f[3, 0] * g[0, 2] + (1/12) *$
 $\text{Pi} * f[0, 3] * f[1, 1] * g[1, 1] - (1/32) * \text{Pi} * f[0, 3] * g[1, 1] * g[2, 0] +$
 $(3/8) * \text{Pi} ^2 * f[0, 2] * f[1, 2] * g[0, 2] - (5/24) * \text{Pi} * f[1, 1] * f[4, 0] +$
 $(1/24) * \text{Pi} * f[1, 1] * f[2, 2] + (1/8) * \text{Pi} * f[0, 4] * f[1, 1] + (13/72) * \text{Pi} * f$
 $[1, 1] * f[2, 0] ^3 + (35/144) * \text{Pi} * f[0, 2] ^3 * f[1, 1] + (3/32) * \text{Pi} ^2 * g[0,$
 $2] ^2 * g[1, 1] ^2 + (3/8) * \text{Pi} ^2 * f[0, 2] ^2 * g[0, 2] ^2 + (5/12) * \text{Pi} * g[0, 2]$

```

^2*g[2, 1]-(7/24)*Pi*g[0, 2]^3*g[1, 1]-(17/96)*Pi*f[2, 0]*f[3, 0]
*g[1, 1]-(17/48)*Pi*f[2, 0]*f[2, 1]*g[2, 0]-(1/18)*Pi*f[0, 2]*f
[2, 0]^2*g[0, 2]+(7/96)*Pi*f[0, 2]*f[1, 1]^2*g[0, 2]+(35/72)*Pi*f
[0, 2]^2*f[2, 0]*g[0, 2]+(3/16)*Pi^2*g[0, 2]*g[1, 1]^2*g[2, 0]-
(5/32)*Pi*g[0, 3]*g[1, 2]+(5/144)*Pi*f[0, 2]^2*g[2, 1]-(7/24)*Pi*
g[0, 2]*g[3, 1]+(3/16)*Pi^2*f[1, 1]*f[1, 2]*f[2, 0]+(1/12)*Pi*f
[0, 2]*g[2, 2]+(1/8)*Pi*f[0, 2]*f[3, 1]+(35/72)*Pi*f[0, 2]^2*f[1,
1]*f[2, 0]-(5/144)*Pi*f[0, 2]^2*g[1, 1]*g[2, 0]+(3/16)*Pi^2*f[0,
2]*f[1, 1]*g[2, 1]+(9/16)*Pi^2*f[0, 2]*f[1, 1]*g[0, 3]+(5/32)*Pi*
f[1, 1]*g[0, 2]*g[0, 3]-(3/8)*Pi^2*f[1, 2]*f[2, 0]*g[2, 0]-
(29/288)*Pi*f[2, 0]*g[1, 1]*g[2, 1]-(9/16)*Pi^2*g[0, 2]*g[0, 3]*g
[1, 1]-(1/96)*Pi*f[2, 0]*g[0, 3]*g[1, 1]+(1/4)*Pi*f[0, 2]*f[1, 1]
*g[0, 2]*g[2, 0]+(1/24)*Pi*f[1, 1]*f[2, 0]*g[0, 2]*g[2, 0]-
(11/48)*Pi*f[1, 1]*g[0, 2]*g[1, 1]*g[2, 0]-(3/8)*Pi^2*f[0, 2]*f
[1, 1]*f[2, 0]*g[2, 0]-(3/16)*Pi^2*f[0, 2]*f[1, 1]*g[1, 1]*g[2,
0]-(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[1, 1]*g[2, 0]+(1/16)*Pi*f[0, 2]*
f[1, 1]*f[2, 0]*g[1, 1]+(3/8)*Pi^2*f[0, 2]*f[1, 1]*f[2, 0]*g[0,
2]-(3/16)*Pi^2*f[0, 2]*f[1, 1]*g[0, 2]*g[1, 1]-(3/4)*Pi^2*f[0, 2]
*f[2, 0]*g[0, 2]*g[2, 0]-(3/8)*Pi^2*f[0, 2]*g[0, 2]*g[1, 1]*g[2,
0]-(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[0, 2]*g[1, 1]+(3/8)*Pi^2*f[2, 0]
*g[0, 2]*g[1, 1]*g[2, 0]-(11/72)*Pi*f[0, 2]*f[2, 0]*g[0, 2]*g[1,
1]+(31/96)*Pi*f[1, 1]*f[3, 0]*g[0, 2]+(11/36)*Pi*f[1, 2]*f[2, 0]
^2+(1/3)*Pi*f[2, 0]^2*g[0, 3]-(3/8)*Pi*g[0, 2]*g[1, 3]-(3/32)*Pi*
g[1, 2]*g[2, 1]+(1/32)*Pi*f[0, 2]*f[1, 1]*g[1, 2]+(5/6)*Pi*f[0,
2]*g[0, 4]+(3/16)*Pi^2*f[1, 1]*f[2, 0]*g[2, 1]+(9/16)*Pi^2*g[0,
2]*g[2, 1]-(9/8)*Pi^2*f[2, 0]*g[0, 3]*g[2, 0]-(1/96)*Pi*f[1, 1]*f
[2, 0]*g[1, 2]-(3/8)*Pi^2*f[2, 0]*g[2, 0]*g[2, 1]+(7/8)*Pi*g[0,
2]^2*g[0, 3]+(2/3)*Pi*f[3, 0]*g[0, 2]^2+(35/72)*Pi*f[0, 2]^3*g[0,
2]-(13/288)*Pi*g[0, 2]*g[1, 1]^3+(1/24)*Pi*f[1, 2]*g[0, 2]^2+
(3/32)*Pi*f[1, 1]^2*g[0, 3]+(1/32)*Pi*f[1, 1]^2*g[2, 1]+(13/96)*
Pi*f[1, 1]^2*f[3, 0]-(3/16)*Pi^2*f[1, 2]*g[0, 2]*g[1, 1]+(3/8)*
Pi^2*f[0, 2]*g[0, 2]*g[2, 1]:

```

Coordinate change

Equation (21)

```

> ff:=exp(a1*u+b1*v)-exp(a2*u+b2*v);
> ff:=simplify(subs({a1=0,b1=0},ff));
      ff := 1 - ea2u + b2v
(1.1.1)

```

```

> gg:=exp(a3*u+b3*v)-exp(a4*u+b4*v);
      gg := ea3u + b3v - ea4u + b4v
(1.1.2)

```

Equilibrium (0,0)

```

> simplify(subs({u=0,v=0},ff),subs({u=0,v=0},gg));
      0, 0
(1.1.3)

```

```

> Jac:=VectorCalculus[Jacobian]([ff,gg],[u,v]);
> J:=simplify(subs({u=0,v=0},Jac),symbolic);
      J :=  $\begin{bmatrix} -a2 & -b2 \\ a3 - a4 & b3 - b4 \end{bmatrix}$ 
(1.1.4)

```

```

> tr:=LinearAlgebra[Trace](J);
      tr := -a2 + b3 - b4
(1.1.5)

```

```

> a2subs:=a2=solve(tr,a2);
      a2subs := a2 = b3 - b4
(1.1.6)

```

```

> detJ:=subs(a2subs,LinearAlgebra[Determinant](J));
(1.1.7)

```

$$\det J := -(b3 - b4) b3 + (b3 - b4) b4 + b2 a3 - b2 a4 \quad (1.1.7)$$

> wsubs:=w=sqrt(detJ);

$$wsubs := w = \sqrt{-(b3 - b4) b3 + (b3 - b4) b4 + b2 a3 - b2 a4} \quad (1.1.8)$$

> fff:=simplify(subs(a2subs,subs({v=- (a1-a2)*u/(b1-b2) - v*w/(b1-b2)},ff/w)));

$$fff := \frac{1 - e^{\frac{u(b3 - b4)b1 - b2(a1u + vw)}{b1 - b2}}}{w} \quad (1.1.9)$$

> ggg:=subs(a2subs,subs({v=- (a1-a2)*u/(b1-b2) - v*w/(b1-b2)},-(a1-a2)*ff/w^2-(b1-b2)*gg/w^2));

$$ggg := -\frac{(a1 - b3 + b4) \left(1 - e^{\frac{(b3 - b4)u + b2 \left(-\frac{(a1 - b3 + b4)u}{b1 - b2} - \frac{vw}{b1 - b2} \right)}{w^2}} \right)}{w^2} \quad (1.1.10)$$

$$- \frac{1}{w^2} \left((b1 - b2) \left(e^{\frac{a3u + b3 \left(-\frac{(a1 - b3 + b4)u}{b1 - b2} - \frac{vw}{b1 - b2} \right)}{w^2}} \right) - e^{\frac{a4u + b4 \left(-\frac{(a1 - b3 + b4)u}{b1 - b2} - \frac{vw}{b1 - b2} \right)}{w^2}} \right)$$

Linear part after coordinate change

> fff:=simplify(subs({a1=0,b1=0},fff));

$$fff := \frac{1 - e^{vw}}{w} \quad (1.1.11)$$

> ggg:=simplify(subs({a1=0,b1=0},ggg));

$$ggg := \frac{1}{w^2} \left(b2 e^{\frac{a3 b2 u - b3^2 u + b3 b4 u + b3 v w}{b2}} - b2 e^{\frac{a4 b2 u - b3 b4 u + b4^2 u + b4 v w}{b2}} \right) \quad (1.1.12)$$

$$- (-1 + e^{vw}) (b3 - b4)$$

> simplify(coeftayl(fff,[u,v]=[0,0],[1,0]));

$$0 \quad (1.1.13)$$

> simplify(coeftayl(fff,[u,v]=[0,0],[0,1]));

$$-1 \quad (1.1.14)$$

> simplify(coeftayl(ggg,[u,v]=[0,0],[1,0]));

$$\frac{b2(a3 - a4) - (b3 - b4)^2}{w^2} \quad (1.1.15)$$

> simplify(subs(wsubs,%));

$$1 \quad (1.1.16)$$

> simplify(coeftayl(ggg,[u,v]=[0,0],[0,1]));

$$0 \quad (1.1.17)$$

Higher taylor coefficients after coordinate change

> for k from 2 to 5 do

 for i from 0 to k do

 ggco[i,k-i]:=simplify(coeftayl(ggg,[u,v]=[0,0],[i,k-i]));

 ffco[i,k-i]:=simplify(coeftayl(fff,[u,v]=[0,0],[i,k-i]));


```
end:
```

```
end:
```

```
> subf:=seq(seq(f[i,k-i]=ffco[i,k-i],i=0..k),k=2..5):
```

```
> subg:=seq(seq(g[i,k-i]=ggco[i,k-i],i=0..k),k=2..5):
```

Case distinctions for first and second focal value

```
> L1:=simplify(subs({subf,subg},L[1])):
```

```
> L1:=simplify(subs(wsubs,L1)):
```

```
> L2:=simplify(subs({subf,subg},L[2])):
```

```
> L2:=simplify(subs(wsubs,L2)):
```

```
> L1;
```

$$-\frac{1}{8} \frac{\pi \left((-a^3 + a^4) b^4 - b^2 a^4 \right) b^3 + a^3 b^2 (a^4 + b^4) (b^3 - b^4)}{\sqrt{b^2 (a^3 - a^4) - (b^3 - b^4)^2} b^2} \quad (1.2.1)$$

```
L1=0, case (a) = (i)
```

```
> subs(b3=b4,detJ);
```

$$a^3 b^2 - a^4 b^2 \quad (1.2.2)$$

```
> subs(b3=b4,L2);
```

$$0 \quad (1.2.3)$$

```
L1=0, case (b)
```

```
> b2sol:=solve(L1,b2);
```

$$b2sol := \frac{b^3 b^4 (a^3 - a^4)}{a^3 a^4 + a^3 b^4 - a^4 b^3} \quad (1.2.4)$$

```
> b2subs:=b2=b2sol;
```

$$b2subs := b^2 = \frac{b^3 b^4 (a^3 - a^4)}{a^3 a^4 + a^3 b^4 - a^4 b^3} \quad (1.2.5)$$

```
> DD:=denom(b2sol);
```

$$DD := a^3 a^4 + a^3 b^4 - a^4 b^3 \quad (1.2.6)$$

```
> NN:=numer(b2sol);
```

$$NN := b^3 b^4 (a^3 - a^4) \quad (1.2.7)$$

```
L2=0
```

```
> L2b2subs:=simplify(subs(b2subs,L2));
```

$$L2b2subs := -\frac{1}{288} \left((b^3 - b^4) (a^4 + b^4) (a^4 - b^3 + b^4) (a^3 - b^3) (a^3 - b^3 \right. \quad (1.2.8)$$

$$\left. + b^4) (a^3 b^4 - a^4 b^3)^2 \pi \right) /$$

$$\left(\sqrt{\frac{(a^3 b^4 - a^4 b^3) (-b^3^2 + (a^3 + 2 b^4) b^3 - b^4 (a^4 + b^4))}{-b^3 a^4 + a^3 (a^4 + b^4)}} b^3 (-b^3 a^4 \right.$$

$$\left. + a^3 (a^4 + b^4) b^4 \right)$$

```
(r1)
```

```
> subs(b4=-a4,b2subs);
```

$$b^2 = a^3 - a^4 \quad (1.2.9)$$

(r2)

$$\text{> subs (b3=a3 , b2subs) ;} \quad b2 = a3 - a4 \quad (1.2.10)$$

(v)

$$\text{> simplify (subs (b3=a4+b4 , b2subs)) ;} \quad b2 = b4 \quad (1.2.11)$$

(moreover, using trace condition)

$$\text{> subs (b3=a4+b4 , a2subs) ;} \quad a2 = a4 \quad (1.2.12)$$

(iv)

$$\text{> simplify (subs (b4=b3-a3 , b2subs)) ;} \quad b2 = b3 \quad (1.2.13)$$

(moreover, using trace condition)

$$\text{> subs (b4=b3-a3 , a2subs) ;} \quad a2 = a3 \quad (1.2.14)$$

$(a3 b4 - a4 b3) = 0$ yields $\det J = 0$

$$a3 b4 - a4 b3 \quad (1.2.15)$$

$$\text{> simplify (subs (b2subs , detJ)) ;} \quad \frac{(a3 b4 - a4 b3) (-b3^2 + (a3 + 2 b4) b3 - b4 (a4 + b4))}{-b3 a4 + a3 (a4 + b4)} \quad (1.2.16)$$

L1=0, case (c): DD=0, NN=0

$$\text{> DD;} \quad a3 a4 + a3 b4 - a4 b3 \quad (1.2.17)$$

$$\text{> NN;} \quad b3 b4 (a3 - a4) \quad (1.2.18)$$

(c1) $b3 = 0$

$$b3 = 0 \quad (1.2.19)$$

$$\text{> factor (subs (b3=0 , DD)) ;} \quad a3 (a4 + b4) \quad (1.2.20)$$

subcases of (c1):

(c1a) = (ii)

$$\text{> subs ({b3=0 , a3=0} , L2) ;} \quad 0 \quad (1.2.21)$$

(c1b)

$$\text{> simplify (subs ({b3=0 , b4=-a4} , L2)) ;} \quad -\frac{1}{288} \frac{(a4 + b2) (a3 - a4) (a3 - a4 - b2) a4^2 a3 \pi}{\sqrt{b2 (a3 - a4) - a4^2 b2}} \quad (1.2.22)$$

$(a4 + b2)$: subcase of (r2)

$$a4 + b2 \quad (1.2.23)$$

$(a3 - a4)$:

$$a3 - a4 \quad (1.2.24)$$

$$\text{> subs ({b3=0 , a3=a4} , detJ) ;}$$

$$-b^4 \quad (1.2.25)$$

$$(a^3 - a^4 - b^2): \text{subcase of (r1)} \quad a^3 - a^4 - b^2 \quad (1.2.26)$$

$$a^3: \text{subcase of (c1a)=(ii)} \quad a^3 \quad (1.2.27)$$

$$a^4: \text{subcase of (c1a)=(ii)} \quad a^4 \quad (1.2.28)$$

$$(c2) \\ > \text{factor(subs(b4=0, DD))}; \quad a^4 (a^3 - b^3) \quad (1.2.29)$$

$$\text{subcases of (c2):} \\ (c2a) = (iii) \\ > \text{subs}\{b4=0, a4=0\}, L2); \quad 0 \quad (1.2.30)$$

$$(c2b) b^4 = 0 \quad b^4 = 0 \quad (1.2.31)$$

$$> \text{simplify(subs}\{b4=0, b3=a3\}, L2)); \\ \frac{1}{288} \frac{(a^3 - b^2) (a^3 - a^4) (a^3 - a^4 - b^2) a^4 a^3^2 \pi}{\sqrt{b^2 (a^3 - a^4) - a^3^2} b^2} \quad (1.2.32)$$

$$(a^3 - b^2): \text{subcase of (r1)} \quad a^3 - b^2 \quad (1.2.33)$$

$$(a^3 - a^4): \quad a^3 - a^4 \quad (1.2.34)$$

$$> \text{subs}\{b4=0, a3=a4\}, \text{detJ}); \quad -b^3^2 \quad (1.2.35)$$

$$(a^3 - a^4 - b^2): \text{subcase of (r2)} \quad a^3 - a^4 - b^2 \quad (1.2.36)$$

$$a^3: \text{subcase of (c2a)=(iii)} \quad a^3 \quad (1.2.37)$$

$$a^4: \text{subcase of (c2a)=(iii)} \quad a^4 \quad (1.2.38)$$

First focal value for Proposition 8

$$> \text{simplify}(L1/\text{Pi} * 8 * \text{sqrt}(\text{detJ})); \\ - \frac{((-b^3 a^4 + a^3 (a^4 + b^4)) b^2 - b^3 b^4 (a^3 - a^4)) (b^3 - b^4)}{b^2} \quad (1.3.1)$$

$$> l1 := \text{subs}\{a2=a2-a1, a3=a3-a1, a4=a4-a1, b2=b2-b1, b3=b3-b1, b4=b4-b1\}, \%); \\ l1 := - \frac{1}{b^2 - b^1} (((- (b^3 - b^1) (a^4 - a^1) + (a^3 - a^1) (a^4 - a^1 + b^4 - b^1)) (b^2 - b^1)) (b^2 - b^1)) \quad (1.3.2)$$

$$\left[\left[-b_1 - (b_3 - b_1)(b_4 - b_1)(a_3 - a_4)(b_3 - b_4) \right] \right]$$